FEM-Design

Verification Examples

version 1.6
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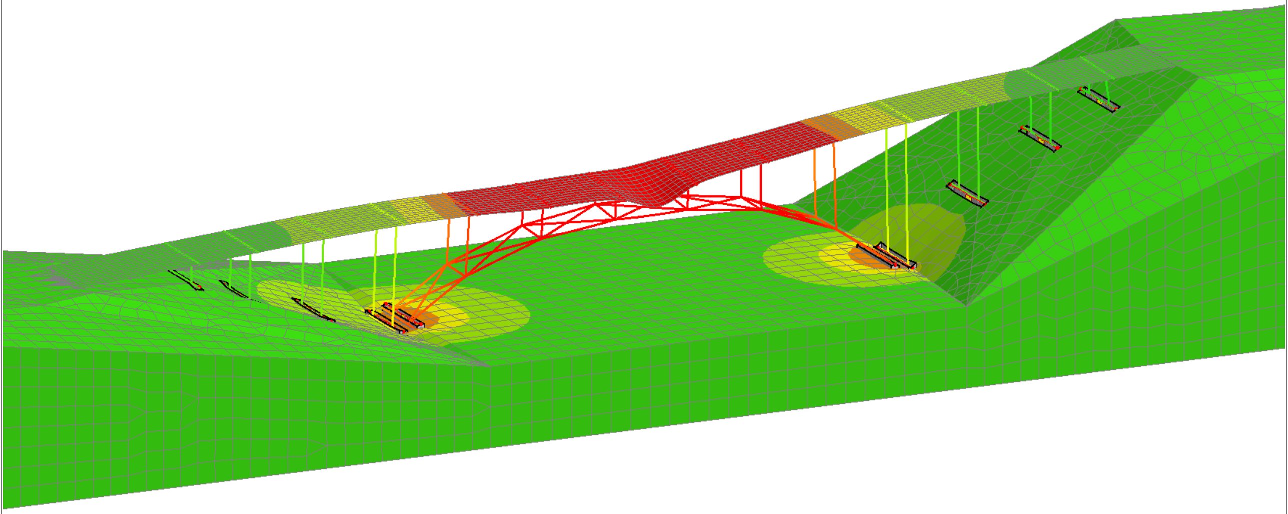
Motto:

„There is singularity between linear and nonlinear world."
(Dr. Imre Bojtár)



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Verification Examples

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Edited by
Zoltán I. Bocskai, Ph.D.

Contents

In this verification handbook we highlighted the analytical results with green and the finite element results with blue background for better comparison. The analytical closed formulas are highlighted with a black frame. The comparisons between the hand calculations and FEM-Design calculations are highlighted with yellow.

If the finite element mesh is not mentioned during the example it means that the automatically generated mesh was used.

WARNING:

We are continuously developing this verification book therefore some discrepancy in the numbering of the chapters or some missing examples can occur.

1 Linear static calculations

1.1 Beam with two point loading at one-third of its span

Fig. 1.1.1 left side shows the simple supported problem. The loads, the geometry and material properties are as follows:

Force	F = 150 kN
Length	L = 6 m
Cross section	Steel I beam HEA 300
The second moment of inertia in the relevant direction	$I_1 = 1.8264 \cdot 10^{-4}\ m^4$
The shear correction factor in the relevant direction	$\rho_2 = 0.21597$
The area of the cross section	$A = 112.53\ cm^2$
Young's modulus	E = 210 GPa
Shear modulus	G = 80.769 GPa

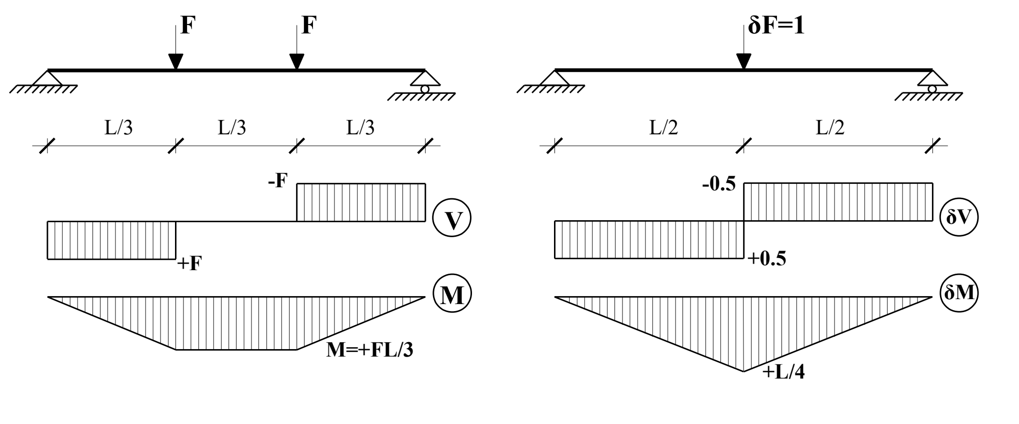

Figure 1.1.1 – The beam theory and the application of a virtual force

The deflection of the mid-span based on the hand calculation (based on virtual force theorem [1], see Fig. 1.1.1 right side also):

$$e=\frac{2M}{EI}\left[\frac{L}{3}\frac{2}{3}\frac{2}{3}\frac{L}{4}\frac{1}{2}+\frac{L}{6}\frac{2}{3}\frac{L}{4}+\frac{L}{6}\frac{1}{3}\frac{L}{4}\frac{1}{2}\right]+\frac{2F}{\rho GA}\left[0.5\frac{L}{3}\right]=\frac{23}{648}\frac{FL^3}{EI}+\frac{FL}{3\rho GA}$$

$$e=\frac{23}{648}\frac{150\cdot 6^3}{210000000\cdot 1.8264\cdot 10^{-4}}+\frac{150\cdot 6}{3\cdot 0.21597\cdot 80769000\cdot 0.011253}=0.03151\ \text{m}=31.51\ \text{mm}$$

The first part of this equation comes from the bending deformation and the second part comes from the consideration of the shear deformation as well, because FEM-Design is using Timoshenko beam theory (see the Scientific Manual).

The deflection and the bending moment at the mid-span based on the linear static calculation with three 2-noded beam elements (Fig. 1.1.2 and Fig. 1.1.3):

$e_{FEM} = 31.51\ \text{mm}$ and the bending moment $M_{FEM} = 300\ \text{kNm}$

The theoretical solution in this case (three 2-noded beam elements) must be equal to the finite element solution because with three beam elements the shape functions order coincides with the order of the theoretical function of the deflection (the solution of the differential equations).

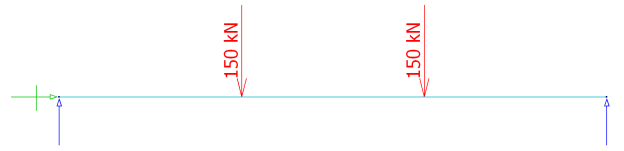

Figure 1.1.2 – The finite element model

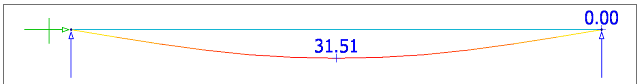

Figure 1.1.3 – The mid-span deflection [mm]

Therefore the difference between the results of the two calculations is zero.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/1.1 Beam with two point loading at one-third of its span.str

1.2 Calculation of a circular plate with concentrated force at its center

In this chapter a circular steel plate with a concentrated force at its center will be analyzed. First of all the maximum deflection (translation) of the plate will be calculated at its center and then the bending moments in the plate will be presented.

Two different boundary conditions will be applied at the edge of the plate. In the first case the edge is clamped (Case I.) and in the second case is simply supported (Case II.), see Fig. 1.2.1.

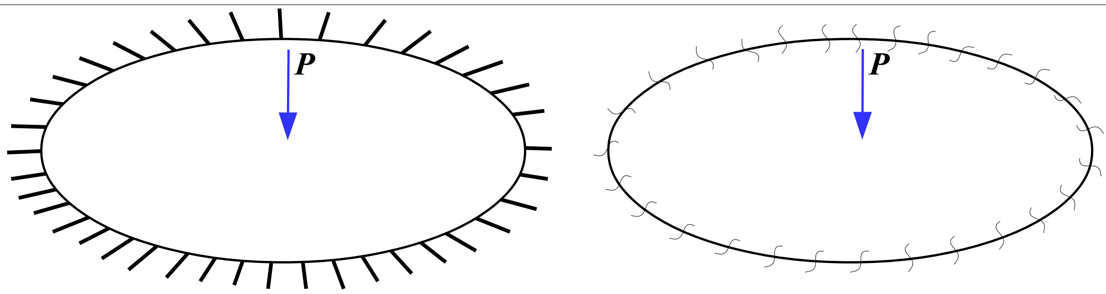

Figure 1.2.1 – Clamped (Case I.) and simply supported (Case II.) circular plate with concentrated force

The input parameters are as follows:

The concentrated force	P = 10 kN
The thickness of the plate	h = 0.05 m
The radius of the circular plate	R = 5 m
The elastic modulus	E = 210 GPa
The Poisson's ratio	ν = 0.3

The ratio between the diameter and the thickness is 2R/h = 200. It means that based on the geometry the shear deformation only have negligible effects on the maximum deflections. It is important because FEM-Design uses the Mindlin plate theory (considering the shear deformation, see Scientific Manual for more details), but in this case the solution of Kirchhoff's plate theory and the finite element result must be close to each other based on the mentioned ratio.

The analytical solution of Kirchhoff's plate theory is given in a closed form [2][3].

Case I.:

For the clamped case the maximum deflection at the center is:

$$w_{cl}=\frac{P R^2}{16\pi\left(\frac{E h^3}{12(1-\nu^2)}\right)}$$

The reaction force at the edge:

$$Q_r=\frac{P}{2\pi R}$$

And the bending moment in the tangential direction at the edge:

$$M_{cl}=\frac{P}{4\pi}$$

With the given input parameters the results based on the analytical and the finite element results (with the default finite element mesh size, see Fig. 1.2.2) are:

$$w_{cl}=\frac{10\cdot 5^2}{16\pi\left(\frac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}=0.002069\,\mathrm{m}=2.069\,\mathrm{mm}$$

$$w_{clFEM}=2.04\,\mathrm{mm}$$

$$Q_r=\frac{10}{2\pi 5}=0.318\frac{\mathrm{kN}}{\mathrm{m}}$$

$$Q_{rFEM}=0.318\frac{\mathrm{kN}}{\mathrm{m}}$$

$$M_{cl}=\frac{P}{4\pi}=\frac{10}{4\pi}=0.796\frac{\mathrm{kNm}}{\mathrm{m}}$$

$$M_{clFEM}=0.796\frac{\mathrm{kNm}}{\mathrm{m}}$$

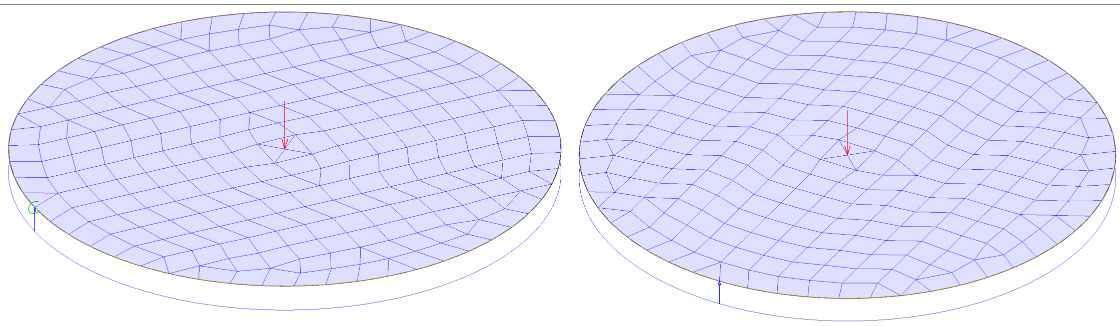

Figure 1.2.2 – The clamped (Case I.) and the simply supported (Case II.) plate with the default mesh

Case II.:

For the simply supported case the maximum deflection in the center is:

$$w_{ss}=\frac{PR^2}{16\pi\left(\frac{Eh^3}{12(1-\nu^2)}\right)}\left(\frac{3+\nu}{1+\nu}\right)$$

The reaction force at the edge:

$$Q_r=\frac{P}{2\pi R}$$

With the given input parameters the results based on the analytical and the finite element results (with the default finite element mesh size, see Fig. 1.2.2) are:

$$w_{ss}=\frac{10\cdot 5^2}{16\pi\left(\frac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}\left(\frac{3+0.3}{1+0.3}\right)=0.005252\,\mathrm{m}=5.252\,\mathrm{mm}$$

$$w_{ssFEM}=5.00\,\mathrm{mm}$$

$$Q_r=\frac{10}{2\pi 5}=0.318\frac{\mathrm{kN}}{\mathrm{m}}$$

$$Q_{rFEM}=0.318\frac{\mathrm{kN}}{\mathrm{m}}$$

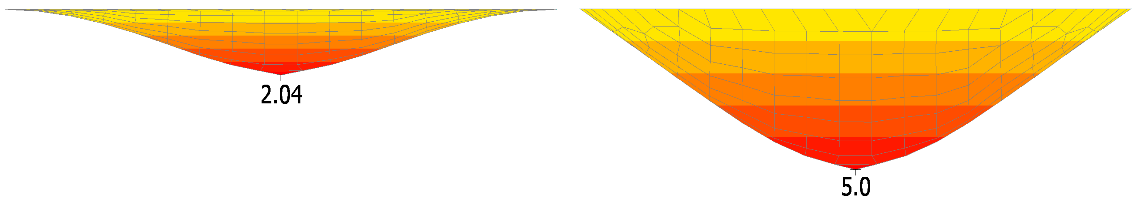

Figure 1.2.3 – The deflected shape of Case I. (clamped) and Case II. (simply supported) with the default mesh [mm]

Fig. 1.2.3 shows the two deflected shape in side view. The different boundary conditions are obvious based on the two different displacement shape. The differences between the analytical solutions and finite element solutions are less than 5% but the results could be more accurate if the applied mesh is more dense than the default size.

Based on the analytical solution the bending moments in plates under the concentrated loads are infinite. It means that if more and more dense mesh will be applied the bending moment under the concentrated load will be greater and greater. Thus the following diagram and table (Fig. 1.2.4 and Table 1.2.1) shows the convergence analysis of Case I. respect to the deflection and bending moment. The deflection converges to the analytical solution (w_{cl} = 2.07 mm) and the bending moment converges to infinite.

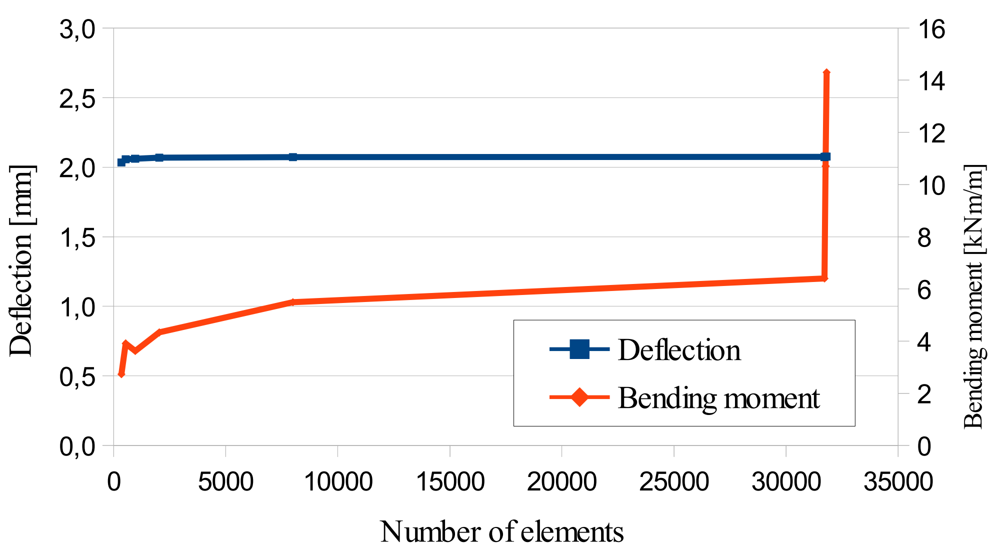

Figure 1.2.4 – Convergence analysis regarding to deflection and bending moment

Number of elements [pcs.]	Deflection [mm]	Bending moment [kNm/m]	Average element size [m]
341	2,034	2,73	0,5
533	2,057	3,91	0,4
957	2,060	3,62	0,3
2035	2,068	4,33	0,2
7994	2,072	5,49	0,1
31719	2,073	6,40	0,05
31772	2,075	10,70	Local refinement 1
31812	2,076	14,30	Local refinement 2

Table 1.2.1 – The convergence analysis

Download links to the example files:

Clamped:

http://download.strusoft.com/FEM-Design/inst170x/models/1.2 Calculation of a circular plate with concentrated force at its center clamped.str

Simply supported:

http://download.strusoft.com/FEM-Design/inst170x/models/1.2 Calculation of a circular plate with concentrated force at its center simplysup.str

1.3 A simply supported square plate with uniform load

In this example a simply supported concrete square plate will be analyzed. The external load is a uniform distributed load (see Fig. 1.3.1). We compare the maximum displacements and maximum bending moments of the analytical solution of Kirchhoff's plate theory and finite element results.

The input parameters are in this table:

The intensity of the uniform load	p = 40 kN/m^2
The thickness of the plate	h = 0.25 m
The edge of the square plate	a = 5 m
The elastic modulus	E = 30 GPa
Poisson's ratio	ν = 0.2

The ratio between the span and the thickness is a/h = 20. It means that based on the geometry the shear deformation may have effect on the maximum deflection. It is important because FEM-Design uses the Mindlin plate theory (considering the shear deformation, see Scientific Manual for more details), therefore in this case the results of Kirchhoff's theory and the finite element result could be different from each other due to the effect of shear deformations.

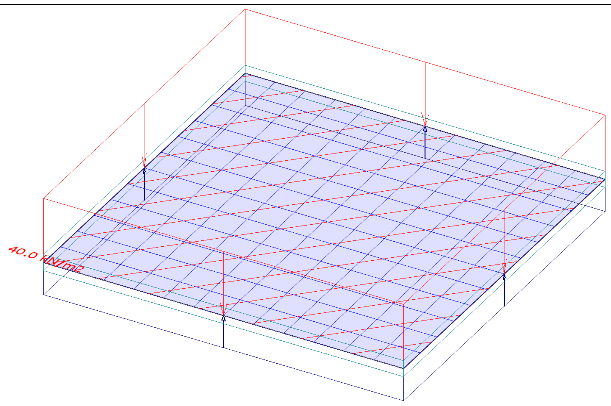

Figure 1.3.1 – The square plate with simply supported edges, uniform load and the default mesh

Based on Kirchhoff's plate theory [2][3] the maximum deflection is in the center of the simply supported square plate and its intensity can be given with the following closed form:

$$w_{max} = 0.00416 \frac{p\,a^4}{\left(\dfrac{E\,h^3}{12(1-\nu^2)}\right)}$$

The maximum bending moment in the plate if the Poisson's ratio ν = 0.2:

$$\boxed{M_{max} = 0.0469\, p\, a^2}$$

According to the input parameters and the analytical solutions the results of this problem are the following:

The deflection at the center of the plate:

$$w_{max} = 0.00416 \frac{40 \cdot 5^4}{\left(\frac{30000000 \cdot 0.25^3}{12\left(1-0.2^2\right)}\right)} = 0.002556\,\mathrm{m} = 2.556\,\mathrm{mm} \qquad w_{maxFEM} = 2.632\,\mathrm{mm}$$

The bending moment at the center of the plate:

$$M_{max} = 0.0469 \cdot 40 \cdot 5^2 = 46.9 \frac{\mathrm{kNm}}{\mathrm{m}} \qquad M_{maxFEM} = 45.97 \frac{\mathrm{kNm}}{\mathrm{m}}$$

Next to the analytical solutions the results of the FE calculations are also indicated (see Fig. 1.3.2 and 1.3.3). The difference is less than 3% and it also comes from the fact that FEM-Design considers the shear deformation (Mindlin plate theory).

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/1.3 A simply supported square plate with uniform load.str

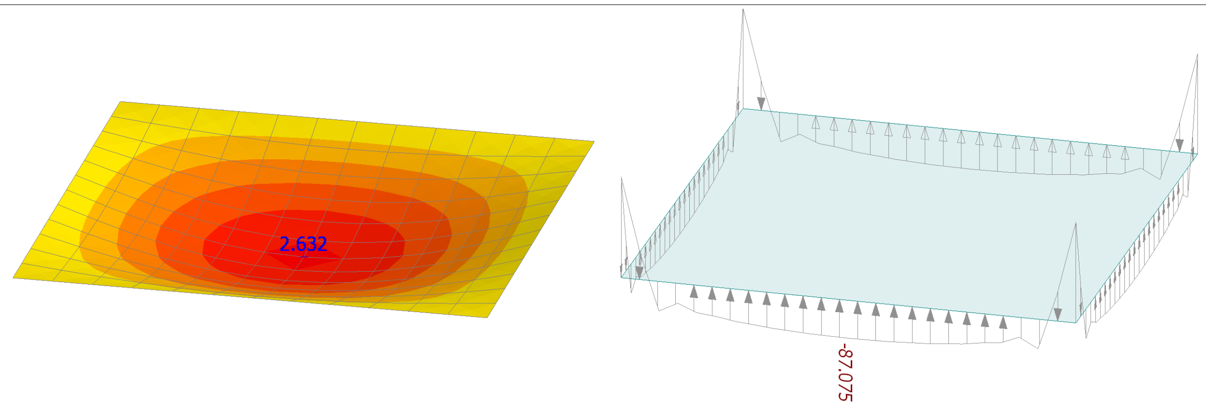

Figure 1.3.2 – The deflected shape [mm] *and the reaction forces* [kN/m] *with the default mesh*

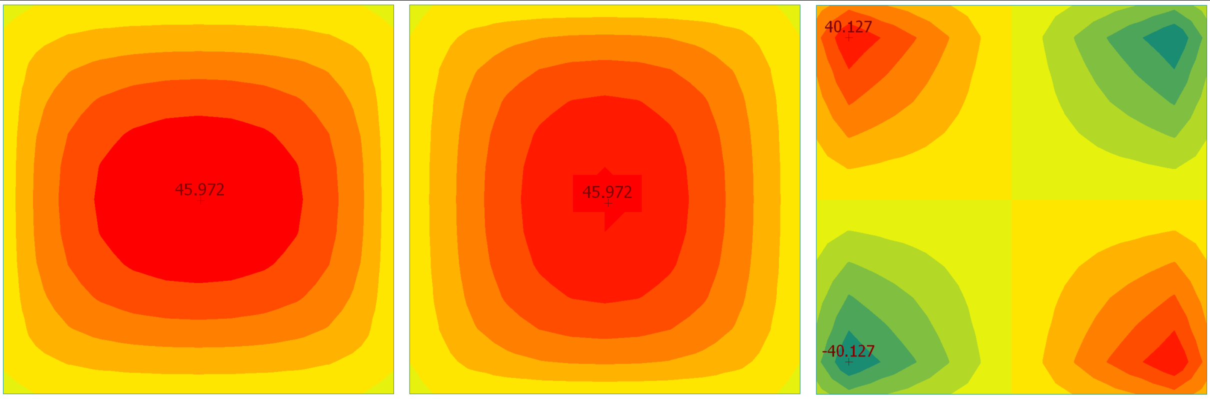

Figure 1.3.3 – The internal forces; $m_x – m_y – m_{xy}$ [kNm/m]

1.4 Peak smoothing of the bending moments in a flat slab

Let's consider a flat slab with 8m · 8m raster for the supporting columns (see Fig. 1.4.1). With the aim of Ref. [16] if the flat slab field assumed to infinite (with the proper consideration of the boundary conditions (line supports on the edges)) we can get “precise” results for the bending moment with three different consideration of the supporting effect of the columns. The load is a constant distributed load (p=20 kN/m^2). The thickness of the slab is 40 cm, the columns are 80cm/80cm, the Young's modulus is E_{cm}=31 GPa, the Poisson's ratio ν=0.167. We neglect the creep effect.

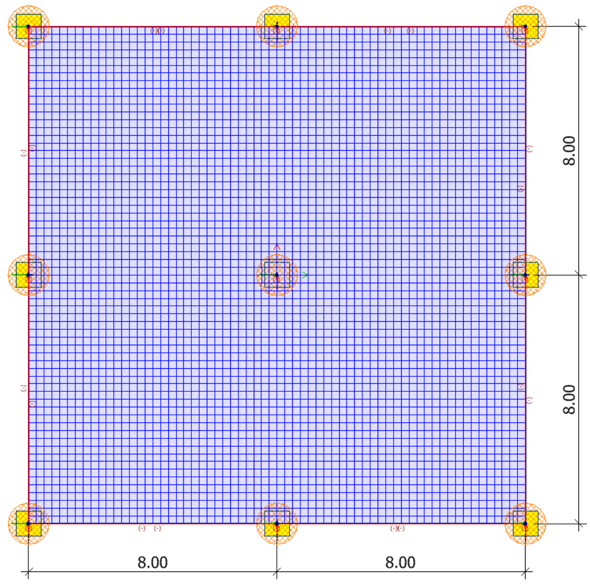

Figure 1.4.1 – The slab with 8m ·8m raster for the columns with constant distributed load

According to Ref. [16] the first modelling condition (Type a)) for a supporting column is a vertical point support (see Fig. 1.4.2). The reaction at this point support is:

$$R_a = p \cdot L \cdot L = 20 \cdot 8 \cdot 8 = 1280\,\text{kN}$$

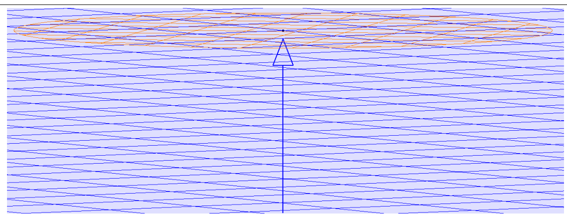

Figure 1.4.2 – Type a) supporting condition

According to Ref. [16] the second modelling condition (Type b)) for a supporting column is a constant distributed reaction along the cross-section of the column (see Fig. 1.4.3).

$$r_b = \frac{R_a}{d^2} = \frac{1280}{0.8^2} = 2000\,\mathrm{kN/m^2}$$

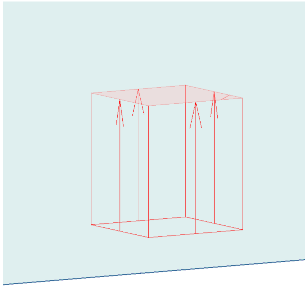

Figure 1.4.3 – Type b) supporting condition

According to Ref. [16] the third modelling condition (Type c)) for a supporting column is a constant distributed reaction (along the cross-section of the column) for half of the resultant reaction force and concentrated reactions at the corner of the column with half of the resultant reaction (one concentrated reaction represent the quarter of the half of the resultant reaction) (see Fig. 1.4.4).

$$r_c = \frac{r_b}{2} = \frac{2000}{2} = 1000\,\mathrm{kN/m^2} \quad ; \quad R_c = \frac{R_a}{2 \cdot 4} = \frac{1280}{2 \cdot 4} = 160\,\mathrm{kN}$$

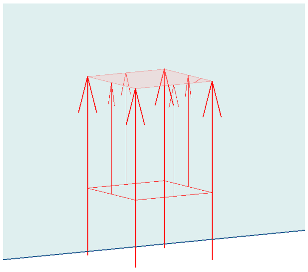

Figure 1.4.4 – Type c) supporting condition

In Ref. [16] there are results for the bending moment distribution of the slab (at the line along the columns and at middle line of the one slab field) for the three different types of the indicated column reaction conditions.

First of all we will modelling the exactly same column supporting conditions in FEM-Design (Type a), Type b) and Type c)) and then we will use the different peak smoothing options in FEM-Design with Type a) support condition.

Ref. [16] states that the moment distribution in case of Type b) and c) are closer to the reality. We will compare these results with FEM-Design different peak smoothing option results with the application of Type a) support condition.

We compared and analyzed the m_y values by the different support conditions/options and indicated the bending moments at the section above the columns and at the section at the middle of one slab field (see Fig. 1.4.5).

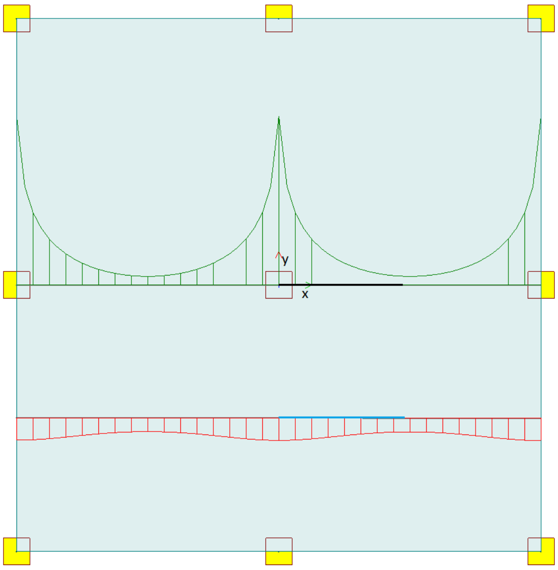

Figure 1.4.5 – The considered sections

In FEM-Design we used the following settings:

Calculation was performed with “fine” finite element group. The element size was 0.25 m on the slab (we didn't use any refinement, see Fig. 1.4.1 and 1.4.6). By the peak smoothing consideration we used the following settings (see Fig. 1.4.7).

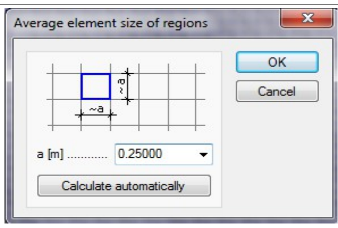

Figure 1.4.6 – The average element size setting

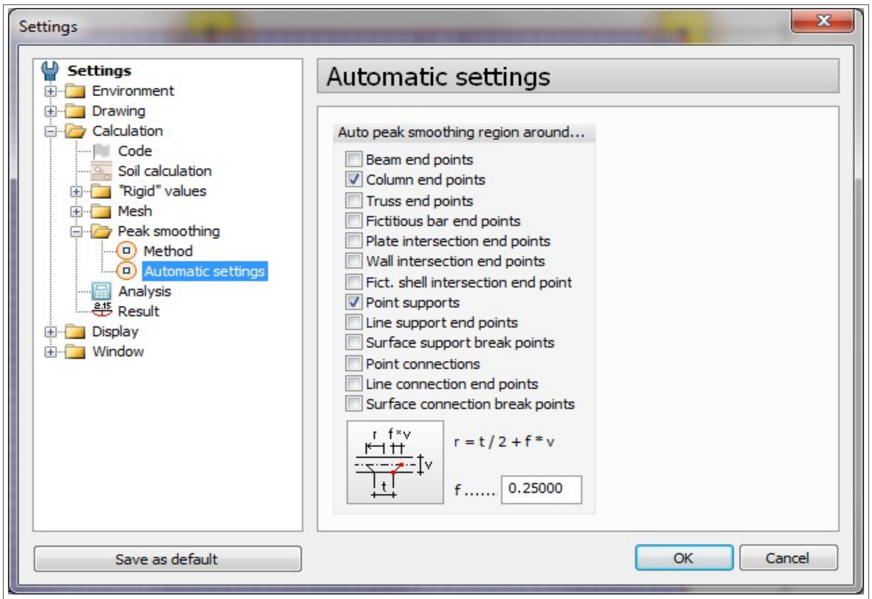

Figure 1.4.7 – The adjusted settings in this example

Fig. 1.4.8 shows the m_y bending moment results at the section at the middle of one slab field along the indicated light blue line (see Fig. 1.4.5).

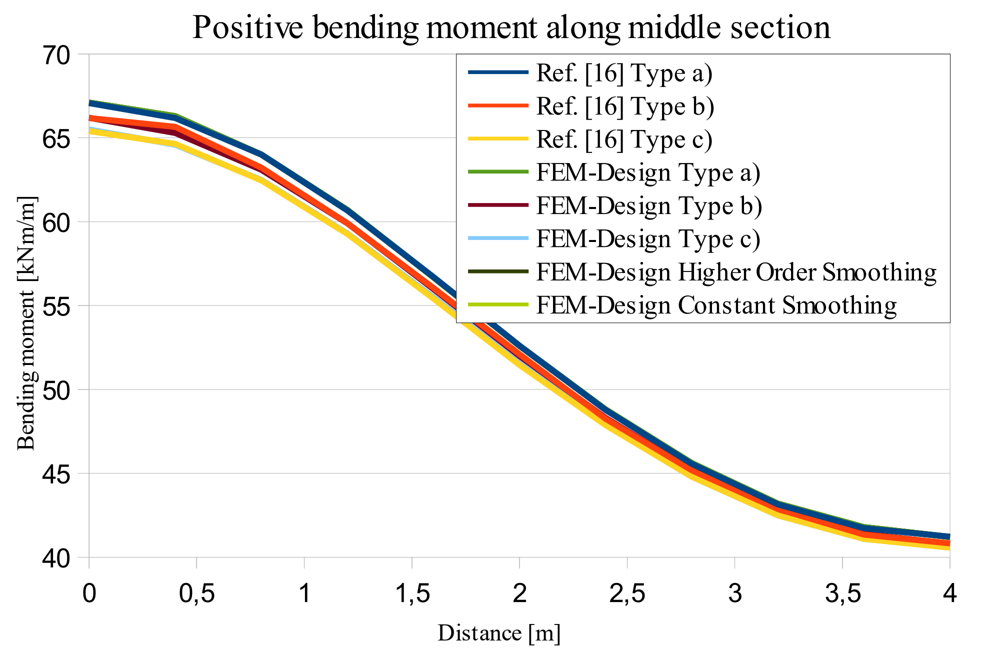

Figure 1.4.8

Fig. 1.4.9 shows the m_y bending moment results at the section above the columns along the indicated black line (see Fig. 1.4.5).

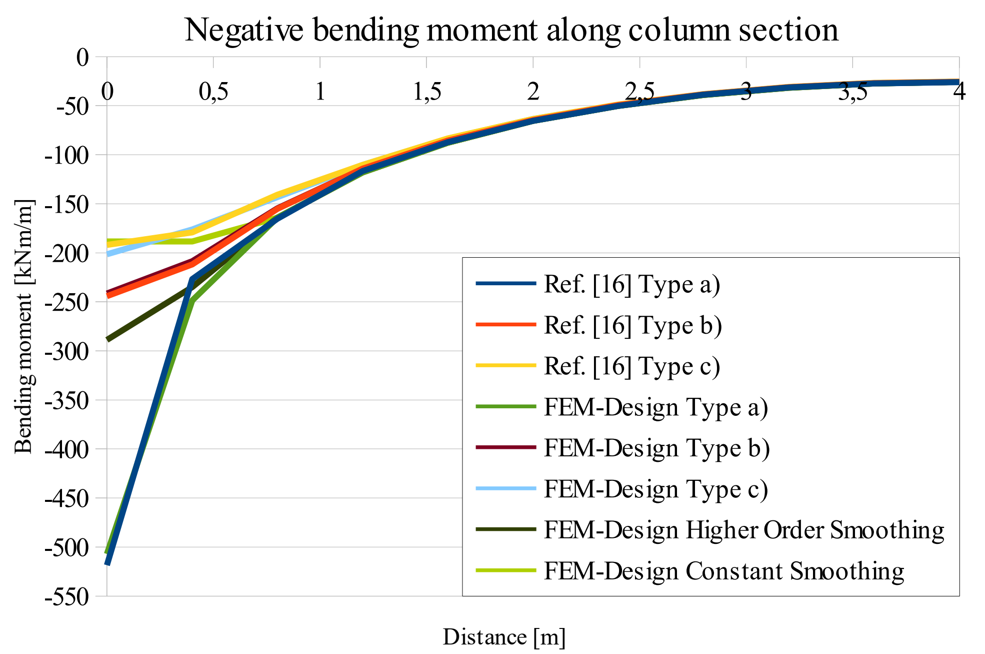

Figure 1.4.9

We can say that Ref. [16] results are identical with the FEM-Design results without using the peak smoothing functions in the program (see Fig. 1.4.8-9). From Fig. 1.4.8 it is obvious that the bending moment results at the section at the middle of one slab field is almost independent from the support condition type.

From Fig. 1.4.9 it is obvious that the negative moment results at the section above the columns is highly depend on the supporting condition type. The theoretical solution above the support by Type a) would be infinite if we would use infinitely small element size (see also the bending moment result in Chapter 1.2 under concentrated point load).

If we check the Higher Order Smoothing results in FEM-Design with Type a) support condition we can say that these results are close to the results from Ref. [16] support condition Type b) (see Fig. 1.4.9).

If we check the Constant Smoothing results in FEM-Design with Type a) support condition we can say that these result are very close to the results from Ref. [16] support condition Type c).

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst180x/models/1.4 Peak smoothing.str

2 Second order analysis

2.1 A column with vertical and horizontal loads

We would like to analyze the following column (see Fig. 2.1.1) with second order theory. First of all we make a hand calculation with third order theory according to Ref. [6] and [8] with stability functions. After this step we compare the results with FEM-Design. In this moment we need to consider that in FEM-Design second order analysis is implemented and the hand calculation will be based on third order theory therefore the final results won't be exactly the same. By FEM-Design calulation we splitted the column into three bar elements thus the finite element number of the bars was three for more precise results.

The input parameters:

Elastic modulus	E = 30 GPa
Normal force	P = 2468 kN
Horizontal load	q = 10 kN/m
Cross section	0.2 m x 0.4 m (rectangle)
Second moment of inertia in the relevant direction	$I_2 = 0.0002667\ m^4$
Column length	L = 4 m

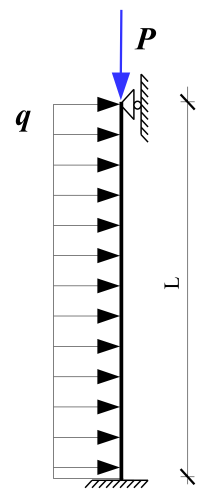

Figure 2.1.1 – The column with vertical and horizontal loads

According to Ref. [6] and [8] first of all we need to calculate the following assistant quantities:

$$\rho=\frac{P}{P_E}=\frac{P}{\left(\frac{\pi^2 EI}{L^2}\right)}=\frac{2468}{\left(\frac{\pi^2 30000000\cdot 0.0002667}{4^2}\right)}=0.500$$

The constants based on this value for the appropriate stability functions:

$$s=3.294 \quad ; \quad c=0.666 \quad ; \quad f=1.104$$

With these values the bending moments and the shear forces based on third order theory and FEM-Design calculation:

$$M_{clamped}=f(1+c)\frac{qL^2}{12}=1.104(1+0.666)\frac{10\cdot 4^2}{12}=24.52\,\text{kNm}$$ $$M_{2ndFEMclamped}=25.55\,\text{kNm}$$

$$M_{roller}=0.0\,\text{kNm}$$ $$M_{2ndFEMroller}=0.0\,\text{kNm}$$

$$V_{clamped}=\left[1+\frac{f(1+c)}{6}\right]\left(\frac{qL}{2}\right)=\left[1+\frac{1.104(1+0.666)}{6}\right]\left(\frac{10\cdot 4}{2}\right)=26.13\,\text{kN}$$

$$V_{2ndFEMclamped}=26.38\,\text{kN}$$

$$V_{roller}=\left[1-\frac{f(1+c)}{6}\right]\left(\frac{qL}{2}\right)=\left[1-\frac{1.104(1+0.666)}{6}\right]\left(\frac{10\cdot 4}{2}\right)=13.87\,\text{kN}$$

$$V_{2ndFEMroller}=13.61\,\text{kNm}$$

The differences are less than 5% between the hand and FE calculations.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/2.1 A column with vertical and horizontal loads.str

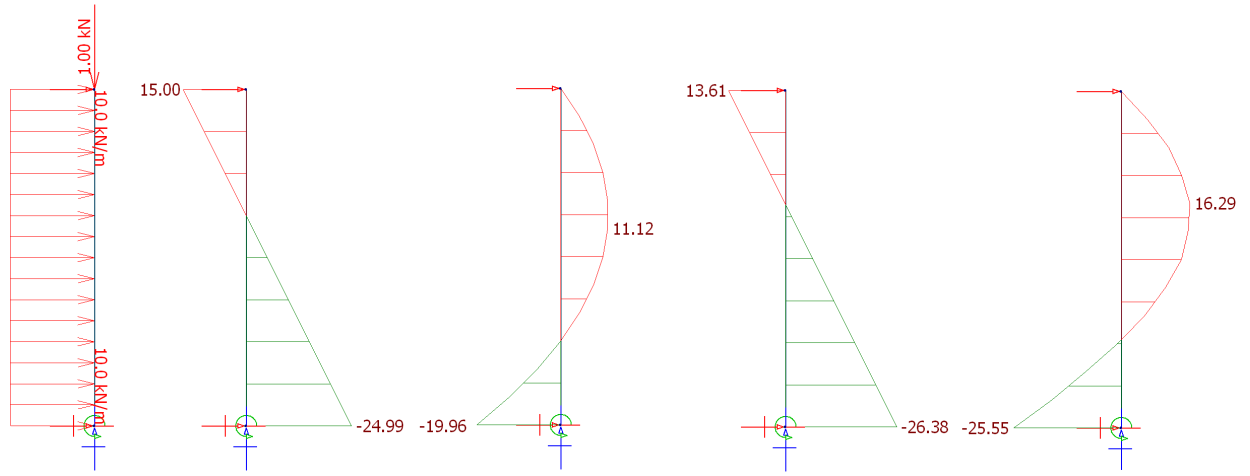

Figure 2.1.2 – The shear [kN] *and bending moment* [kNm] *diagram with 1st and 2nd order theory*

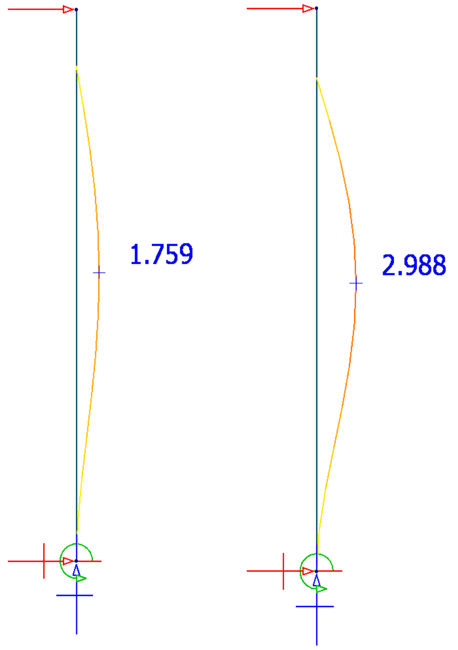

Figure 2.1.3 – The lateral translations [mm] *with 1st and 2nd order theory*

2.2 A plate with in-plane and out-of-plane loads

In this chapter we will analyze a rectangular plate with single supported four edges. The load is a specific normal force at the shorter edge and a lateral distributed total load perpendicular to the plate (see Fig. 2.2.1). The displacement and the bending moment are the question based on a 2nd order analysis. First of all we calculate the results with analytical solution and then we compare the results with FE calculations.

In this case the material and the geometric properties are the following:

The thickness of the plate	h = 0.05 m
The dimensions of the plate	a = 8 m; b = 6 m
The elastic modulus	E = 210 GPa
Poisson's ratio	ν = 0.3
The specific normal force	n_x = 1000 kN/m
The lateral distributed load	q_z = 10 kN/m^2

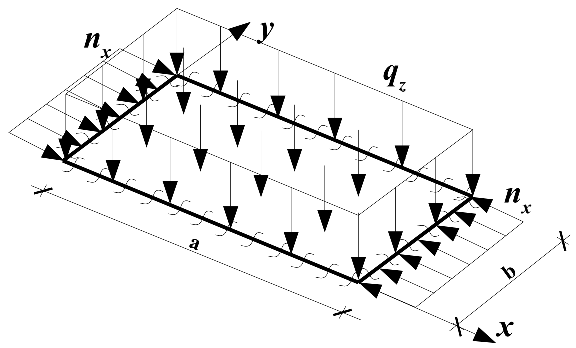

Figure 2.2.1 – The single supported edges, the lateral distributed load and the specific normal force

The maximum displacement and moments based on the 1st order linear calculation:

$$w_{max}=35.38\,\text{mm} \quad , \quad m_{x,max}=18.05\,\frac{\text{kNm}}{\text{m}} \quad , \quad m_{y,max}=25.62\,\frac{\text{kNm}}{\text{m}} \quad , \quad m_{xy,max}=13.68\,\frac{\text{kNm}}{\text{m}}$$

Based on Chapter 3.2 the critical specific normal force for this example is:

$$n_{cr}=2860\,\frac{\text{kN}}{\text{m}}$$

If the applied specific normal force is not so close to the critical value (now it is lower than the half of the critical value) we can assume the second order displacements and internal forces based on the linear solutions with the following formulas (with blue highlight we indicated the results of the FE calculation):

$$w_{max,2nd} = w_{max} \frac{1}{\left(1-\frac{n_x}{n_{cr}}\right)} = 35.38 \frac{1}{\left(1-\frac{1000}{2860}\right)} = 54.41 \text{ mm} \quad , \qquad w_{max,2nd,FEM} = 54.69 \text{ mm}$$

$$m_{x,max,2nd} = m_{x,max} \frac{1}{\left(1-\frac{n_x}{n_{cr}}\right)} = 18.05 \frac{1}{\left(1-\frac{1000}{2860}\right)} = 27.76 \frac{\text{kNm}}{\text{m}} \quad , \quad m_{x,max,2nd,FEM} = 28.50 \frac{\text{kNm}}{\text{m}}$$

$$m_{y,max,2nd} = m_{y,max} \frac{1}{\left(1-\frac{n_x}{n_{cr}}\right)} = 25.62 \frac{1}{\left(1-\frac{1000}{2860}\right)} = 39.40 \frac{\text{kNm}}{\text{m}} \quad , \quad m_{y,max,2nd,FEM} = 40.30 \frac{\text{kNm}}{\text{m}}$$

$$m_{xy,max,2nd} = m_{xy,max} \frac{1}{\left(1-\frac{n_x}{n_{cr}}\right)} = 13.68 \frac{1}{\left(1-\frac{1000}{2860}\right)} = 21.04 \frac{\text{kNm}}{\text{m}} \quad , \quad m_{xy,max,2nd,FEM} = 20.53 \frac{\text{kNm}}{\text{m}}$$

The differences are less than 3% between the hand and FEM-Design calculations.

Figure 2.2.2 shows the problem in FEM-Design with the default mesh.

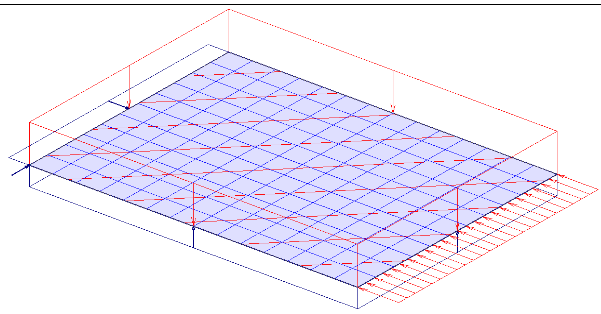

Figure 2.2.2 – The single supported slab with in-plane and out-of-plane loads

The following figures show the moment distribution in the plate and the displacements with FEM-Design according to 1st and 2nd order theory.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/2.2 A plate with in-plane and out-of-plane loads.str

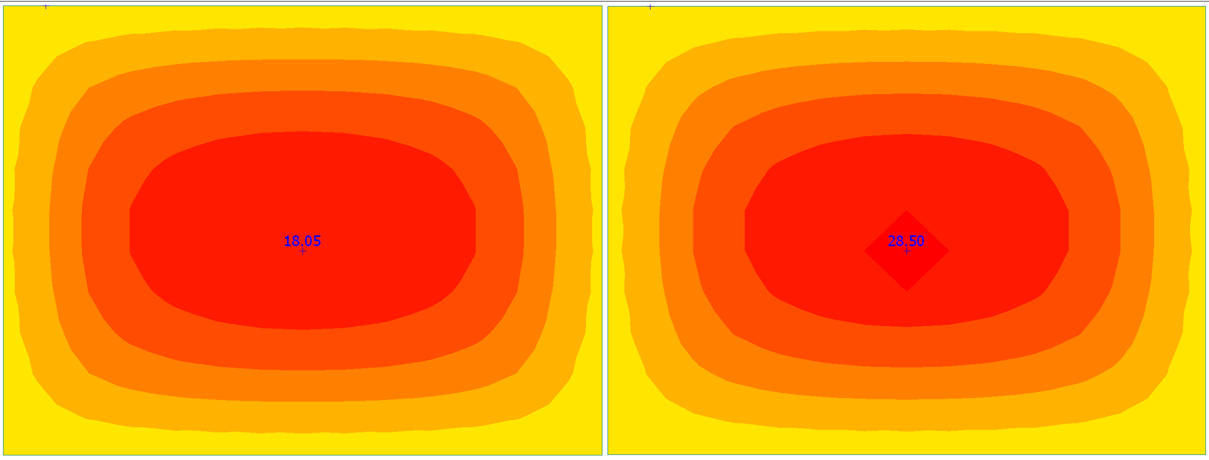

Figure 2.2.3 – The m_x [kNm/m] *moment with 1st and 2nd order analysis*

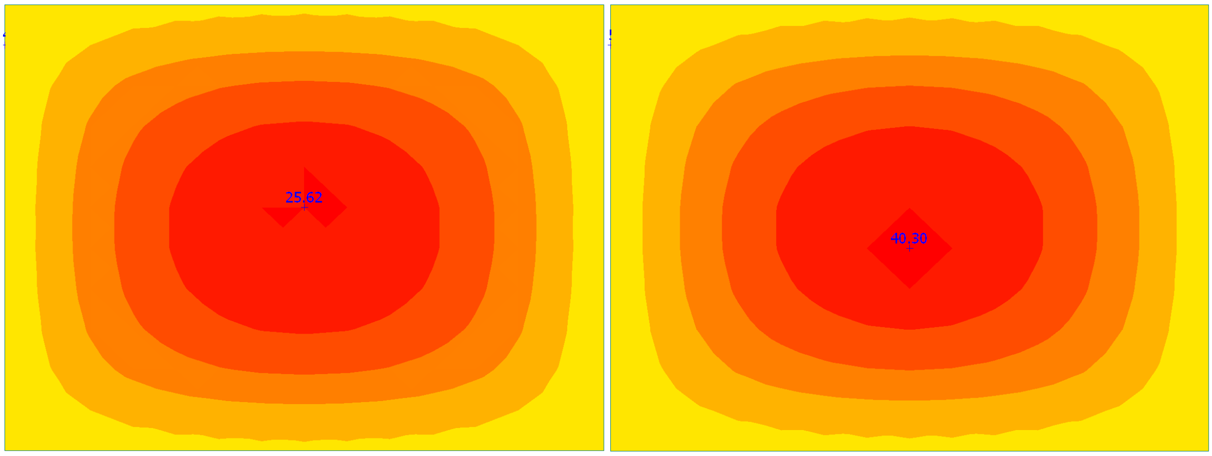

Figure 2.2.4 – The m_y [kNm/m] *moment with 1st and 2nd order analysis*

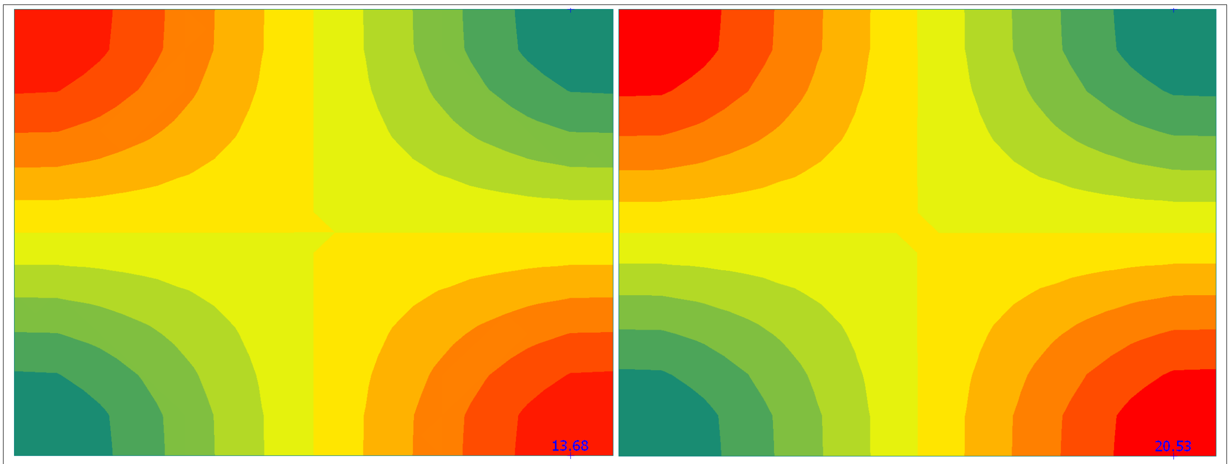

Figure 2.2.5 – The m_{xy} [kNm/m] *moment with 1st and 2nd order analysis*

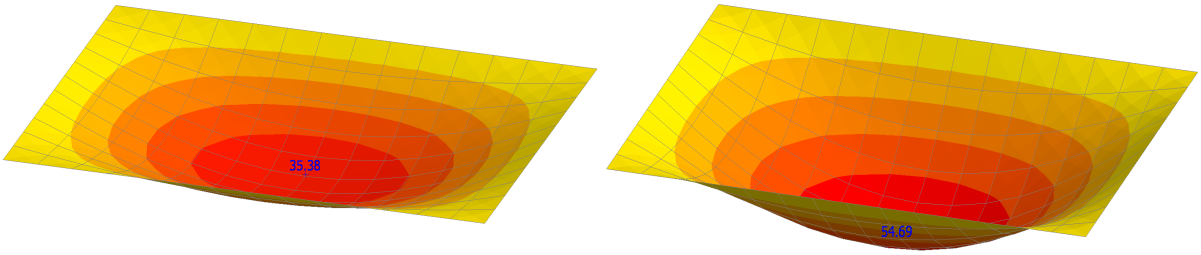

Figure 2.2.6 – The vertical translations [mm] *with 1st and 2nd order analysis*

3 Stability analysis

3.1 Flexural buckling analysis of a beam modell with different boundary conditions

The cross section is a rectangular section	see Fig. 3.1.1
The material	C 20/25 concrete
The elastic modulus	E = 30 GPa
The second moment of inertia about the weak axis	$I_2 = 2.667 \cdot 10^{-4}\ m^4$
The length of the column	L = 4 m
The boundary conditions	see Fig. 3.1.1

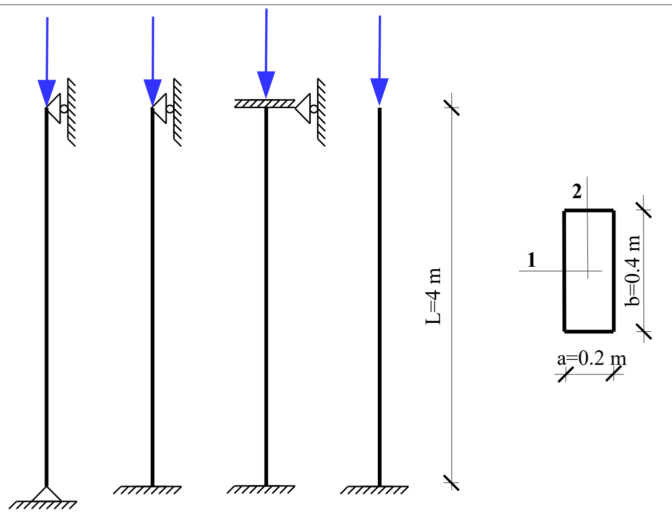

Figure 3.1.1 – The buckling problem with the different boundary conditions and the cross section

The critical load parameters according to the Euler's theory are as follows and next to the analytical solutions [1] the relevant results of the FEM-Design calculation can be seen. By the calculation we splitted the beams to five finite elements to get more accurate buckling mode shapes (see Fig. 3.1.2).

Pinned-pinned boundary condition:

$$F_{cr1} = \frac{\pi^2 E I_2}{L^2} = 4934.8\,\text{kN}$$

$$F_{crFEM1} = 4910.9\,\text{kN}$$

Fixed-pinned boundary condition:

$$F_{cr2}=\frac{\pi^2 E I_2}{(0.6992\,L)^2}=10094.1\,\text{kN}$$

$$F_{crFEM2}=9974.6\,\text{kN}$$

Fixed-fixed boundary condition:

$$F_{cr3}=\frac{\pi^2 E I_2}{(0.5\,L)^2}=19739.2\,\text{kN}$$

$$F_{crFEM3}=19318.7\,\text{kN}$$

Fixed-free boundary condition:

$$F_{cr4}=\frac{\pi^2 E I_2}{(2\,L)^2}=1233.7\,\text{kN}$$

$$F_{crFEM4}=1233.1\,\text{kN}$$

The differences between the two calculations are less than 3% but keep in mind that FEM-Design considers the shear deformation therefore we can be sure that the Euler's results give a bit higher critical values in these cases. Fig. 3.1.2 shows the first mode shapes of the problems with the different boundary conditions.

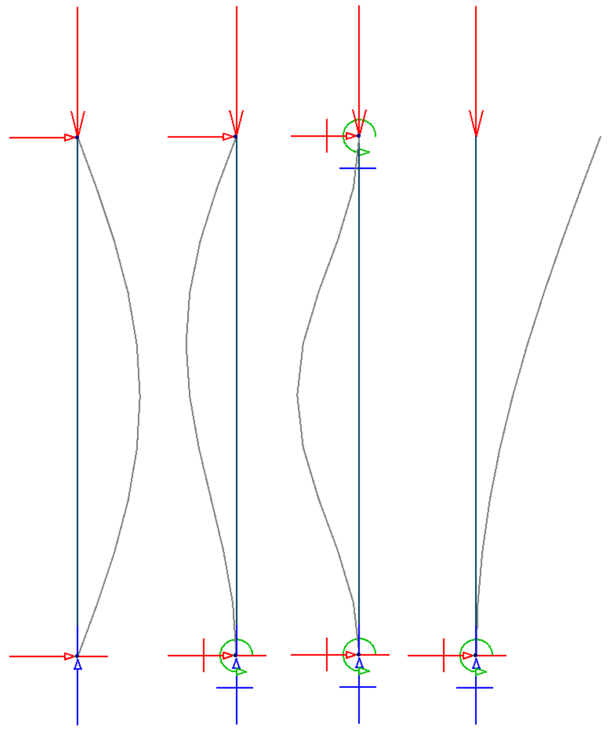

Figure 3.1.2 – The buckling mode shapes for different boundary conditions pinned-pinned; fixed-pinned; fixed-fixed; fixed-free

Download links to the example files:

http://download.strusoft.com/FEM-Design/inst170x/models/3.1 Flexural buckling analysis of a beam modell with different boundary conditions fixed-free.str

http://download.strusoft.com/FEM-Design/inst170x/models/3.1 Flexural buckling analysis of a beam modell with different boundary conditions pinned-pinned.str

3.2 Buckling analysis of a plate with shell modell

In this chapter we will analyze a rectangular plate with simply supported four edges. The load is a specific normal force at the shorter edge (see Fig. 3.2.1). The critical force parameters are the questions due to this edge load, therefore it is a stability problem of a plate.

In this case the material and the geometric properties are the following:

The thickness of the plate	h = 0.05 m
The dimensions of the plate	a = 8 m; b = 6 m
The elastic modulus	E = 210 GPa
Poisson's ratio	ν = 0.3

The solutions of the differential equation of the plate buckling problem are as follows [6]:

$$\boxed{n_{cr}=\left(\frac{mb}{a}+\frac{n^2 a}{mb}\right)^2 \frac{\pi^2\left(\frac{E h^3}{12(1-\nu^2)}\right)}{b^2}} \quad , \quad m=1,2,3\ldots \quad , \quad n=1,2,3\ldots$$

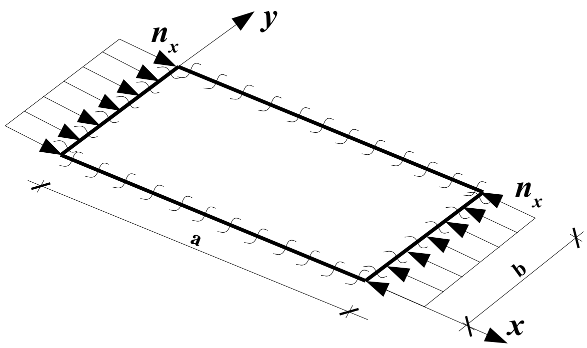

Figure 3.2.1 – The simply supported edges and the specific normal force

Figure 3.2.2 shows the problem in FEM-Design with the default mesh.

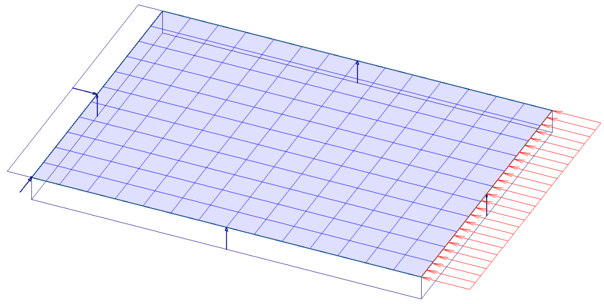

Figure 3.2.2 – The stability problem of a plate with simply supported edges

According to the analytical solution the first five critical load parameters are:

$m=1 \quad , \quad n=1$

$$n_{cr1}=\left(\frac{1\cdot 6}{8}+\frac{1^2 8}{1\cdot 6}\right)^2 \frac{\pi^2\left(\frac{210000000\cdot 0.05^3}{12\left(1-0.3^2\right)}\right)}{6^2}=2860.36\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$$n_{crFEM1}=2862.58\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$m=2 \quad , \quad n=1$

$$n_{cr2}=\left(\frac{2\cdot 6}{8}+\frac{1^2 8}{2\cdot 6}\right)^2 \frac{\pi^2\left(\frac{210000000\cdot 0.05^3}{12\left(1-0.3^2\right)}\right)}{6^2}=3093.77\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$$n_{crFEM2}=3109.96\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$m=3 \quad , \quad n=1$

$$n_{cr3}=\left(\frac{3\cdot 6}{8}+\frac{1^2 8}{3\cdot 6}\right)^2 \frac{\pi^2\left(\frac{210000000\cdot 0.05^3}{12\left(1-0.3^2\right)}\right)}{6^2}=4784.56\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$$n_{crFEM3}=4884.90\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$m=4 \quad , \quad n=1$

$$n_{cr4}=\left(\frac{4\cdot 6}{8}+\frac{1^2 8}{4\cdot 6}\right)^2 \frac{\pi^2\left(\frac{210000000\cdot 0.05^3}{12\left(1-0.3^2\right)}\right)}{6^2}=7322.53\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$$n_{crFEM4}=7655.58\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$m=3 \quad , \quad n=2$

$$n_{cr5}=\left(\frac{3\cdot 6}{8}+\frac{2^2 8}{3\cdot 6}\right)^2 \frac{\pi^2\left(\frac{210000000\cdot 0.05^3}{12\left(1-0.3^2\right)}\right)}{6^2}=10691.41\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$$n_{crFEM5}=10804.62\,\frac{\mathrm{kN}}{\mathrm{m}}$$

Next to these values we indicated the critical load parameters what were calculated with FEM-Design.

The difference between the calculations less than 5%.

Figure 3.2.3 shows the first five stability mode shapes of the rectangular simply supported plate.

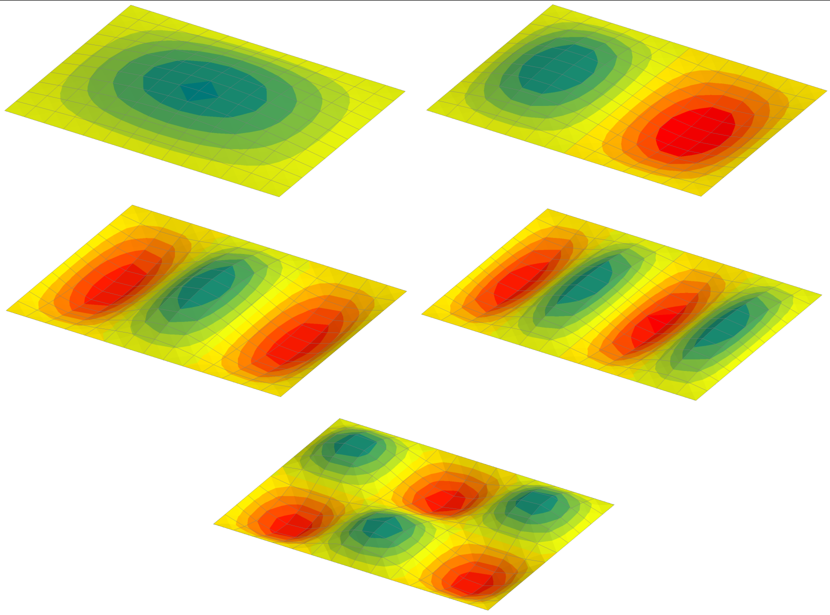

Figure 3.2.3 – The first five stability mode shapes of the described problem

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/3.2 Buckling analysis of a plate with shell modell.str

3.3 Lateral torsional buckling of an I section with shell modell

The purpose of this example is calculate the lateral torsional critical moment of the following simply supported beam (see Fig. 3.3.1).

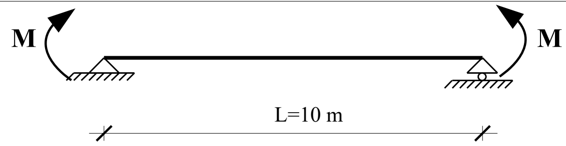

Figure 3.3.1 – The static frame of a simply supported beam loaded with bending moments at both ends

The length of the beam	L = 10 m
The cross section	see Fig. 3.3.2
The warping constant of the section	I_w = 125841 cm^6
The St. Venant torsional inertia	I_t = 15.34 cm^4
The minor axis second moment of area	I_z = 602.7 cm^4
The elastic modulus	E = 210 GPa
The shear modulus	G = 80.77 GPa

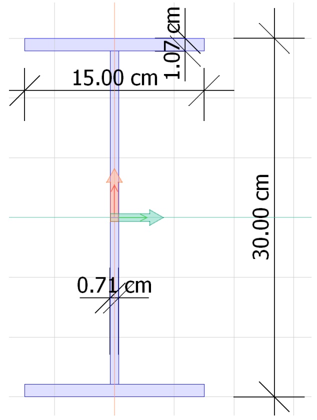

Figure 3.3.2 – The dimensions of the double symmetric cross section

In this case the critical moment can be calculated with the following formula based on the analytical solution [6]:

$$M_{cr} = \frac{\pi^2 E I_z}{L^2} \sqrt{\frac{I_w}{I_z} + \frac{L^2 G I_t}{\pi^2 E I_z}}$$

$$M_{cr} = \frac{\pi^2 \cdot 21000 \cdot 602.7}{1000^2} \sqrt{\frac{125841}{602.7} + \frac{1000^2 \cdot 8077 \cdot 15.34}{\pi^2 \cdot 21000 \cdot 602.7}} = 4328\,\text{kNcm}$$

In FEM-Design a shell modell was built to analyze this problem. The moment loads in the shell model were considered with line loads at the end of the flanges (see Fig. 3.3.3).

The supports provides the simple supported beam effects with a fork support for the shell model (see Fig. 3.3.3).

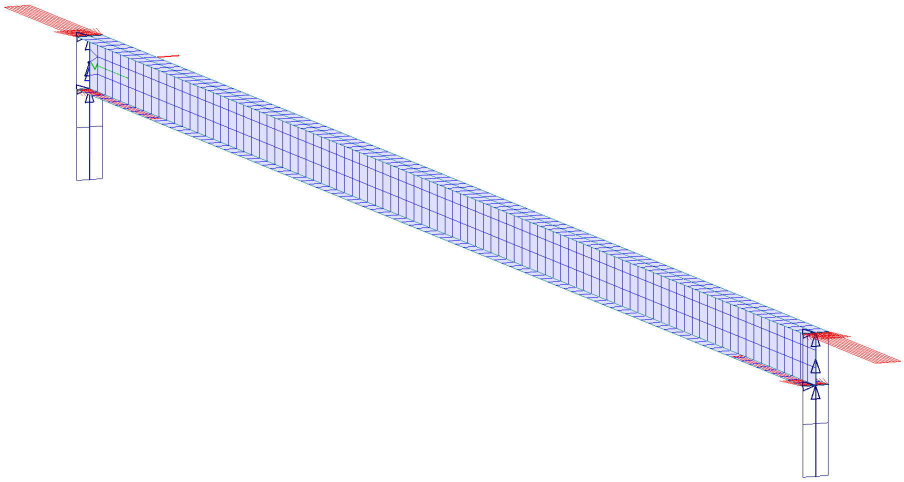

Figure 3.3.3 – The FEM model with the supports and the loads (moments) at the ends

From the FEM-Design stability calculation the critical moment value for this lateral torsional buckling problem is:

$$M_{crFEM} = 4363\,\text{kNcm}$$

The critical shape is in Fig 3.3.4. The finite element mesh size was provided based on the automatic mesh generator of FEM-Design.

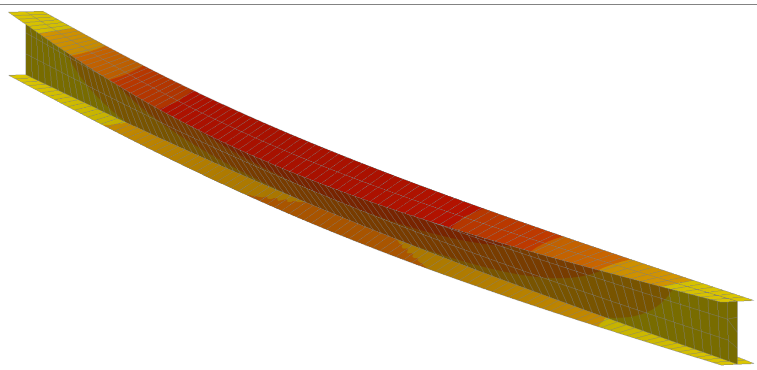

Figure 3.3.4 – The critical mode shape of the problem

The difference between the two calculated critical moments is less than 1%.

Download link to the example file:

FEM-Design file:

http://download.strusoft.com/FEM-Design/inst170x/models/3.3 Lateral torsional buckling of an I section with shell modell.str

Section Editor file for cross-sectional properties:

http://download.strusoft.com/FEM-Design/inst170x/models/3.3 Lateral torsional buckling of an I section with shell modell.sec

3.4 Lateral torsional buckling of a cantilever with elongated rectangle section

The purpose of this example is calculate the critical force at the end of a cantilever beam (see Fig. 3.4.1). If the load is increasing the state of the cantilever will be unstable due to lateral torsional buckling.

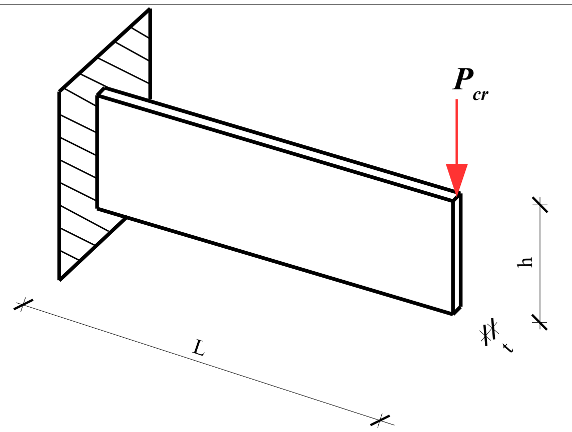

Figure 3.4.1 – The cantilever beam with concentrated load

The input parameters:

The length of the beam is	L = 10 m
The cross section	t = 40 mm; h = 438 mm; see Fig. 3.4.1
The St. Venant torsional inertia	$I_t = 8806246\ mm^4$
The minor axis second moment of area	$I_z = 2336000\ mm^4$
The elastic modulus	E = 210 GPa
The shear modulus	G = 80.77 GPa

In this case (elongated rectangle cross section with cantilever boundary condition) the critical concentrated force at the end can be calculated with the following formula based on analytical solution [ask for the reference from Support team]:

$$P_{cr} = \frac{4.01\, E\, I_z}{L^2} \sqrt{\frac{G\, I_t}{E\, I_z}}$$

$$P_{cr} = \frac{4.01 \cdot 210000 \cdot 2336000}{10000^2} \sqrt{\frac{80770 \cdot 8806246}{210000 \cdot 2336000}} = 23687\ \mathrm{N} = 23.69\ \mathrm{kN}$$

In FEM-Design a shell modell was built to analyze this problem. The concentrated load at the end of the cantilever was considered at the top of the beam (see Fig. 3.4.2).

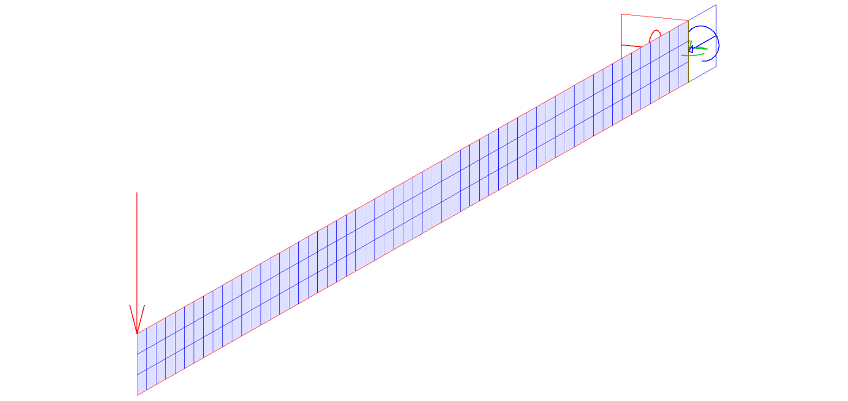

Figure 3.4.2 – The FE model of the cantilever beam with the default mesh

With the FEM-Design stability calculation the critical concentrated force value for this lateral torsional buckling problem is:

$$P_{cr\mathrm{FEM}} = 24.00\,\mathrm{kN}$$

The critical shape is in Fig 3.4.3. The finite element mesh size was provided based on the automatic mesh generator of FEM-Design.

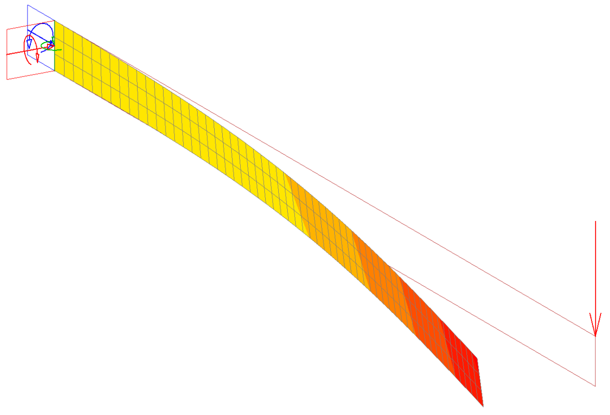

Figure 3.4.3 – The critical mode shape of the problem

The difference between the two calculated critical load parameters is less than 2%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/3.4 Lateral torsional buckling of a cantilever with elongated rectangle section.str

4 Calculation of eigenfrequencies with linear dynamic theory

4.1 Continuous mass distribution on a cantilever column

Column height	H = 4 m
The cross section	square with 0.4 m edge
The second moment of inertia	$I = 0.002133\ m^4$
The area of the cross section	$A = 0.16\ m^2$
The shear correction factor	ρ = 5/6 = 0.8333
The elastic modulus	E = 30 GPa
The shear modulus	G = 12.5 GPa
The specific self-weight of the column	$\gamma = 25\ kN/m^3$
The mass of the column	m = 1.631 t

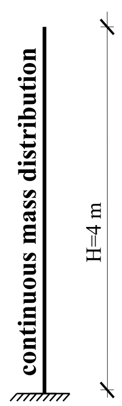

Figure 4.1.1 – The cantilever with continuous mass distribution

Based on the analytical solution [4] the angular frequencies for this case is:

$$\boxed{\omega_B = \mu_{Bi}\sqrt{\frac{EI}{m H^3}}} \quad ; \quad \mu_{B1} = 3.52 ; \mu_{B2} = 22.03 ; \mu_{B3} = 61.7$$

if only the bending deformations are considered.

The angular frequencies are [4]:

$$\boxed{\omega_S=\mu_{Si}\sqrt{\frac{\rho GA}{m\,H}}} \quad ; \quad \mu_{S1}=0.5\,\pi\,;\,\mu_{S2}=1.5\,\pi\,;\,\mu_{S3}=2.5\,\pi$$

if only the shear deformations are considered.

Based on these two equations (considering bending and shear deformation) using the Föppl theorem the angular frequency for a continuous mass distribution column is:

$$\boxed{\frac{1}{\omega_n^2}=\frac{1}{\omega_B^2}+\frac{1}{\omega_S^2}}$$

Based on the given equations the first three angular frequencies separately for bending and shear deformations are:

$$\omega_{B1}=3.52\sqrt{\frac{30000000\cdot 0.002133}{1.631\cdot 4^3}}=87.16\frac{1}{\mathrm{s}}$$

$$\omega_{B2}=22.03\sqrt{\frac{30000000\cdot 0.002133}{1.631\cdot 4^3}}=545.4\frac{1}{\mathrm{s}}$$

$$\omega_{B3}=61.7\sqrt{\frac{30000000\cdot 0.002133}{1.631\cdot 4^3}}=1527.7\frac{1}{\mathrm{s}}$$

$$\omega_{S1}=0.5\,\pi\sqrt{\frac{0.8333\cdot 12500000\cdot 0.16}{1.631\cdot 4}}=793.9\frac{1}{\mathrm{s}}$$

$$\omega_{S2}=1.5\,\pi\sqrt{\frac{0.8333\cdot 12500000\cdot 0.16}{1.631\cdot 4}}=2381.8\frac{1}{\mathrm{s}}$$

$$\omega_{S3}=2.5\,\pi\sqrt{\frac{0.8333\cdot 12500000\cdot 0.16}{1.631\cdot 4}}=3969.6\frac{1}{\mathrm{s}}$$

According to the Föppl theorem the resultant first three angular frequencies of the problem are:

$$\omega_{n1}=86.639\frac{1}{\mathrm{s}} \quad , \quad \omega_{n2}=531.64\frac{1}{\mathrm{s}} \quad , \quad \omega_{n3}=1425.8\frac{1}{\mathrm{s}}$$

And based on these results the first three eigenfrequencies are ($f=\omega/(2\pi)$):

$$f_{n1}=13.789\frac{1}{\mathrm{s}} \quad , \quad f_{n2}=84.613\frac{1}{\mathrm{s}} \quad , \quad f_{n3}=226.923\frac{1}{\mathrm{s}}$$

In FEM-Design to consider the continuous mass distribution 200 beam elements were used for the cantilever column. The first three planar mode shapes are as follows according to the FE calculation:

$$f_{FEM1} = 13.780 \frac{1}{s} \quad , \quad f_{FEM2} = 83.636 \frac{1}{s} \quad , \quad f_{FEM3} = 223.326 \frac{1}{s}$$

The first three mode shapes can be seen in Fig. 4.1.2.

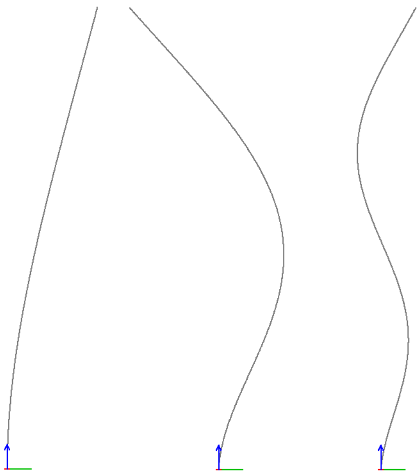

Figure 4.1.2 – The first three mode shapes for the cantilever with continuous mass distribution

The differences between the analytical and FE solutions are less than 2%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/4.1 Continuous mass distribution on a cantilever column.str

4.2 Free vibration shapes of a clamped circular plate due to its self-weight

In the next example we will analyze a circular clamped plate. The eigenfrequencies are the question due to the self-weight of the slab.

In this case the material and the geometric properties are the following:

The thickness of the plate	h = 0.05 m
The radius of the circular plate	R = 5 m
The elastic modulus	E = 210 GPa
Poisson's ratio	ν = 0.3
The density	$\rho = 7.85\ t/m^3$

The solution of the dynamic differential equation for the first two angular frequencies of a clamped circular plate are [5]:

$$\boxed{\omega_{nm}=\frac{\pi^2}{R^2}\beta_{nm}^{\ 2}\sqrt{\frac{\left(\frac{E\,h^3}{12(1-\nu^2)}\right)}{\rho\,h}}}\quad,\quad \beta_{10}=1.015\quad,\quad \beta_{11}=1.468$$

Figure 4.2.1 shows the problem in FEM-Design with the clamped edges and with the default mesh.

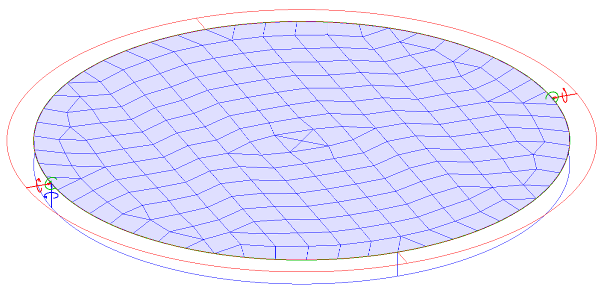

Figure 4.2.1 – The clamped circular plate and the default finite element mesh

According to the analytical solution the first two angular frequencies are:

$$\omega_{10}=\frac{\pi^2}{5^2}1.015^2\sqrt{\frac{\left(\frac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{7.85\cdot 0.05}}=31.83\frac{1}{s} \quad , \quad f_{10}=5.066\frac{1}{s} \quad , \quad f_{10FEM}=5.129\frac{1}{s}$$

$$\omega_{11}=\frac{\pi^2}{5^2}1.468^2\sqrt{\frac{\left(\frac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{7.85\cdot 0.05}}=66.58\frac{1}{s} \quad , \quad f_{11}=10.60\frac{1}{s} \quad , \quad f_{11FEM}=10.731\frac{1}{s}$$

Based on the angular frequencies we can calculate the eigenfrequencies in a very easy way. Next to these values we indicated the eigenfrequencies which were calculated with FEM-Design.

The differences between the calculations are less than 2%.

Figure 4.2.2 shows the first two vibration mode shapes of the circular clamped plate.

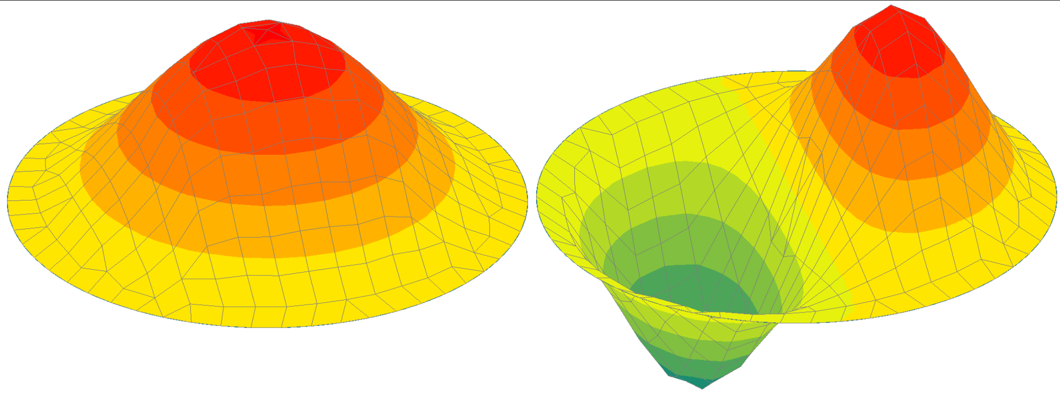

Figure 4.2.2 – The first two vibration shape mode of a clamped circular plate

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/4.2 Free vibration shapes of a clamped circular plate due to its self-weight.str

5 Seismic calculation

5.1 Lateral force method with linear shape distribution on a cantilever

Inputs:

Column height	H = 10 m
The cross section	square with 0.4 m edge
The second moment of inertia	I = 0.002133 m^4
The elastic modulus	E = 31 GPa
The concentrated mass points	10 pieces of 1.0 t (see Fig. 5.1.1)
The total mass	m = 10.0 t

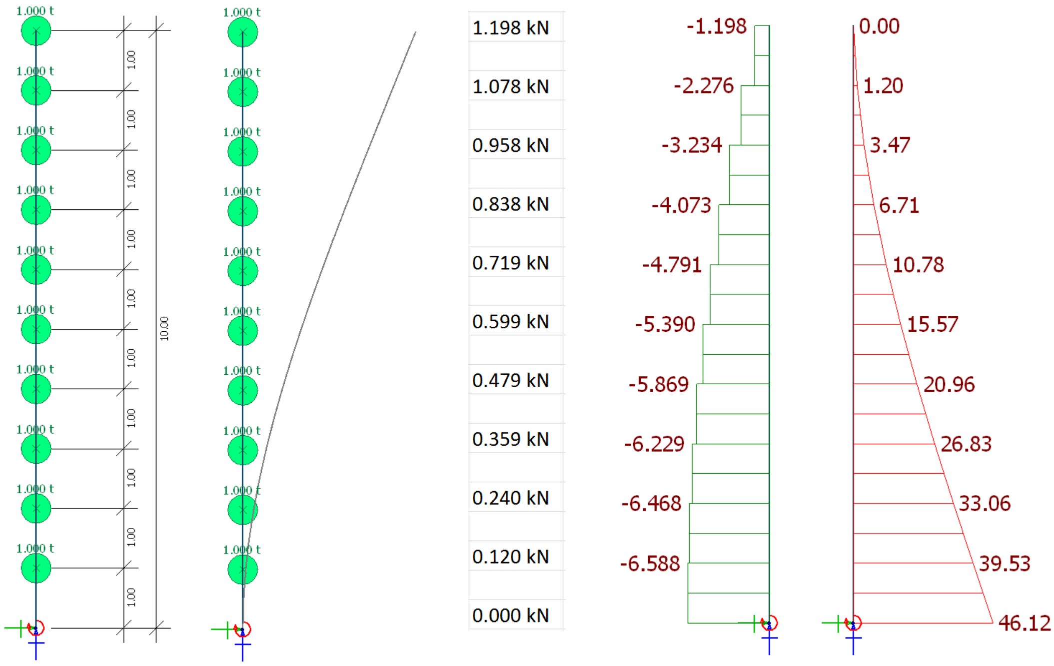

Figure 5.1.1 – The cantilever column with the concentrated mass points, the first vibration shape [T=0.765 s], *the equivalent forces* [kN], *the shear force diagram* [kN] *and the bending moment diagram* [kNm] *with FEM-Design*

First of all based on a hand calculation we determine the first fundamental period:

The first fundamental period of a cantilever column (length *H*) with a concentrated mass at the end (*m* mass) and *EI* bending stiffness [4]:

$$T_i = \frac{2\pi}{\sqrt{\frac{3EI}{m_i H_i^3}}}$$

The fundamental period separately for the mass points from bottom to top:

$$T_1=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 1^3}}}=0.01411\,\mathrm{s}\quad;\quad T_2=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 2^3}}}=0.03990\,\mathrm{s}\quad;$$

$$T_3=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 3^3}}}=0.07330\,\mathrm{s}\quad;\quad T_4=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 4^3}}}=0.1129\,\mathrm{s}\quad;$$

$$T_5=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 5^3}}}=0.1577\,\mathrm{s}\quad;\quad T_6=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 6^3}}}=0.2073\,\mathrm{s}\quad;$$

$$T_7=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 7^3}}}=0.2613\,\mathrm{s}\quad;\quad T_8=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 8^3}}}=0.3192\,\mathrm{s}\quad;$$

$$T_9=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 9^3}}}=0.3809\,\mathrm{s}\quad;\quad T_{10}=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 10^3}}}=0.4461\,\mathrm{s}\quad.$$

The approximated period based on these values according to the Dunkerley summary and the result of FE calculation:

$$T_{HC}=\sqrt{\sum_{i=1}^{10} T_i^2}=0.7758\,\mathrm{s}\qquad T_{FEM}=0.765\,\mathrm{s}$$

The difference between the hand calculation and FEM-Design calculation is less than 2%, for further information on the fundamental period calculation see Chapter 4.

The base shear force according to the fundamental period of vibration (see Fig. 5.1.1) and the response spectrum (see Fig. 5.1.2):

$$F_b=S_d(T_1)\,m\,\lambda=0.6588\cdot 10\cdot 1.0=6.588\,\mathrm{kN}$$

We considered the response acceleration based on the period from FE calculation to get a more comparable results at the end. Thus the equivalent forces on the different point masses are:

$$F_i=F_b\frac{z_i m_i}{\sum z_j m_j}=6.588\frac{z_i m_i}{1\cdot 1+2\cdot 1+3\cdot 1+4\cdot 1+5\cdot 1+6\cdot 1+7\cdot 1+8\cdot 1+9\cdot 1+10\cdot 1}=6.588\frac{z_i m_i}{55}$$

The equivalent forces from the bottom to the top on each point mass:

$F_1 = 0.120\,\text{kN}$; $F_2 = 0.240\,\text{kN}$; $F_3 = 0.359\,\text{kN}$; $F_4 = 0.479\,\text{kN}$; $F_5 = 0.599\,\text{kN}$;
$F_6 = 0.719\,\text{kN}$; $F_7 = 0.838\,\text{kN}$; $F_8 = 0.958\,\text{kN}$; $F_9 = 1.078\,\text{kN}$; $F_{10} = 1.198\,\text{kN}$

These forces are identical with the FEM-Design calculation and the shear force and bending moment diagrams are also identical with the hand calculation.

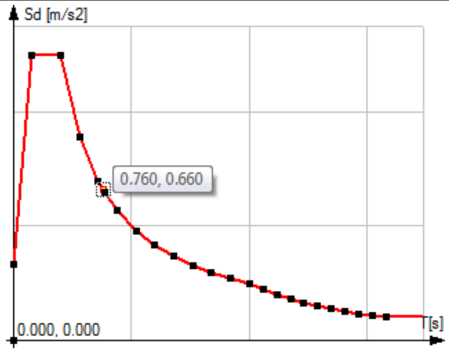

Figure 5.1.2 – The response spectrum [T = 0.765 s; S_d = 0.6588 m/s^2]

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/5.1 Lateral force method with linear shape distribution on a cantilever.str

5.2 Lateral force method with fundamental mode shape distribution on a cantilever

Inputs:

Column height	H = 10 m
The cross section	square with 0.4 m edge
The second moment of inertia	I = 0.002133 m^4
The elastic modulus	E = 31 GPa
The concentrated mass points	10 pieces of 1.0 t (see Fig. 5.1.1)
The total mass	m = 10.0 t

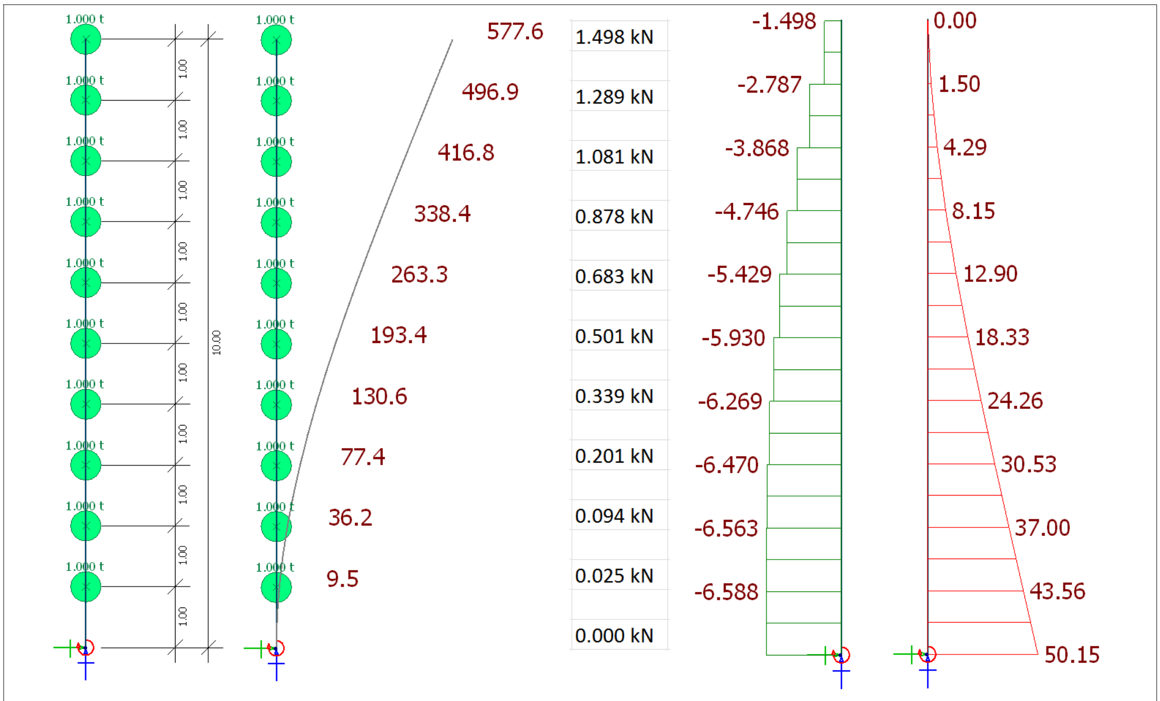

Figure 5.2.1 – The cantilever column with the concentrated mass points, the first vibration shape with the value of the eigenvector [T=0.765 s], *the equivalent forces* [kN], *the shear force diagram* [kN] *and the bending moment diagram* [kNm] *with FEM-Design*

The base shear force according to the fundamental period of vibration (see Fig. 5.2.1) and the response spectrum (see Fig. 5.2.2):

$$F_b = S_d(T_1)\, m\, \lambda = 0.6588 \cdot 10 \cdot 1.0 = 6.588\,\text{kN}$$

We considered the response acceleration based on the period from FE calculation to get a more comparable results at the end. Thus the equivalent forces on the different point masses are:

$$F_i = F_b \frac{s_i m_i}{\sum s_j m_j} =$$

$$= 6.588 \frac{s_i m_i}{9.5 \cdot 1 + 36.2 \cdot 1 + 77.4 \cdot 1 + 130.6 \cdot 1 + 193.4 \cdot 1 + 263.3 \cdot 1 + 338.4 \cdot 1 + 416.8 \cdot 1 + 496.9 \cdot 1 + 577.6 \cdot 1} = 6.588 \frac{s_i m_i}{2540.1}$$

The equivalent forces from the bottom to the top on each point mass:

$F_1 = 0.0246\,\mathrm{kN}$; $F_2 = 0.0939\,\mathrm{kN}$; $F_3 = 0.201\,\mathrm{kN}$; $F_4 = 0.339\,\mathrm{kN}$; $F_5 = 0.502\,\mathrm{kN}$; $F_6 = 0.683\,\mathrm{kN}$; $F_7 = 0.878\,\mathrm{kN}$; $F_8 = 1.081\,\mathrm{kN}$; $F_9 = 1.289\,\mathrm{kN}$; $F_{10} = 1.498\,\mathrm{kN}$

These forces are identical with the FEM-Design calculation and the shear force and bending moment diagrams are also identical with the hand calculation.

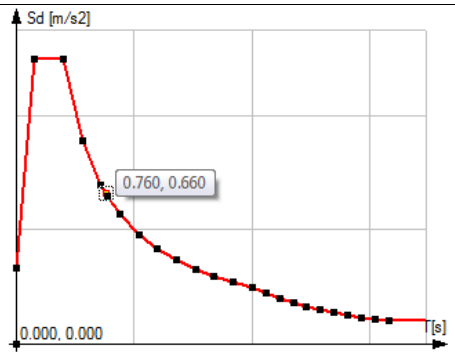

Figure 5.2.2 – The response spectrum [T = 0.765 s; S_d = 0.6588 m/s^2]

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/5.2 Lateral force method with fundamental mode shape distribution on a cantilever.str

5.3 Modal analysis of a concrete frame building

In this chapter we show a worked example for modal analysis on a concrete frame building according to EN 1998-1:2008 with hand calculation and compare the results with FEM-Design. This example is partly based on [4]. The geometry, the dimensions, the material and the bracing system are in Fig. 5.3.1-3 and in the following table. As a bracing system we used trusses with very large normal stiffness (*EA*) to reach pure eigenvectors by the fundamental period calculation (see Fig. 5.3.2 and Fig. 5.3.5).

Inputs:

Column height/Total height	h = 3.2 m; H = 2·3.2=6.4 m
The cross sections	Columns: 30/30 cm; Beams: 30/50 cm
The second moment of inertia	$I_c = 0.000675\ m^4$; $I_b = 0.003125\ m^4$
The elastic modulus	E = 28.80 GPa
The concentrated mass points	12 pieces of 13.358 t on 1st storey and 12 pieces of 11.268 t on 2nd storey (see Fig. 5.3.2)
The total mass	1st storey: $m_1 = 160.3$ t 2nd storey: $m_2 = 135.2$ t total mass: M = 295.5 t
Reduction factor for elastic modulus considering the cracking according to EN 1998-1:2008	$\alpha = 0.5$
Behaviour factors	$q = 1.5$, $q_d = 1.5$
Accidental torsional effect was not considered	$\xi = 0.05$ (damping factor)

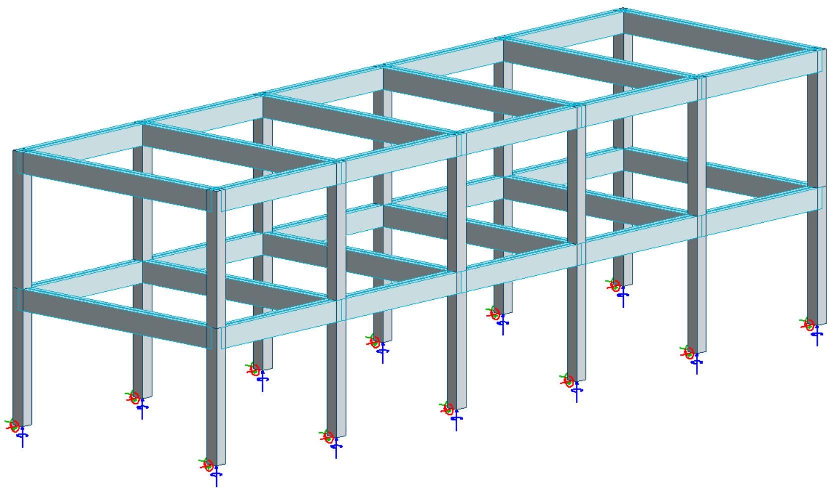

Figure 5.3.1 – The concrete frame building with the columns and beams

The first exercise is the determination of the fundamental periods and mode shapes. There are several hand calculation modes to get these values but in this chapter the details of the modal analysis are important therefore we considered the first two fundamental periods based on FEM-Design calculation (see Fig. 5.3.5). See the details and example on the eigenfrequency calculation in Chapter 4.

The dead loads and the live loads are considered in the mass points (see Fig. 5.3.2 and the input table).

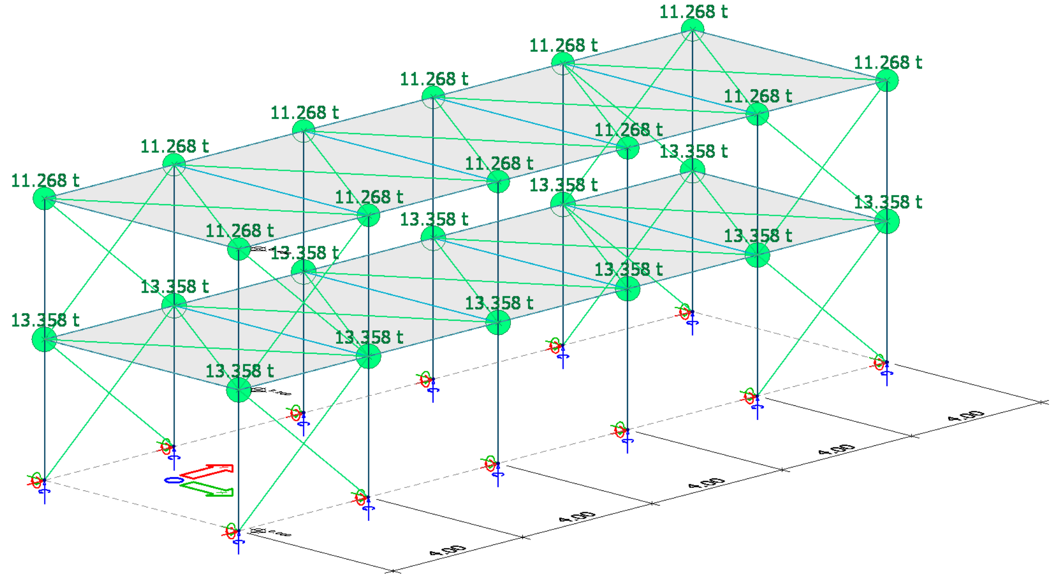

Figure 5.3.2 – The frame building with the masses and bracings

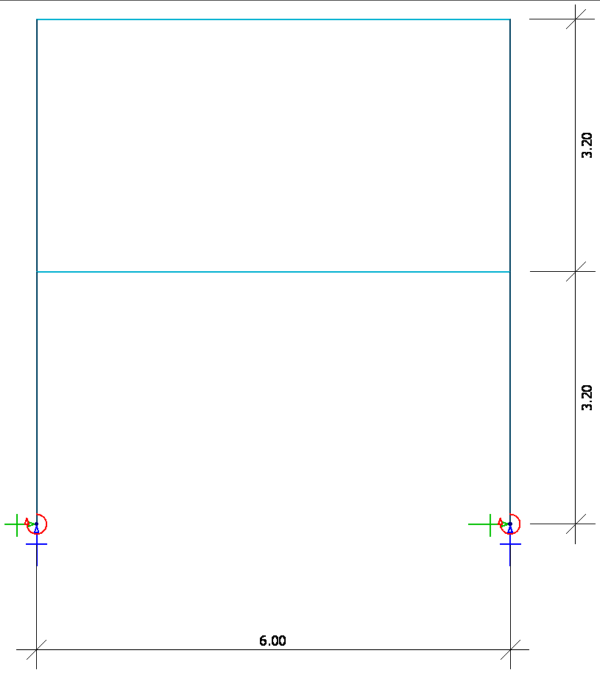

Figure 5.3.3 – The side view of the building

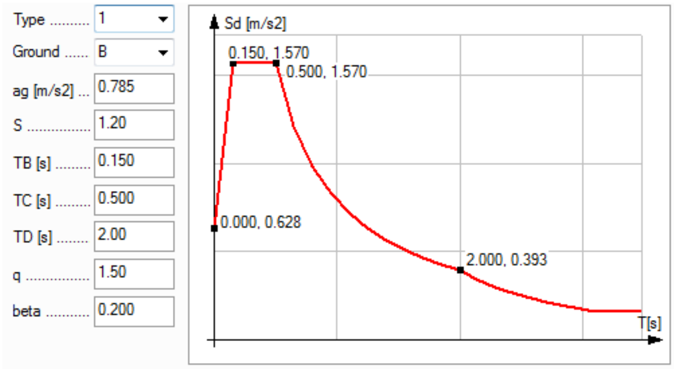

Figure 5.3.4 – The considered design response specra according to EN 1998-1:2008

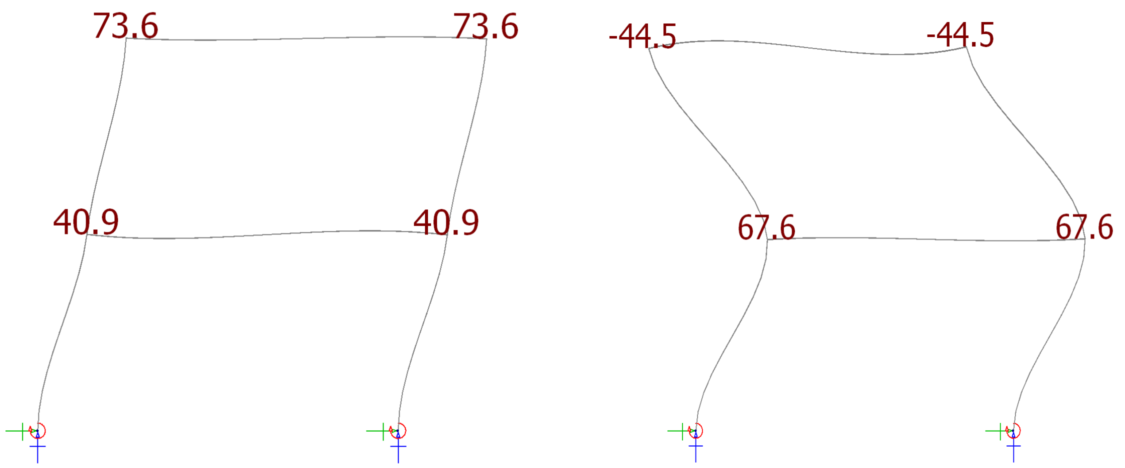

Figure 5.3.5 – The first two fundamental mode shapes [-], $T_1 = 0.704$ s; $T_2 = 0.252$ s
For this figure we splitted the beams/columns to 5-5 finite elements for better visualization of the eigenvectors but the default mesh was used during the whole FEM calculation

According to the fundamental periods in Fig. 5.3.5 the response accelerations from Fig. 5.3.4 are:

$$T_1 = 0.704\,\mathrm{s} \qquad S_{d1} = 1.115\,\frac{\mathrm{m}}{\mathrm{s}^2} \quad ;$$

$$T_2 = 0.252\,\mathrm{s} \qquad S_{d2} = 1.57\,\frac{\mathrm{m}}{\mathrm{s}^2} \quad .$$

The second step is to calculate the effective modal masses based on this formula:

$$\boxed{m_i^* = \frac{\left(\mathbf{\Phi}_\mathbf{i}^\mathrm{T}\mathbf{m}\boldsymbol{\iota}\right)^2}{\mathbf{\Phi}_\mathbf{i}^\mathrm{T}\mathbf{m}\mathbf{\Phi}_\mathbf{i}}}$$

During the hand calculation we assume that the structure is a two degrees of freedom system in the x direction with the two storeys, because the first two modal shapes are in the same plane see Fig. 5.3.5. Thus we only consider the seismic loads in one direction because in this way the hand calculation is more comprehensible.

$$m_1^* = \frac{\left(\begin{bmatrix} 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2}{\begin{bmatrix} 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 40.9 \\ 73.6 \end{bmatrix}} = 272.3\,\mathrm{t} \quad ; \quad \frac{m_1^*}{M} = \frac{272.3}{295.5} = 92.1\,\%$$

$$m_2^* = \frac{\left(\begin{bmatrix} 67.6 & -44.5 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2}{\begin{bmatrix} 67.6 & -44.5 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 67.6 \\ -44.5 \end{bmatrix}} = 23.23\,\mathrm{t} \quad ; \quad \frac{m_2^*}{M} = \frac{23.23}{295.5} = 7.9\,\%$$

According to the assumption of a two degrees of freedom system the sum of the effective modal masses is equal to the total mass:

$$\frac{m_1^*}{M} + \frac{m_2^*}{M} = \frac{272.3}{295.5} + \frac{23.23}{295.5} = 100.0\,\%$$

Calculation of the base shear forces:

$$F_{b1} = S_{d1} m_1^* = 1.115 \cdot 272.3 = 303.6\,\mathrm{kN} \quad ; \quad F_{b2} = S_{d2} m_2^* = 1.570 \cdot 23.23 = 36.5\,\mathrm{kN}$$

The equivalent forces come from this formula:

$$\mathbf{p}_\mathbf{i} = \mathbf{m}\, \mathbf{\Phi}_\mathbf{i} \frac{\mathbf{\Phi}_\mathbf{i}^{\mathrm{T}} \mathbf{m}\, \boldsymbol{\iota}}{\mathbf{\Phi}_\mathbf{i}^{\mathrm{T}} \mathbf{m}\, \mathbf{\Phi}_\mathbf{i}} S_{di}$$

The equivalent forces at the storeys respect to the mode shapes considering the mentioned two degrees of freedom model:

$$\mathbf{p}_1 = \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 40.9 \\ 73.6 \end{bmatrix} \frac{\begin{bmatrix} 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 40.9 \\ 73.6 \end{bmatrix}} 1.115 = \begin{bmatrix} 120.6 \\ 183.0 \end{bmatrix} \mathrm{kN}$$

$$\mathbf{p}_2=\begin{bmatrix}160.3 & 0\\ 0 & 135.2\end{bmatrix}\begin{bmatrix}67.6\\ -44.5\end{bmatrix}\frac{\begin{bmatrix}67.6 & -44.5\end{bmatrix}\begin{bmatrix}160.3 & 0\\ 0 & 135.2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}}{\begin{bmatrix}67.6 & -44.5\end{bmatrix}\begin{bmatrix}160.3 & 0\\ 0 & 135.2\end{bmatrix}\begin{bmatrix}67.6\\ -44.5\end{bmatrix}}1.570=\begin{bmatrix}81.98\\ -44.52\end{bmatrix}\mathrm{kN}$$

The equivalent forces on one frame from the six (see Fig. 5.3.1):

$$\mathbf{p}_{f1}=\begin{bmatrix}120.6/6\\ 183.0/6\end{bmatrix}=\begin{bmatrix}20.10\\ 30.50\end{bmatrix}\mathrm{kN}$$

$$\mathbf{p}_{f2}=\begin{bmatrix}81.98/6\\ -44.52/6\end{bmatrix}=\begin{bmatrix}13.66\\ -7.420\end{bmatrix}\mathrm{kN}$$

The shear forces between the storeys respect to the two different mode shapes:

$$\mathbf{V}_1=\begin{bmatrix}20.1+30.5\\ 30.5\end{bmatrix}=\begin{bmatrix}50.6\\ 30.5\end{bmatrix}\mathrm{kN}\qquad \mathbf{V}_2=\begin{bmatrix}13.66-7.42\\ -7.42\end{bmatrix}=\begin{bmatrix}6.24\\ -7.42\end{bmatrix}\mathrm{kN}$$

The shear forces in the columns respect to the two different mode shapes:

$$\mathbf{V}_{c1}=\begin{bmatrix}50.6/2\\ 30.5/2\end{bmatrix}=\begin{bmatrix}25.30\\ 15.25\end{bmatrix}\mathrm{kN}\qquad \mathbf{V}_{c2}=\begin{bmatrix}6.24/2\\ -7.42/2\end{bmatrix}=\begin{bmatrix}3.13\\ -3.71\end{bmatrix}\mathrm{kN}$$

The bending moments in the columns respect to the two different mode shapes from the relevant shear forces (by the hand calculation we assumed zero bending moment points in the middle of the columns between the storeys):

$$\mathbf{M}_{c1}=\begin{bmatrix}25.30\cdot 3.2/2\\ 15.25\cdot 3.2/2\end{bmatrix}=\begin{bmatrix}40.48\\ 24.40\end{bmatrix}\mathrm{kNm}\qquad \mathbf{M}_{c2}=\begin{bmatrix}3.13\cdot 3.2/2\\ -3.71\cdot 3.2/2\end{bmatrix}=\begin{bmatrix}5.008\\ -5.936\end{bmatrix}\mathrm{kNm}$$

The bending moments in the beams respect to the two different mode shapes:

$$\mathbf{M}_{b1}=\begin{bmatrix}40.48+24.40\\ 24.40\end{bmatrix}=\begin{bmatrix}64.88\\ 24.40\end{bmatrix}\mathrm{kNm}\qquad \mathbf{M}_{b2}=\begin{bmatrix}5.008-5.936\\ -5.936\end{bmatrix}=\begin{bmatrix}-0.928\\ -5.936\end{bmatrix}\mathrm{kNm}$$

The SRSS summation on the internal forces:

$$\mathbf{V_c}=\begin{bmatrix}\sqrt{25.30^2+3.13^2}\\\sqrt{15.25^2+(-3.71)^2}\end{bmatrix}=\begin{bmatrix}25.49\\15.69\end{bmatrix}\text{kN}\qquad \mathbf{M_c}=\begin{bmatrix}\sqrt{40.48^2+5.008^2}\\\sqrt{24.40^2+5.936^2}\end{bmatrix}=\begin{bmatrix}40.79\\25.11\end{bmatrix}\text{kNm}$$

$$\mathbf{M_b}=\begin{bmatrix}\sqrt{64.88^2+(-0.928)^2}\\\sqrt{24.40^2+(-5.936)^2}\end{bmatrix}=\begin{bmatrix}64.89\\25.11\end{bmatrix}\text{kNm}$$

The CQC summation on the internal forces:

$$\alpha_{12}=\frac{T_2}{T_1}=\frac{0.252}{0.704}=0.358$$

$$r_{12}=\frac{8\xi^2(1+\alpha_{12})\alpha_{12}^{3/2}}{\left(1-\alpha_{12}^2\right)^2+4\xi^2\alpha_{12}(1+\alpha_{12})^2}=\frac{8\cdot 0.05^2(1+0.358)0.358^{3/2}}{\left(1-0.358^2\right)^2+4\cdot 0.05^2\cdot 0.358(1+0.358)^2}=0.007588$$

$$\mathbf{r}=\begin{bmatrix}1 & 0.007588\\0.007588 & 1\end{bmatrix}$$

And based on these values the results of the CQC summation:

$$\mathbf{V_c}=\begin{bmatrix}\sqrt{25.30^2+3.13^2+2\cdot 25.3\cdot 3.13\cdot 0.007588}\\\sqrt{15.25^2+(-3.71)^2+2\cdot 15.25\cdot(-3.71)\cdot 0.007588}\end{bmatrix}=\begin{bmatrix}25.52\\15.67\end{bmatrix}\text{kN}$$

$$\mathbf{M_c}=\begin{bmatrix}\sqrt{40.48^2+5.008^2+2\cdot 40.48\cdot 5.008\cdot 0.007588}\\\sqrt{24.40^2+5.936^2+2\cdot 24.40\cdot 5.936\cdot 0.007588}\end{bmatrix}=\begin{bmatrix}40.83\\25.16\end{bmatrix}\text{kNm}$$

$$\mathbf{M_b}=\begin{bmatrix}\sqrt{64.88^2+(-0.928)^2+2\cdot 64.88\cdot(-0.928)\cdot 0.007588}\\\sqrt{24.40^2+(-5.936)^2+2\cdot 24.40\cdot(-5.936\cdot 0.007588)}\end{bmatrix}=\begin{bmatrix}64.88\\25.07\end{bmatrix}\text{kNm}$$

The following displacements come from the FEM-Design calculation on the complete frame structure to ensure the comprehensible final results on the P-Δ effect.

The displacements at the storeys respect to the two different mode shapes considering the displacement behaviour factor:

$$\mathbf{u_1}=q_d\begin{bmatrix}9.54\\17.15\end{bmatrix}=1.5\begin{bmatrix}9.54\\17.15\end{bmatrix}=\begin{bmatrix}14.31\\25.73\end{bmatrix}\text{mm}\qquad \mathbf{u_2}=q_d\begin{bmatrix}0.818\\-0.540\end{bmatrix}=1.5\begin{bmatrix}0.818\\-0.540\end{bmatrix}=\begin{bmatrix}1.227\\-0.810\end{bmatrix}\text{mm}$$

Based on these values the storey drifting respect to the two different mode shapes:

$$\Delta_1=\begin{bmatrix}14.31\\25.73-14.31\end{bmatrix}=\begin{bmatrix}14.31\\11.42\end{bmatrix}\text{mm}\qquad \Delta_2=\begin{bmatrix}1.227\\-0.810-1.227\end{bmatrix}=\begin{bmatrix}1.227\\-2.037\end{bmatrix}\text{mm}$$

SRSS summation on the story drifting:

$$\Delta=\begin{bmatrix}\sqrt{14.31^2+1.227^2}\\\sqrt{11.42^2+(-2.027)^2}\end{bmatrix}=\begin{bmatrix}14.36\\11.60\end{bmatrix}\text{mm}$$

P-Δ effect checking on the total building:

$$\mathbf{P}_{\mathbf{tot}}=\begin{bmatrix}(m_1+m_2)g\\m_2 g\end{bmatrix}=\begin{bmatrix}(160.3+135.2)9.81\\135.2\cdot 9.81\end{bmatrix}=\begin{bmatrix}2899\\1326\end{bmatrix}\text{kN}$$

$$\mathbf{V}_{\mathbf{tot}}=\begin{bmatrix}6\cdot\sqrt{50.6^2+6.24^2}\\6\cdot\sqrt{30.5^2+(-7.42)^2}\end{bmatrix}=\begin{bmatrix}305.9\\188.3\end{bmatrix}\text{kN}$$

$$\theta_1=\frac{P_{tot1}\Delta_1}{V_{tot1}h}=\frac{2899\cdot 14.36}{305.9\cdot 3200}=0.0425\quad ;\quad \theta_2=\frac{P_{tot2}\Delta_2}{V_{tot2}h}=\frac{1326\cdot 11.60}{188.3\cdot 3200}=0.0255$$

After the hand calculation let's see the results from the FEM-Design calculation and compare them to each other. Fig. 5.3.6 shows the effective modal masses from the FE calculation. Practically these values coincide with the hand calculation.

Shape no.	T	mx'	mx'
[-]	[s]	[%]	[t]
1	0.704	92.2	272.374
2	0.252	7.8	23.139

Figure 5.3.6 – The first two fundamental periods and the effective modal masses from FEM-Design

Fig. 5.3.7 and the following table shows the equivalent resultant shear forces and the base shear forces respect to the first two mode shapes. The differences between the two calculations are less than 2%.

	Storey 1 equivalent resultant [kN]		Storey 2 equivalent resultant [kN]		Base shear force [kN]	
	Hand	FEM	Hand	FEM	Hand	FEM
Mode shape 1	120.6	121.9	81.98	81.80	303.6	306.9
Mode shape 2	183.0	185.0	– 44.52	– 45.47	36.50	36.33

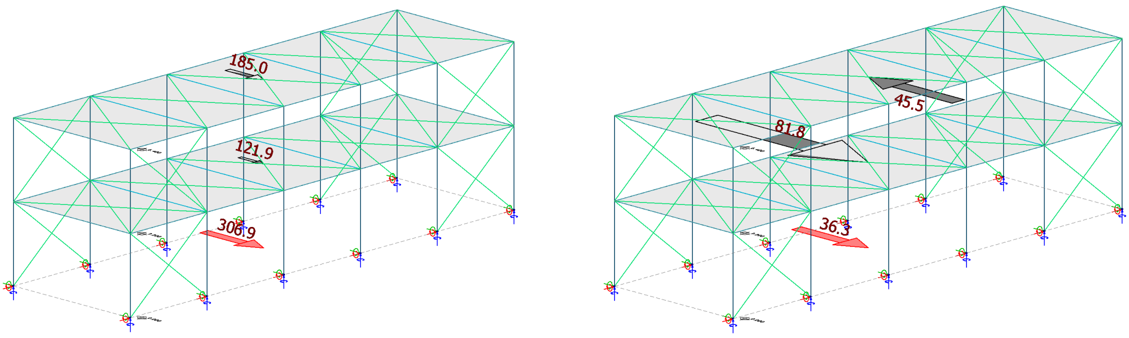

Figure 5.3.7 – The equivalent forces respect to the storeys and the base shear forces for the first two mode shapes [kN]

Fig. 5.3.8-9 and the following table shows the internal forces after the different summation methods (SRSS and CQC). The differences between the two calculation methods are less than 2 %. Here by the moments the difference between the hand and FEM calculations (10%) comes from the simplified moment hand calculation.

	Column shear force [kN]		Column bending moment [kNm]		Beam bending moment [kNm]	
	Storey 1	Storey 2	Storey 1	Storey 2	Storey 1	Storey 2
SRSS Hand	25.49	15.69	40.79	25.11	64.89	25.11
SRSS FEM	25.78	15.89	(37.18+45.32)/2= 41.25	27.51	59.33	27.51
CQC Hand	25.50	15.67	40.83	25.16	64.88	25.07
CQC FEM	25.80	15.86	(37.21+45.36)/2= 41.29	27.46	59.32	27.46

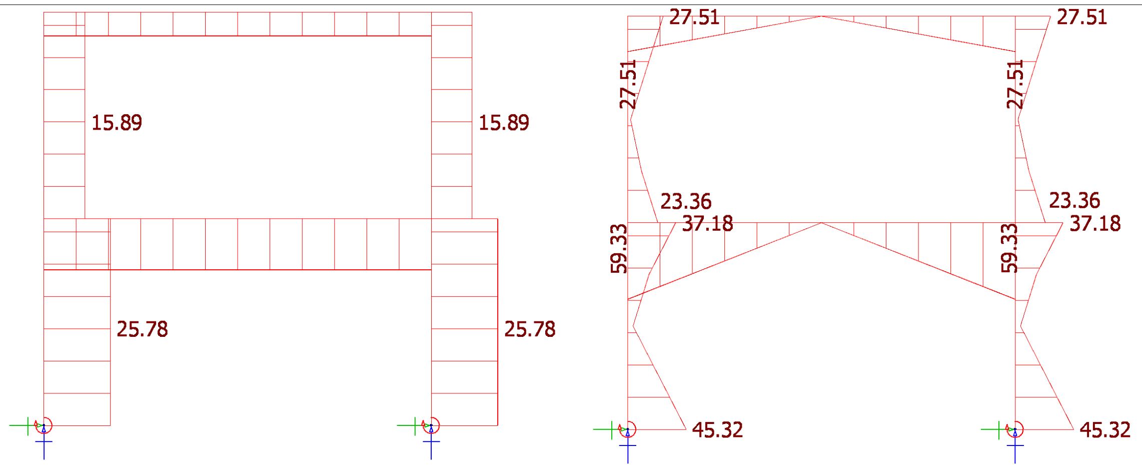

Figure 5.3.8 – The shear force [kN] *and bending moment diagram* [kNm] *after the SRSS summation rule*

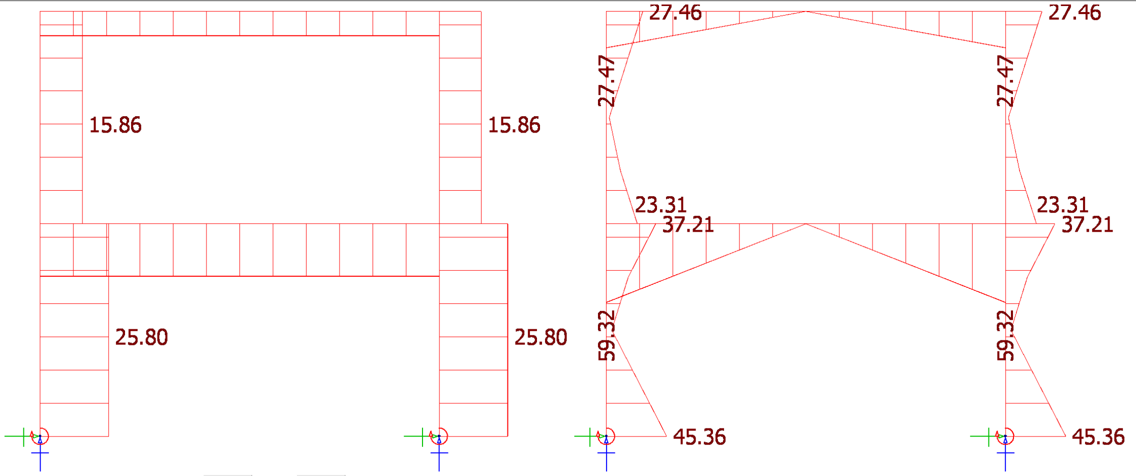

Figure 5.3.9 – The shear force [kN] *and bending moment diagram* [kNm] *after the CQC summation rule*

Fig. 5.3.10 shows the Θ values from FEM-Design. The differences between the hand calculation and FEM-Design calculations are less than 3%.

Storey	Theta x
1	0.0420
2	0.0253

Figure 5.3.10 – The θ values at the different storeys from FEM-Design

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http://download.strusoft.com/FEM-Design/inst170x/models/5.3 Modal analysis of a concrete frame building.str

6 Calculation considering diaphragms

6.1. A simple calculation with diaphragms

If we apply two diaphragms on the two storeys of the building from Chapter 5.3 then the eigenfrequencies and the periods will be the same what we indicated in Chapter 5.3 with the bracing system.

6.2. The calculation of the shear center

In this example we show that how can we calculate the shear center of a storey based on the FEM-Design calculation. We analyzed a bottom fixed cantilever structure made of three concrete shear walls which are connected to each other at the edges (see Fig. 6.2.1). The diaphragm is applied at the top plane of the structure (see also Fig. 6.2.1 right side). If the height of the structure is high enough then the shear center will be on the same geometry point where it should be when we consider the complete cross section of the shear walls as a "thin-walled" "C" cross section (see Fig. 6.2.1 left side). Therefore we calculate by hand the shear center of the "C" profile assumed to be a thin-walled cross section then compare the results what we can get from FEM-Design calculation with diaphragms.

Secondly we calculate the idealized bending stiffnesses in the principal rigidity directions by hand and compare the results what we can calculate with FEM-Design results.

Inputs:

Height of the walls	H = 63 m
The thickness of the walls	t = 20 cm
The width of wall number 1 and 3	$w_1 = w_3 = 4.0$ m
The width of wall number 2	$w_2 = 6.0$ m
The applied Young's modulus of concrete	E = 9.396 GPa

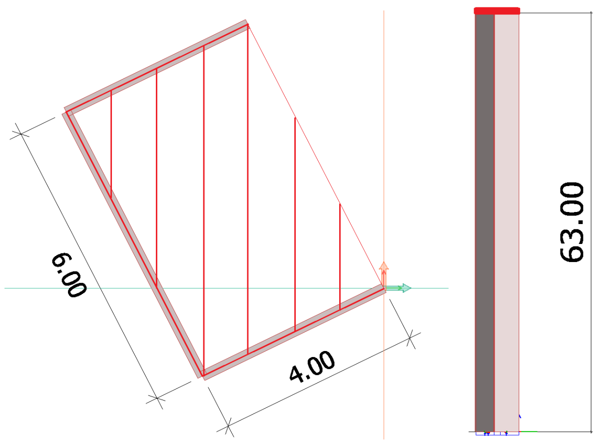


Figure 6.2.1 – The geometry of the bracing core and the height of the bottom fixed structure (the diaphragm is lying on the top plane, see the red line and hatch)

First of all let's see Fig. 6.2.2. The applied cross section is a symmetric cross section. In the web the shear stress distribution comes from the shear formula regarding bending with shear (see Fig. 6.2.2) therefore it is a second order polynom. In the flanges the shear stress distribution is linear according to the thin walled theory. With the resultant of these shear stress distribution (see Fig. 6.2.2, V_1, V_2 and V_3) the position of the shear center can be calculated based on the statical (equilibrium) equations.

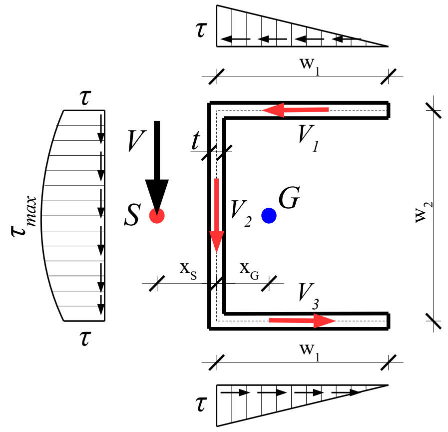


Figure 6.2.2 – The shear stress distribution in a thin-walled cross section if the shear force acting on the shear center

The shear stress values (see Fig. 6.2.2):

$$\tau = \frac{V\,S}{I\,t} = \frac{1\cdot(0.2\cdot 4\cdot 3)}{\left(\frac{0.2\cdot 6^3}{12}+\frac{4\cdot 6.2^3}{12}-\frac{4\cdot 5.8^3}{12}\right)\cdot 0.2} = 0.6665\,\frac{\mathrm{kN}}{\mathrm{m}^2}$$

$$\tau_{max} = \frac{V\,S_{max}}{I\,t} = \frac{1\cdot(0.2\cdot 4\cdot 3+0.2\cdot 3\cdot 1.5)}{\left(\frac{0.2\cdot 6^3}{12}+\frac{4\cdot 6.2^3}{12}-\frac{4\cdot 5.8^3}{12}\right)\cdot 0.2} = 0.9164\,\frac{\mathrm{kN}}{\mathrm{m}^2}$$

Based on these stresses the resultant in the flanges and in the web:

$$V_1 = V_3 = \frac{\tau\,t\,w_1}{2} = \frac{0.6665\cdot 0.2\cdot 4}{2} = 0.2667\,\mathrm{kN}$$

$$V_2 = \frac{2}{3}(\tau_{max}-\tau)\,w_2 t + \tau\,w_2 t = \frac{2}{3}(0.9164-0.6665)\,6\cdot 0.2 + 0.6665\cdot 6\cdot 0.2 = 0.9997\,\mathrm{kN}$$

Respect to the equilibrium (sum of the forces):

$$V = 1\,\mathrm{kN} \approx V_2 = 0.9997\,\mathrm{kN}$$

And also respect to the equilibrium (if the external load is acting on the shear center, see Fig.

6.2.2) the sum of the moments:

$$V_1 w_2 = V_2 x_S$$

$$x_S = \frac{V_1 w_2}{V_2} = \frac{0.2667 \cdot 6}{0.9997} = 1.601\,\mathrm{m}$$

Thus the shear center is lying on the symmetry axis and it is x_S=1.601 m from the web (see Fig. 6.2.2). In FEM-Design the global coordinate system does not coincide with the symmetry axis of the structure (see Fig. 6.2.1). Therefore we need no transform the results.

Lets be a selected key node at the diaphragm in the global coordinate system (see Fig. 6.2.1):

$$x_m = 0\,\mathrm{m} \quad ; \quad y_m = 0\,\mathrm{m}$$

Based on the unit forces (1 kN) and moment (1 kNm) on the key node the displacements of the key node are as follows based on the FEM-Design calculation (see the Scientific Manual Calculation considering diaphragm chapter also):

According to unit force on key node in X direction:

$$u_{xx} = 1.5852\,\mathrm{mm} \qquad u_{yx} = 0.72166\,\mathrm{mm} \qquad \varphi_{zx} = 0.29744 \cdot 10^{-4}\,\mathrm{rad}$$

According to unit force on key node in Y direction:

$$u_{xy} = 0.72166\,\mathrm{mm} \qquad u_{yy} = 7.3314\,\mathrm{mm} \qquad \varphi_{zy} = 0.10328 \cdot 10^{-2}\,\mathrm{rad}$$

According to unit moment on key node around Z direction:

$$\varphi_{zz} = 0.16283 \cdot 10^{-3}\,\mathrm{rad}$$

Based on these finite element results the global coordinates of the shear center of the diaphragm are:

$$x_S = x_m - \frac{\varphi_{zy}}{\varphi_{zz}} = 0 - \frac{0.10328 \cdot 10^{-2}}{0.16283 \cdot 10^{-3}} = -6.343\,\mathrm{m}$$

$$y_S = y_m + \frac{\varphi_{zx}}{\varphi_{zz}} = 0 + \frac{0.29744 \cdot 10^{-4}}{0.16283 \cdot 10^{-3}} = +0.1827\,\mathrm{m}$$

In FEM-Design the coordinates of the middle point of the web are (see Fig. 6.2.1):

$$x_{mid} = -4.919\,\mathrm{m} \quad ; \quad y_{mid} = +0.894\,\mathrm{m}$$

With the distance between these two points we get a comparable solution with the hand calculation.

$$x_{SFEM}=\sqrt{(x_S-x_{mid})^2+(y_S-y_{mid})^2}=\sqrt{(-6.343-(-4.919))^2+(0.1827-0.894)^2}=1.592\,\mathrm{m}$$

The difference between FEM and hand calculation is less than 1%.

The gravity center of the cross section (Fig. 6.2.2) can be calculated based on the statical moments. And of course the gravity center lying on the symmetry axis. The distance of the gravity center from the web is:

$$x'_{\mathrm{G}}=\frac{S_y'}{A}=\frac{2(0.2\cdot 4\cdot 2)}{0.2(4+6+4)}=1.143\,\mathrm{m}$$

With the input Young's modulus and with the second moments of inertia the idealized bending stiffnesses in the principal directions can be calculated by hand.

$$E\,I_1=9396\cdot 10^3\cdot\left(\frac{0.2\cdot 6^3}{12}+\frac{4\cdot 6.2^3}{12}-\frac{4\cdot 5.8^3}{12}\right)=1.692\cdot 10^8\,\mathrm{kNm}^2$$

$$E\,I_2=9396\cdot 10^3\cdot\left(2\frac{0.2\cdot 4^3}{12}+2(0.2\cdot 4(2-1.143)^2)+\frac{6\cdot 0.2^3}{12}+0.2\cdot 6(1.143)^2\right)=4.585\cdot 10^7\,\mathrm{kNm}^2$$

With the finite element results we can calculate the translations of the shear center according to the unit forces and moment on the key node (see the former calculation method).

The distances between the shear center and the selected key node are:

$$\Delta x=x_S-x_m=-6.343-0=-6.343\,\mathrm{m}=-6343\,\mathrm{mm}$$

$$\Delta y=y_S-y_m=+0.1827-0=+0.1827\,\mathrm{m}=182.7\,\mathrm{mm}$$

The translations of the shear center are as follows:

$$u_{Sxx}=u_{xx}-\varphi_{zx}\,\Delta y=1.5852-0.29744\cdot 10^{-4}\cdot 182.7=1.5798\,\mathrm{mm}$$

$$u_{Syx}=u_{yx}+\varphi_{zx}\,\Delta x=0.72166+0.29744\cdot 10^{-4}\cdot(-6343)=0.5330\,\mathrm{mm}$$

$$u_{Sxy}=u_{xy}-\varphi_{zy}\,\Delta y=0.72166-0.10328\cdot 10^{-2}\cdot 182.7=0.5330\,\mathrm{mm}$$

$$u_{Syy}=u_{yy}+\varphi_{zy}\,\Delta x=7.3314+0.10328\cdot 10^{-2}\cdot(-6343)=0.7803\,\mathrm{mm}$$

Based on these values the translations of the shear center in the principal directions:

$$u_1=\frac{u_{Sxx}+u_{Syy}}{2}+\sqrt{\left(\frac{u_{Sxx}-u_{Syy}}{2}\right)^2+u_{Sxy}{}^2}=\frac{1.5798+0.7803}{2}+\sqrt{\left(\frac{1.5798-0.7803}{2}\right)^2+0.5330^2}=$$
$$=1.8463\mathrm{mm}$$

$$u_2 = \frac{u_{Sxx}+u_{Syy}}{2} - \sqrt{\left(\frac{u_{Sxx}-u_{Syy}}{2}\right)^2 + u_{Sxy}{}^2} = \frac{1.5798+0.7803}{2} - \sqrt{\left(\frac{1.5798-0.7803}{2}\right)^2 + 0.5330^2} =$$
$$= 0.5138\,\text{mm}$$

According to these values the angles of the principal rigidity directions:

$$\alpha_{1FEM} = \arctan\frac{u_1 - u_{Sxx}}{u_{Sxy}} = \arctan\frac{1.8463-1.5798}{0.533} = 26.57°$$

$$\alpha_{2FEM} = \arctan\frac{u_2 - u_{Sxx}}{u_{Sxy}} = \arctan\frac{0.5138-1.5798}{0.533} = -63.43°$$

The directions coincide with the axes of symmetries (see Fig. 6.2.1-2) which is one of the principal rigidity direction in this case.

Then with FEM-Design results we can calculate the idealized bending stiffnesses of the structure:

$$EI_{1FEM} = \frac{H^3}{3u_2} = \frac{63^3}{3\cdot(0.5138/1000)} \qquad E\,I_{1FEM} = 1.622\cdot10^8\,\text{kNm}^2$$

$$EI_{2FEM} = \frac{H^3}{3u_1} = \frac{63^3}{3\cdot(1.8463/1000)} \qquad E\,I_{2FEM} = 4.514\cdot10^7\,\text{kNm}^2$$

The difference between FEM and hand calculation is less than 4%.

Download link to the example file:

FEM-Design file:

http://download.strusoft.com/FEM-Design/inst170x/models/6.1.1 A simple calculation with diaphragms.str

Section Editor file:

http://download.strusoft.com/FEM-Design/inst170x/models/6.1.2 The calculation of the shear center.sec

7 Calculations considering nonlinear effects

7.1 Uplift calculation

7.1.1 A trusses with limited compression members

In this example a truss will be analyzed. First of all we calculate the normal forces in the truss members and the maximum deflection for the given concentrated loads. After this step we calculate the load multiplier when the vertical truss members reaches its limit compression bearing capacity what we set. See the inputs in the following table. After the hand calulation we compare the results with the FEM-Design nonlinear calculation results.

Inputs:

Column height/Span length	H = 2.0 m; L = 8.0 m
The cross sections	KKR 80x80x6
The area of the cross sections	$A = 1652\ mm^2$
The elastic modulus	E = 210 GPa, structural steel
The concentrated loads	F = 40 kN
Limited compression of the vertical truss members	$P_{cr} = 700$ kN

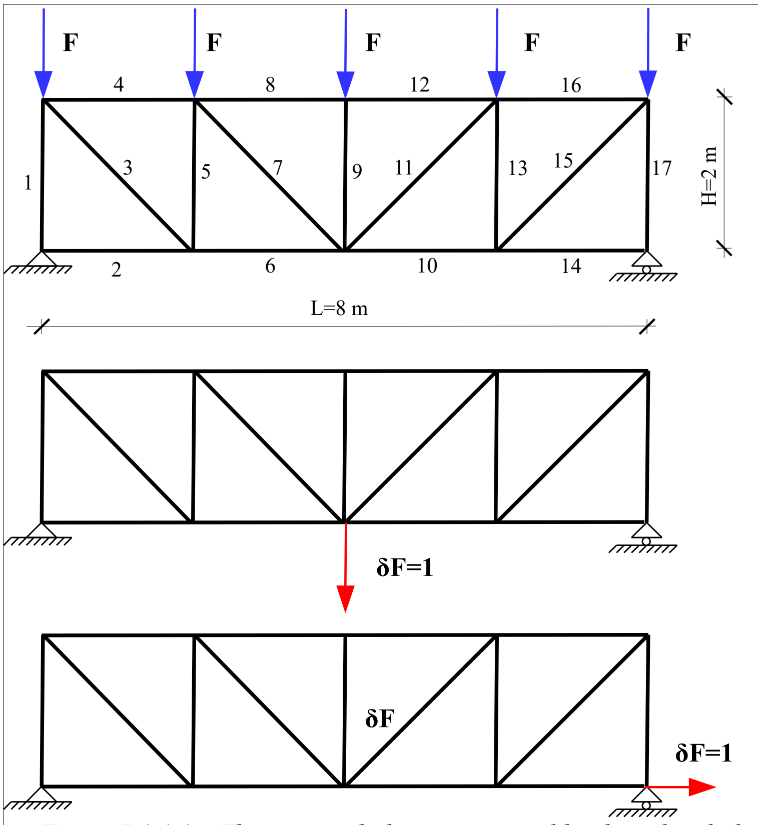


Figure 7.1.1.1 – The truss with the concentrated loads and with the virtual loads for the translation calculation

The normal forces in the truss members based on the hand calculation (without further details) are:

$N_1 = N_{17} = -100\,\text{kN}$; $N_2 = N_{14} = 0\,\text{kN}$; $N_3 = N_{15} = +84.85\,\text{kN}$;

$N_4 = N_5 = N_{13} = N_{16} = -60.00\,\text{kN}$; $N_6 = N_{10} = +60.00\,\text{kN}$;

$N_7 = N_{11} = +28.28\,\text{kN}$; $N_8 = N_{12} = -80.00\,\text{kN}$; $N_9 = -40.00\,\text{kN}$.

The normal forces in the truss members according to the vertical virtual force (see Fig. 7.1.1.1):

$N_{1,1} = N_{1,17} = -0.5\,\text{kN}$; $N_{1,2} = N_{1,14} = \text{N}_{1,9} = 0\,\text{kN}$;

$N_{1,3} = N_{1,15} = N_{1,7} = N_{1,11} = +0.7071\,\text{kN}$;

$N_{1,4} = N_{1,5} = N_{1,13} = N_{1,16} = -0.5\,\text{kN}$; $N_{1,6} = N_{1,10} = +0.5\,\text{kN}$;

$N_{1,8} = N_{1,12} = -1.0\,\text{kN}$.

The normal forces in the truss members according to the horizontal virtual force (see Fig. 7.1.1.1):

$N_{2,2} = N_{2,6} = N_{2,10} = N_{2,14} = +1.0\,\text{kN}$;

$N_{2,1} = N_{2,3} = N_{2,4} = N_{2,5} = N_{2,7} = N_{2,8} = N_{2,9} = N_{2,11} = N_{2,12} = N_{2,13} = N_{2,15} = N_{2,16} = N_{2,17} = 0\,\text{kN}$.

The hand calculation of the vertical translation at the mid-span with the virtual force method:

$$e_z = \frac{1}{EA} \sum_{i=1}^{17} N_i \delta N_{1,i} l_i = 0.003841\,\text{m} = 3.841\,\text{mm}$$

The hand calculation of the horizontal translation at right roller with the virtual force method:

$$e_x = \frac{1}{EA} \sum_{i=1}^{17} N_i \delta N_{2,i} l_i = 0.0006918\,\text{m} = 0.6918\,\text{mm}$$

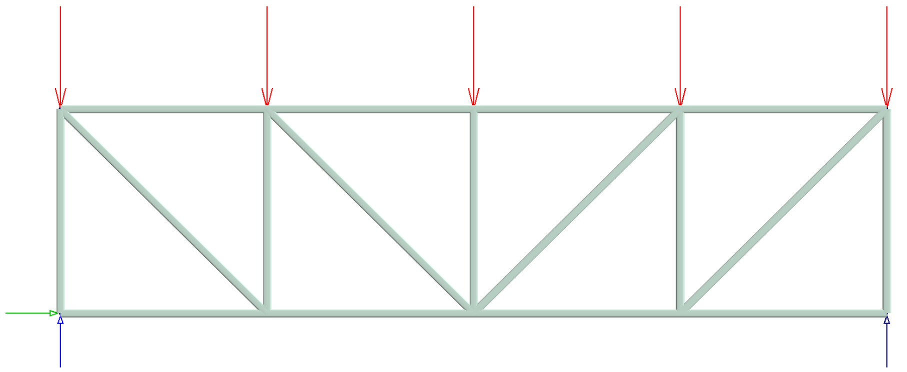


Figure 7.1.1.2 – The truss with the concentrated loads in FEM-Design

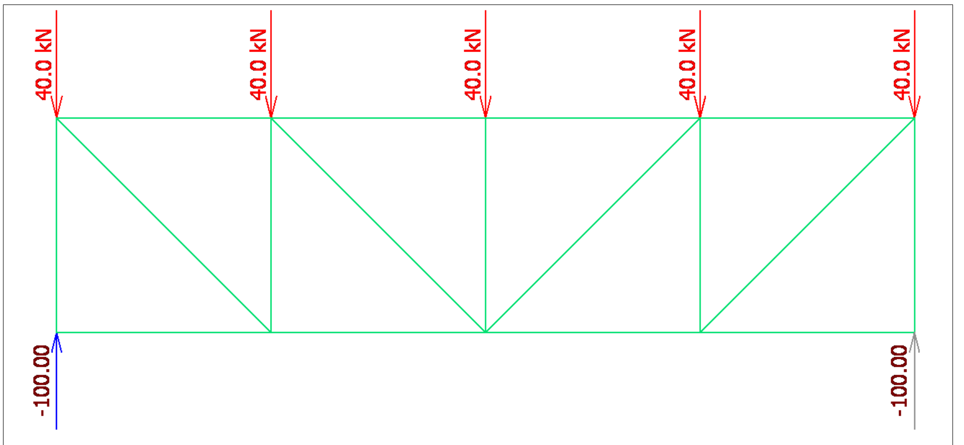


Figure 7.1.1.3 – The reaction forces

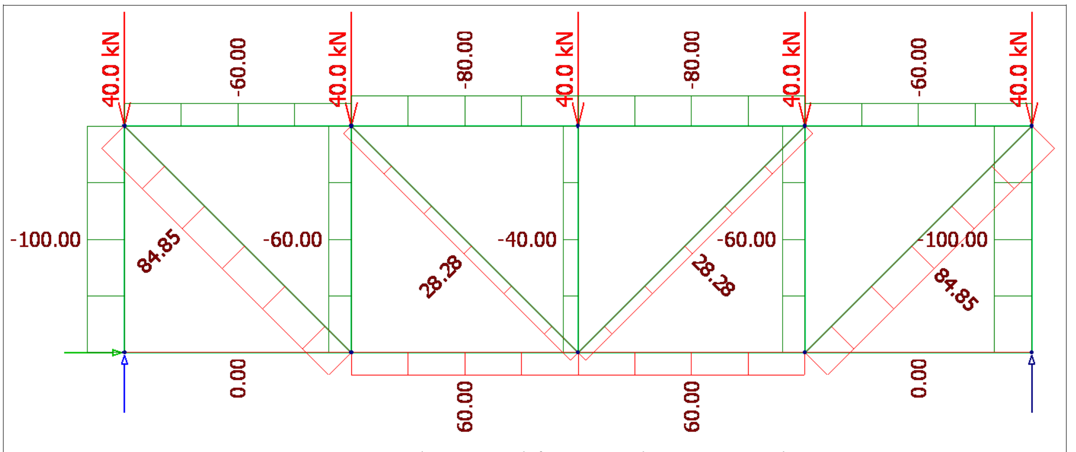


Figure 7.1.1.4 – The normal forces in the truss members

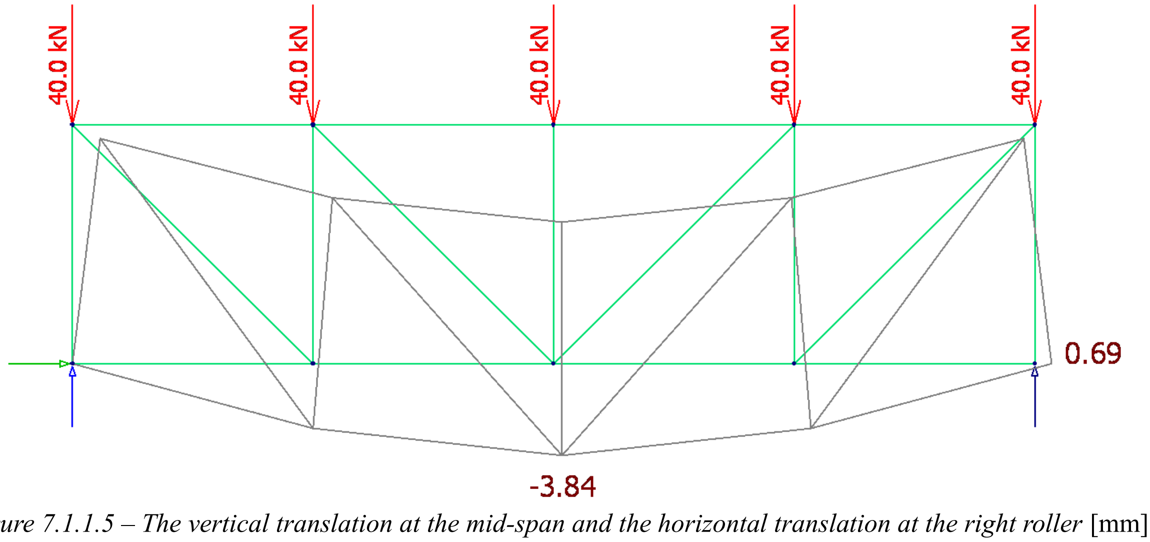


Figure 7.1.1.5 – The vertical translation at the mid-span and the horizontal translation at the right roller [mm]

The translations and the normal forces in the truss members based on the hand calculation are identical with the FEM-Design calculation, see Fig. 7.1.1.2-5.

After this step we would like to know the maximum load multiplier when the vertical truss members reaches its limit compression bearing capacity what we set, P_{cr} = 700 kN. The maximum compression force arises in the side columns, see the hand calculation, N_1 = (–)100 kN. Therefore the load multiplier based on the hand calculation is λ = 7.0.

Let's see the FEM-Design uplift calculation considering the limit compression in the vertical members.

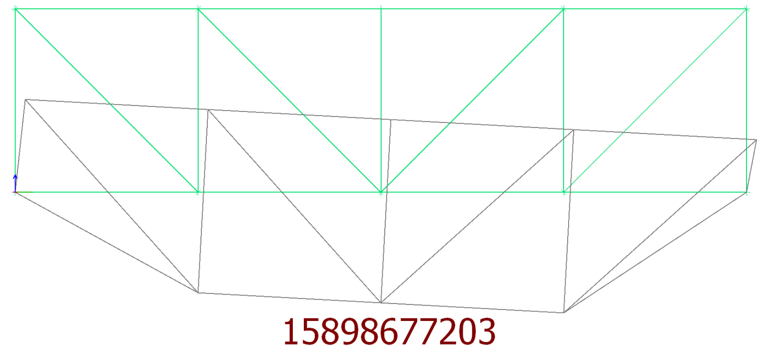


Figure 7.1.1.6 – Large nodal displacements when the side truss members reached the limit compression value [mm]

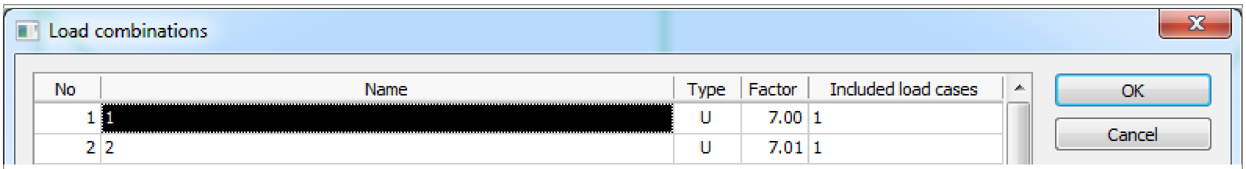


Figure 7.1.1.7 – The two different analyzed load multiplier in FEM-Design

With λ_{FEM} = 7.00 multiplier the FEM-Design analysis gives the accurate result but with λ_{FEM} = 7.01 (see Fig. 7.1.1.7) large nodal displacements occurred, see Fig. 7.1.1.6. Thus by this structure if we neglect the effect of the side members the complete truss became a statically overdetermined structure. FEM-Design solve this problem with iterative solver due to the fact that these kind of problems are nonlinear.

Based on the FEM-Design calculation the load multiplier is identical with the hand calculation.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.1.1 A trusses with limited compression members.str

7.1.2 A continuous beam with three supports

In this example we analyse non-linear supports of a beam. Let's consider a continuous beam with three supports with the following parameters:

Inputs:

Span length	L = 2 m, total length = 2x2 = 4 m
The cross sections	Rectangle: 120x150 mm
The elastic modulus	E = 30 GPa, concrete C20/25
Intensity of distributed load (total, partial)	p = 10 kN/m

In Case I. the distribution of the external load and the nonlinearity of the supports differ from Case II. See the further details below (Fig. 7.1.2.1 and Fig. 7.1.2.8).

a) Case I.

In this case the distributed load is a total load (Fig. 7.1.2.1). In the first part all of the three supports behave the same way for compression and tension. In the second part of this case the middle support only bears tension. We calculated in both cases the deflections, shear forces and bending moments by hand and compared the results with FEM-Design uplift (nonlinear) calculations.

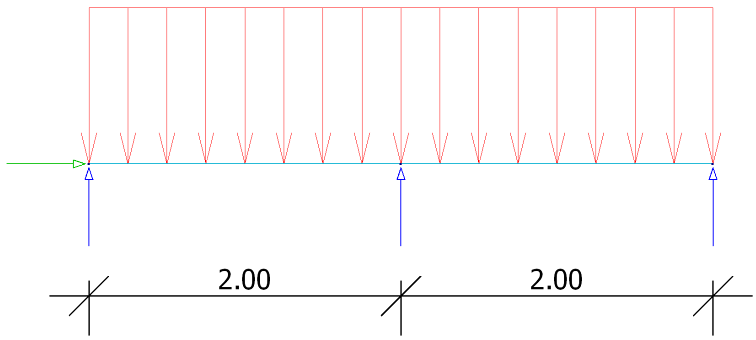


Figure 7.1.2.1 – The beam with three supports and uniform distributed load

In first part of this case the maximum deflection comes from the following formula considering only the bending deformations in the beam:

$$e_{max} = \frac{2.1}{384}\frac{pL^4}{EI} = \frac{2.1}{384}\frac{10 \cdot 2^4}{30000000 \cdot 0.12 \cdot 0.15^3/12} = 0.0008642\,\mathrm{m} = 0.8642\,\mathrm{mm}$$

The relevant results with FEM-Design:

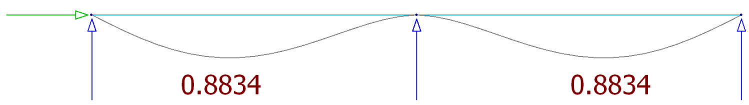


Figure 7.1.2.2 – The deflection [mm] *of the beam with three supports (total load)*

The extremums of the shear force without signs:

$$V_1 = \frac{3}{8} p\,L = \frac{3}{8} 10 \cdot 2 = 7.5\,\mathrm{kN} \quad ; \quad V_2 = \frac{5}{8} p\,L = \frac{5}{8} 10 \cdot 2 = 12.5\,\mathrm{kN}$$

The relevant results with FEM-Design:

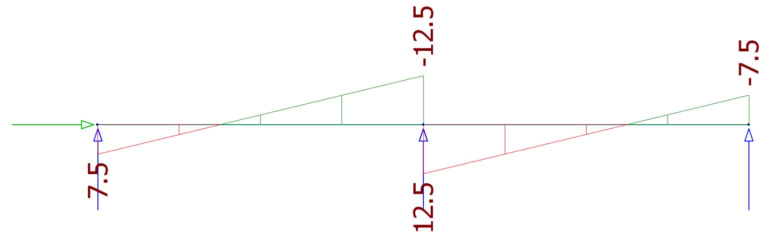


Figure 7.1.2.3 – The shear diagram [kN] *of the beam with three supports (total load)*

The extremums of the bending moment without signs:

$$M_{midspan} = \frac{9}{128} p\,L^2 = \frac{9}{128} 10 \cdot 2^2 = 2.812\,\mathrm{kNm} \quad ; \quad M_{middle} = \frac{1}{8} p\,L^2 = \frac{1}{8} 10 \cdot 2^2 = 5.0\,\mathrm{kNm}$$

The relevant results with FEM-Design:

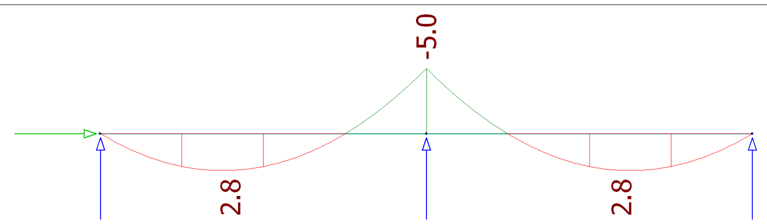


Figure 7.1.2.4 – The bending moment diagram [kNm] *of the beam with three supports (total load)*

When the middle support only bear tension (second part of this case) basically under the total vertical load (Fig. 7.1.2.1) the middle support is not active (support nonlinearity). Therefore it works as a simply supported beam with two supports. The deflection, the shear forces and the bending moments are the following:

The maximum deflection comes from the following formula considering only the bending deformations in the beam:

$$e_{max} = \frac{5}{384} \frac{p(L+L)^4}{EI} = \frac{5}{384} \frac{10 \cdot (2+2)^4}{30000000 \cdot 0.12 \cdot 0.15^3 / 12} = 0.03292\,\mathrm{m} = 32.92\,\mathrm{mm}$$

The relevant results with FEM-Design:

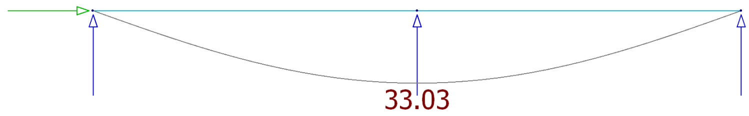


Figure 7.1.2.5 – The deflection [mm] *of the beam when the middle support only bear tension (total load)*

The maximum of the shear force without sign:

$$V=\frac{1}{2}p(L+L)=\frac{1}{2}10(2+2)=20\,\mathrm{kN}$$

The relevant results with FEM-Design:

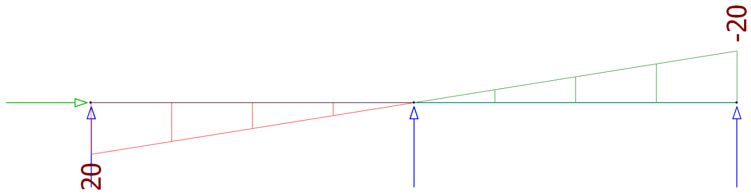


Figure 7.1.2.6 – The shear diagram [kN] *of the beam when the middle support only bear tension (total load)*

The extremum of the bending moment without sign:

$$M_{max}=\frac{1}{8}p(L+L)^2=\frac{1}{8}10\cdot(2+2)^2=20\,\mathrm{kNm}$$

The relevant results with FEM-Design:

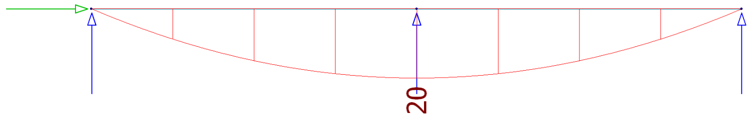


Figure 7.1.2.7 – The bending moment diagram [kNm] *of the beam when the middle support only bear tension (total load)*

The differences between the calculated results by hand and by FEM-Design are less than 2%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.1.2 A continuous beam with three supports case a.str

b) Case II.

In this case the distributed load is a partial load (Fig. 7.1.2.8). In the first part all of the three supports behave the same way for compression and tension. In the second part of this case the right side support only bears compression. We calculate in both cases the deflections, shear forces and bending moments by hand and compared the results with FEM-Design calculations.

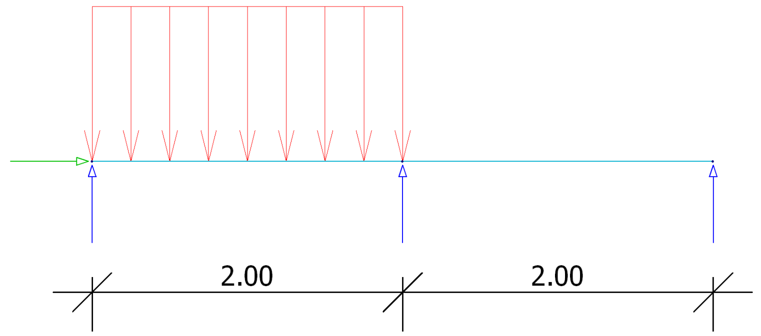


Figure 7.1.2.8 – The beam with three supports and partial load

The extremums of the deflection come from the following formulas considering only the bending deformations in the beam (without signs):

$$e_{max} \approx \frac{2.1}{384}\frac{(p/2)L^4}{EI} + \frac{5}{384}\frac{(p/2)L^4}{EI} = \frac{2.1}{384}\frac{10/2 \cdot 2^4}{EI} + \frac{5}{384}\frac{10/2 \cdot 2^4}{EI} = 0.001461\,\text{m} = 1.461\,\text{mm}$$

$$e_{min} \approx \frac{5}{384}\frac{(p/2)L^4}{EI} - \frac{2}{384}\frac{(p/2)L^4}{EI} = \frac{5}{384}\frac{10/2 \cdot 2^4}{EI} - \frac{2}{384}\frac{10/2 \cdot 2^4}{EI} = 0.0006173\,\text{m} = 0.6173\,\text{mm}$$

The relevant results with FEM-Design:

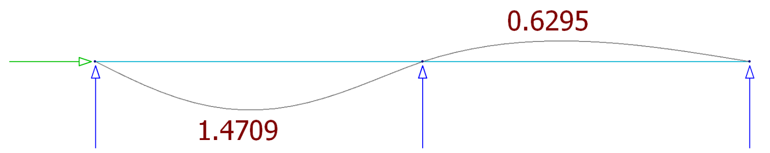


Figure 7.1.2.9 – The deflection [mm] *of the beam with three supports (partial load)*

The extremums of the shear force without signs:

$$V_1 = \frac{7}{16}pL = \frac{7}{16}10 \cdot 2 = 8.75\,\text{kN} \quad ; \quad V_2 = \frac{9}{16}pL = \frac{9}{16}10 \cdot 2 = 11.25\,\text{kN} \quad ;$$

$$V_3 = \frac{1}{16}pL = \frac{1}{16}10 \cdot 2 = 1.25\,\text{kN}$$

The relevant results with FEM-Design:

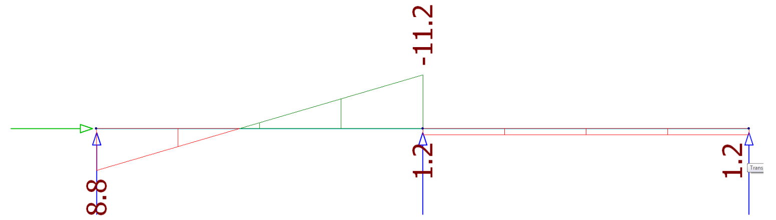


Figure 7.1.2.10 – The shear diagram [kN] *of the beam with three supports (partial load)*

The extremums of the bending moment without signs:

$$M_{midspan}=\frac{\left(\frac{7}{16}pL\right)^2}{2p}=\frac{\left(\frac{7}{16}10\cdot 2\right)^2}{2\cdot 10}=3.828\,\text{kNm} \quad ; \quad M_{middle}=\frac{1}{16}pL^2=\frac{1}{16}10\cdot 2^2=2.5\,\text{kNm}$$

The relevant results with FEM-Design:

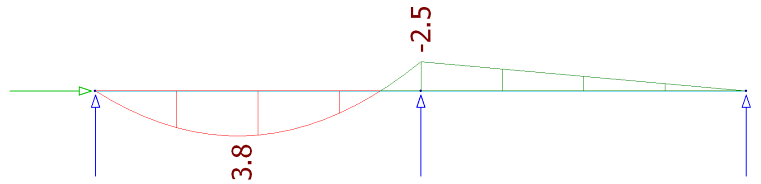


Figure 7.1.2.11 – The bending moment diagram [kNm] *of the beam with three supports (partial load)*

When the right side support only bear compression (second part of this case) basically under the partial vertical load (Fig. 7.1.2.8) the right side support is not active (support nonlinearity). Therefore it works as a simply supported beam with two supports. The deflection, the shear forces and the bending moments are the following:

The maximum deflection comes from the following formula considering only the bending deformations in the beam:

$$e_{midspan}=\frac{5}{384}\frac{pL^4}{EI}=\frac{5}{384}\frac{10\cdot 2^4}{EI}=0.002058\,\text{m}=2.058\,\text{mm}$$

$$e_{right}=\frac{1}{24}\frac{pL^4}{EI}=\frac{1}{24}\frac{10\cdot 2^4}{EI}=0.006584\,\text{m}=6.584\,\text{mm}$$

The relevant results with FEM-Design:

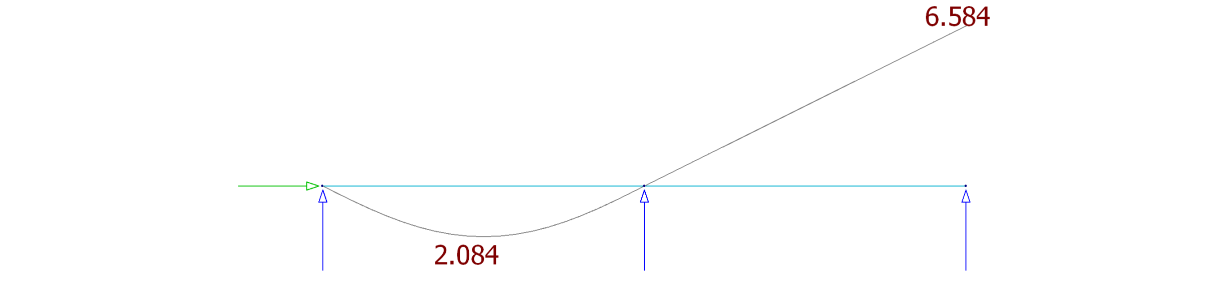


Figure 7.1.2.12 – The deflection [mm] *of the beam when the right support only bear compression (partial load)*

The extremum of the shear force without sign:

$$V=\frac{1}{2}p\,L=\frac{1}{2}10\cdot 2=10\,\text{kN}$$

The relevant results with FEM-Design:

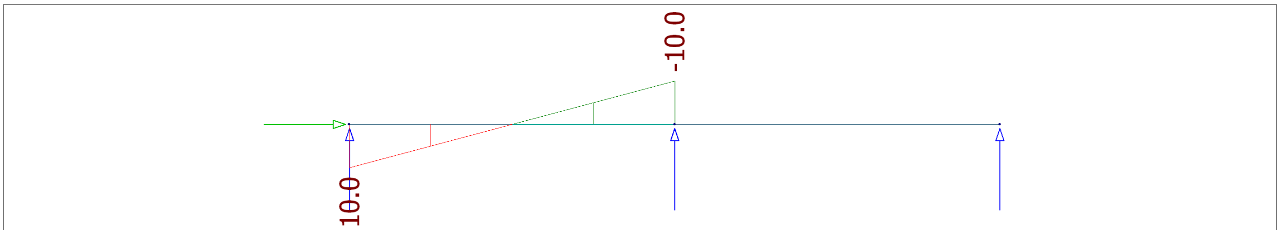


Figure 7.1.2.13 – The shear diagram [kN] *of the beam when the right support only bear compression (partial load)*

The extremum of the bending moment without sign:

$$M_{max}=\frac{1}{8}p\,L^2=\frac{1}{8}10\cdot 2^2=5.0\,\text{kNm}$$

The relevant results with FEM-Design:

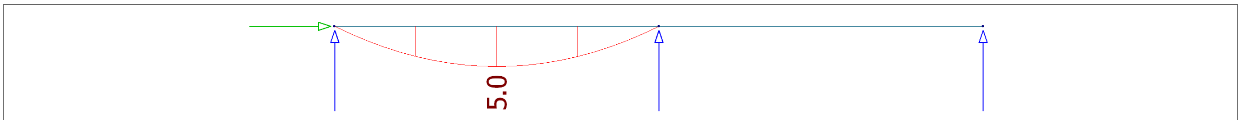


Figure 7.1.2.14 – The bending moment diagram [kNm] *of the beam when the right support only bear compression (partial load)*

The differences between the calculated results by hand and by FEM-Design are less than 2%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.1.2 A continuous beam with three supports case b.str

7.2 Cracked section analysis by reinforced concrete elements

7.2.1 Cracked deflection of a simly supported beam

Inputs:

Span length	$L_{eff} = 7.2$ m
The cross section	Rectangle: b = 300 mm; h = 450 mm
The elastic modulus of concrete	$E_{cm} = 31.476$ GPa, C25/30
The creep factor	$\varphi_{28} = 2.35$
Effective elastic modulus of concrete	$E_{ceff} = E_{cm}/(1+\varphi_{28}) = 9.396$ GPa
Mean tensile strength	$f_{ctm} = 2.565$ MPa
Elastic modulus of steel bars	$E_s = 200$ GPa
Characteristic value of dead load	$g_k = 8.5$ kN/m
Characteristic value of live load	$q_k = 12.0$ kN/m
Live load combination factor	$\psi_2 = 0.6$
Diameter of the longitudinal reinforcement	$\phi_l = 18$ mm
Diameter of the stirrup reinforcement	$\phi_s = 8$ mm
Area of longitudinal reinforcement	$A_l = 5x18^2\pi/4 = 1272.3\ m^2$
Nominal concrete cover	$c_{nom} = 20$ mm
Effective height	$d = h - c_{nom} - \phi_s - \phi_l/2 = 413$ mm
Shrinkage strain	$\varepsilon_{cs} = 0.5$ ‰

The cross sectional properties without calculation details (considering creep effect):

I. stress stadium second moment of inertia	$I_I = 3.075x10^9\ mm^4$
II. stress stadium second moment of inertia	$I_{II} = 2.028x10^9\ mm^4$
I. stress stadium position of neutral axis	$x_I = 256.4$ mm
II. stress stadium position of neutral axis	$x_{II} = 197.3$ mm

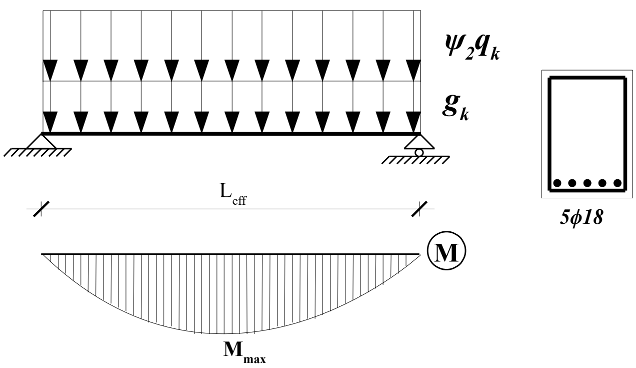


Figure 7.2.1.1 – The simply supported RC beam

The calculation of deflection according to EN 1992-1-1:

The load value for the quasi-permanent load combination:

$$p_{qp}=g_k+\psi_2 q_k=8.5+0.6\cdot 12=15.7\,\frac{\text{kN}}{\text{m}}$$

The maximum deflection with cross sectional properties in Stadium I. (uncracked):

$$w_{k.I}=\frac{5}{384}\frac{p_{qp}L_{eff}^{4}}{E_{ceff}I_I}=\frac{5}{384}\frac{15.7\cdot 7.2^4}{9396000\cdot 0.003075}=0.01901\,\text{m}=19.01\,\text{mm}$$

The maximum deflection with cross sectional properties in Stadium II. (cracked):

$$w_{k.II}=\frac{5}{384}\frac{p_{qp}L_{eff}^{4}}{E_{ceff}I_{II}}=\frac{5}{384}\frac{15.7\cdot 7.2^4}{9396000\cdot 0.002028}=0.02883\,\text{m}=28.83\,\text{mm}$$

The maximum bending moment under the quasi-permanent load (see Fig. 7.2.1.1):

$$M_{max}=\frac{1}{8}p_{qp}L_{eff}^{2}=\frac{1}{8}15.7\cdot 7.2^2=101.74\,\text{kNm}$$

The cracking moment with the mean tensile strength:

$$M_{cr}=f_{ctm}\frac{I_I}{h-x_I}=2565\frac{0.003075}{0.45-0.2564}=40.74\,\text{kNm}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour at the most unfavourable cross-section:

$$\zeta=\max\left[1-0.5\left(\frac{M_{cr}}{M_{max}}\right)^2,0\right]=\max\left[1-0.5\left(\frac{40.74}{101.74}\right)^2,0\right]=0.9198$$

This value is almost 1.0, it means that the final deflection will be closer to the cracked deflection than to the uncracked one.

The final deflection with the aim of interpolation factor:

$$w_k=(1-\zeta)w_{k.I}+\zeta\, w_{k.II}=(1-0.9198)19.01+0.9198\cdot 28.83=28.04\,\text{mm}$$

This deflection is even greater if we are considering the effect of shrinkage.

The curvatures in uncracked and cracked states due to shrinkage:

$$\kappa_{I,cs} = \varepsilon_{cs} \frac{E_s}{E_{c,eff}} \frac{S_{s,I}}{I_I} = \frac{0.5}{1000} \frac{200}{9.396} \frac{1272.3 \cdot (413 - 256.4)}{3.075 \cdot 10^9} = 6.896 \cdot 10^{-4} \frac{1}{\mathrm{m}}$$

$$\kappa_{II,cs} = \varepsilon_{cs} \frac{E_s}{E_{c,eff}} \frac{S_{s,II}}{I_{II}} = \frac{0.5}{1000} \frac{200}{9.396} \frac{1272.3 \cdot (413 - 197.3)}{2.028 \cdot 10^9} = 1.440 \cdot 10^{-3} \frac{1}{\mathrm{m}}$$

The additional deflection in the two different states due to shrinkage:

$$w_{k.I.cs} = \frac{1}{8} L_{eff}{}^2 \kappa_{I,cs} = \frac{1}{8} 7.2^2 \cdot 6.896 \cdot 10^{-4} = 0.004469\,\mathrm{m} = 4.469\,\mathrm{mm}$$

$$w_{k.II.cs} = \frac{1}{8} L_{eff}{}^2 \kappa_{II,cs} = \frac{1}{8} 7.2^2 \cdot 1.440 \cdot 10^{-3} = 0.009331\,\mathrm{m} = 9.331\,\mathrm{mm}$$

The total deflection considering cracking and the effect of shrinkage:

$$w_{k.cs} = (1-\zeta)(w_{k.I} + w_{k.I.cs}) + \zeta (w_{k.II} + w_{k.II.cs}) =$$
$$= (1 - 0.9198)(19.01 + 4.469) + 0.9198(28.83 + 9.331) = 36.98\,\mathrm{mm}$$

First we modelled the beam with beam elements. In FEM-Design we increased the division number of the beam finite elements to ten to get the more accurate results.

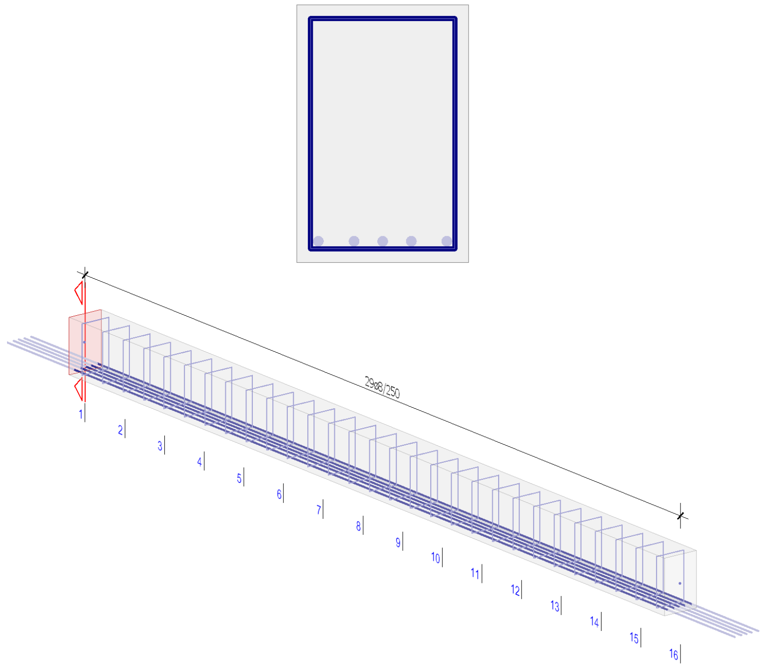


Figure 7.2.1.2 – The cross section of the RC beam in FEM-Design

Fig. 7.2.1.2 shows the applied cross section and reinforcement with the defined input parameters.

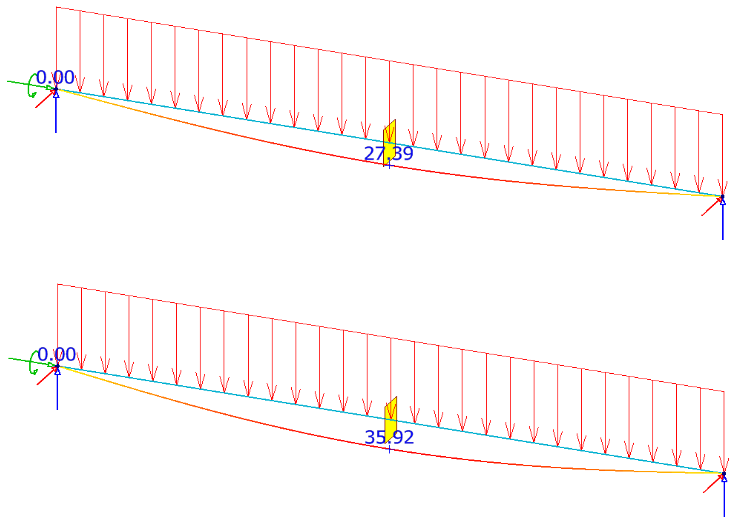


Figure 7.2.1.3 – The deflection [mm] *of the RC bar in FEM-Design with cracked section analysis (without shrinkage [above], with shrinkage [below])*

Fig. 7.2.1.3 shows the deflection after the cracked section analysis without and with considering shrinkage. The deflection of the beam model in FEM-Design:

Cracked section analysis without shrinkage: $w_{kFEM} = 27.39\,\mathrm{mm}$

Cracked section analysis with shrinkage: $w_{k.csFEM} = 35.92\,\mathrm{mm}$

The difference between the hand and FEM-Design calculations is less than 3%, but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result.

Secondly we modelled the beam with shell finite elements. Fig. 7.2.1.4 shows the applied specific reinforcement with the defined input parameters with slab model.

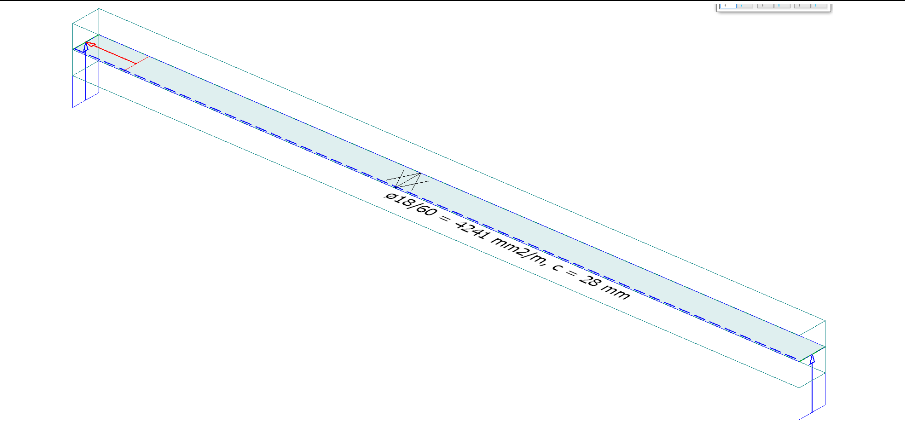


Figure 7.2.1.4 – The specific reinforcement with the shell model in FEM-Design

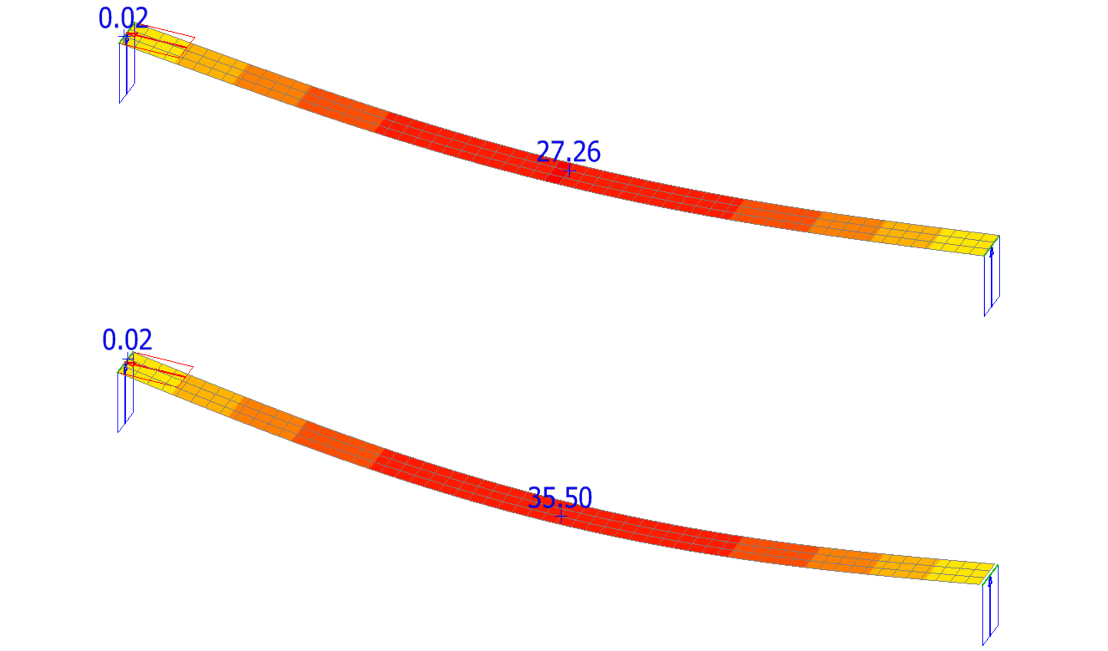


Figure 7.2.1.5 – The deflection [mm] *of the RC shell model in FEM-Design with cracked section analysis (without shrinkage [above], with shrinkage [below])*

Fig. 7.2.1.5 shows the deflection and the finite element mesh after the cracked section analysis without and with considering shrinkage. The deflection of the shell model in FEM-Design:

Cracked section analysis without shrinkage: $w_{kFEM} = 27.26\,mm$

Cracked section analysis with shrinkage: $w_{kFEM} = 35.50\,mm$

The difference between the hand and FEM-Design calculations is less than 4% but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result.

Fig. 7.2.1.6 shows the effect of tension stiffening (without shrinkage effects) in FEM-Design at the relevant SLS load interval. We indicated the load level where the first crack occured.

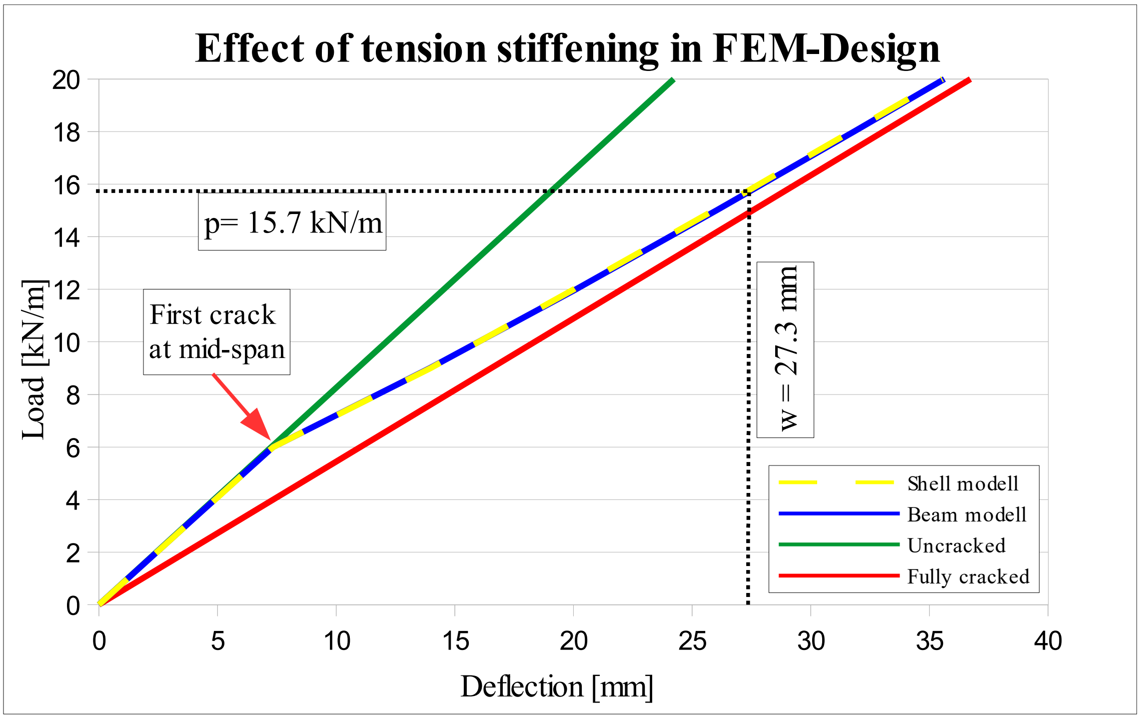


Figure 7.2.1.6 – The effect of tension stiffening by a simply supported beam

Download link to the example files:

Beam model:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.1 Cracked deflection of a simply supported beam.beam.str

Shell model:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.1 Cracked deflection of a simply supported beam.shell.str

7.2.2 Cracked deflection of a statically indeterminate beam

Inputs:

Span length	$L_{eff} = 7.2$ m
The cross section	Rectangle: b = 300 mm; h = 450 mm
The elastic modulus of concrete	$E_{cm} = 31.476$ GPa, C25/30
The creep factor	$\varphi_{28} = 2.35$
Effective elastic modulus of concrete	$E_{ceff} = E_{cm}/(1+\varphi_{28}) = 9.396$ GPa
Mean tensile strength	$f_{ctm} = 2.565$ MPa
Elastic modulus of steel bars	$E_s = 200$ GPa
Characteristic value of dead load	$g_k = 8.5$ kN/m
Characteristic value of live load	$q_k = 12.0$ kN/m
Live load combination factor	$\psi_2 = 0.6$
Diameter of the longitudinal reinforcement	$\phi_l = 18$ mm
Diameter of the stirrup reinforcement	$\phi_s = 8$ mm
Area of longitudinal reinforcement	$A_l = 5x18^2\pi/4 = 1272.3$ m^2
Nominal concrete cover	$c_{nom} = 20$ mm
Effective height	$d = h - c_{nom} - \phi_s - \phi_l/2 = 413$ mm
Shrinkage strain	$\varepsilon_{cs} = 0.5$ ‰

The cross sectional properties without calculation details (considering creep effect):

I. stress stadium second moment of inertia	$I_I = 3.075x10^9$ mm^4
II. stress stadium second moment of inertia	$I_{II} = 2.028x10^9$ mm^4
I. stress stadium position of neutral axis	$x_I = 256.4$ mm
II. stress stadium position of neutral axis	$x_{II} = 197.3$ mm

In this chapter we will calculate the cracked deflection of a statically indeterminate structure (see Fig. 7.2.2.1).

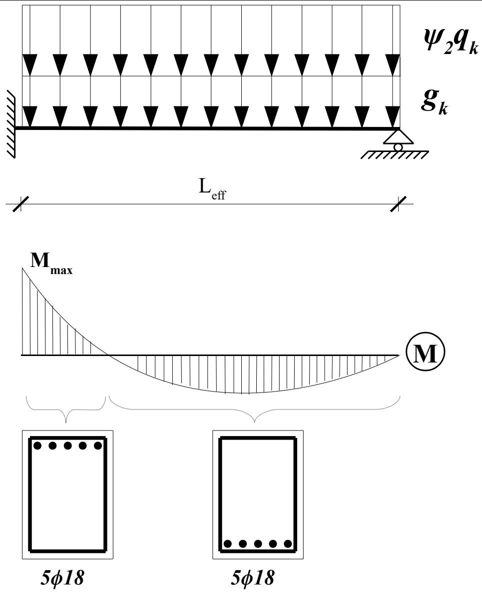


Figure 7.2.2.1 – The fixed end and a roller boundary conditions RC beam (statically indeterminate structure)

The maximum deflection with cross sectional properties in Stadium I. (uncracked):

$$w_{k.I}=\frac{2.1}{384}\frac{p_{qp}L_{eff}^{4}}{E_{ceff}I_{I}}=\frac{2.1}{384}\frac{15.7\cdot 7.2^{4}}{9396000\cdot 0.003075}=0.007986\,\mathrm{m}=7.986\,\mathrm{mm}$$

The maximum deflection with cross sectional properties in Stadium II. (cracked):

$$w_{k.II}=\frac{2.1}{384}\frac{p_{qp}L_{eff}^{4}}{E_{ceff}I_{II}}=\frac{2.1}{384}\frac{15.7\cdot 7.2^{4}}{9396000\cdot 0.002028}=0.01211\,\mathrm{m}=12.11\,\mathrm{mm}$$

The maximum bending moment under the quasi-permanent load at the fixed end (see Fig. 7.2.2.1):

$$M_{max}=\frac{1}{8}p_{qp}L_{eff}^{2}=\frac{1}{8}15.7\cdot 7.2^{2}=101.74\,\mathrm{kNm}$$

The cracking moment with the mean tensile strength:

$$M_{cr}=f_{ctm}\frac{I_I}{h-x_I}=2565\frac{0.003075}{0.45-0.2564}=40.74\,\mathrm{kNm}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour at the most unfavourable cross-section:

$$\zeta = \max\left[1-0.5\left(\frac{M_{cr}}{M_{max}}\right)^2, 0\right] = \max\left[1-0.5\left(\frac{40.74}{101.74}\right)^2, 0\right] = 0.9198$$

This value is almost 1.0, it means that the final deflection will be closer to the cracked deflection than to the uncracked one with the hand calculation.

The final deflection with the aim of interpolation factor:

$$w_k = (1-\zeta) w_{k.I} + \zeta\, w_{k.II} = (1-0.9198)\, 7.986 + 0.9198 \cdot 12.11 = 11.78\,\text{mm}$$

This deflection is even greater if we are considering the effect of shrinkage.

The curvatures in uncracked and cracked states due to shrinkage:

$$\kappa_{I,cs} = \varepsilon_{cs} \frac{E_s}{E_{c,eff}} \frac{S_{s,I}}{I_I} = \frac{0.5}{1000} \frac{200}{9.396} \frac{1272.3 \cdot (413-256.4)}{3.075 \cdot 10^9} = 6.896 \cdot 10^{-4} \frac{1}{\text{m}}$$

$$\kappa_{II,cs} = \varepsilon_{cs} \frac{E_s}{E_{c,eff}} \frac{S_{s,II}}{I_{II}} = \frac{0.5}{1000} \frac{200}{9.396} \frac{1272.3 \cdot (413-197.3)}{2.028 \cdot 10^9} = 1.440 \cdot 10^{-3} \frac{1}{\text{m}}$$

The additional deflection due to shrinkage:

$$w_{k.I.cs} = \frac{1}{12} L_{eff}^{\;2} \kappa_{I,cs} = \frac{1}{12} 7.2^2 \cdot 6.896 \cdot 10^{-4} = 0.002979\,\text{m} = 2.979\,\text{mm}$$

$$w_{k.II.cs} = \frac{1}{12} L_{eff}^{\;2} \kappa_{II,cs} = \frac{1}{12} 7.2^2 \cdot 1.440 \cdot 10^{-3} = 0.006221\,\text{m} = 6.221\,\text{mm}$$

The total deflection considering cracking and the effect of shrinkage:

$$w_{k.cs} = (1-\zeta)(w_{k.I} + w_{k.I.cs}) + \zeta (w_{k.II} + w_{k.II.cs}) =$$
$$= (1-0.9198)(7.986+2.979) + 0.9198(12.11+6.221) = 17.74\,\text{mm}$$

First we modelled the beam with beam elements. In FEM-Design we increased the division number of the beam finite elements to ten to get the more accurate results.

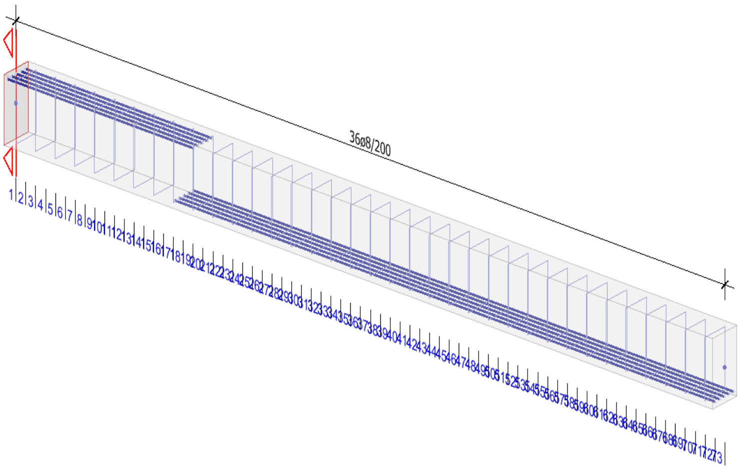


Figure 7.2.2.2 – The cross section of the RC beam in FEM-Design

Fig. 7.2.2.2 shows the applied cross section and reinforcement with the defined input parameters.

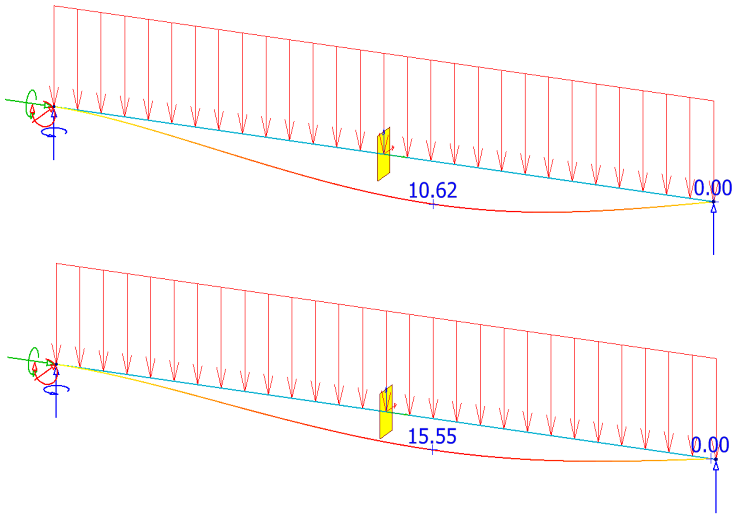


Figure 7.2.2.3 – The deflection [mm] *of the RC bar in FEM-Design with cracked section analysis (without shrinkage [above], with shrinkage [below])*

Fig. 7.2.2.3 shows the deflection after the cracked section analysis without and with considering shrinkage. The deflection of the beam model in FEM-Design:

Cracked section analysis without shrinkage: $w_{kFEM} = 10.62\ \text{mm}$

Cracked section analysis with shrinkage: $w_{k.csFEM} = 15.55\ \text{mm}$

The difference between the hand and FEM-Design calculations is around 10%, but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result and by a statically indeterminate structure it causes greater difference.

Secondly we modelled the beam with shell finite elements. Fig. 7.2.2.4 shows the applied specific reinforcement with the defined input parameters with slab model.

Figure 7.2.2.4 – The specific reinforcement with the shell model in FEM-Design

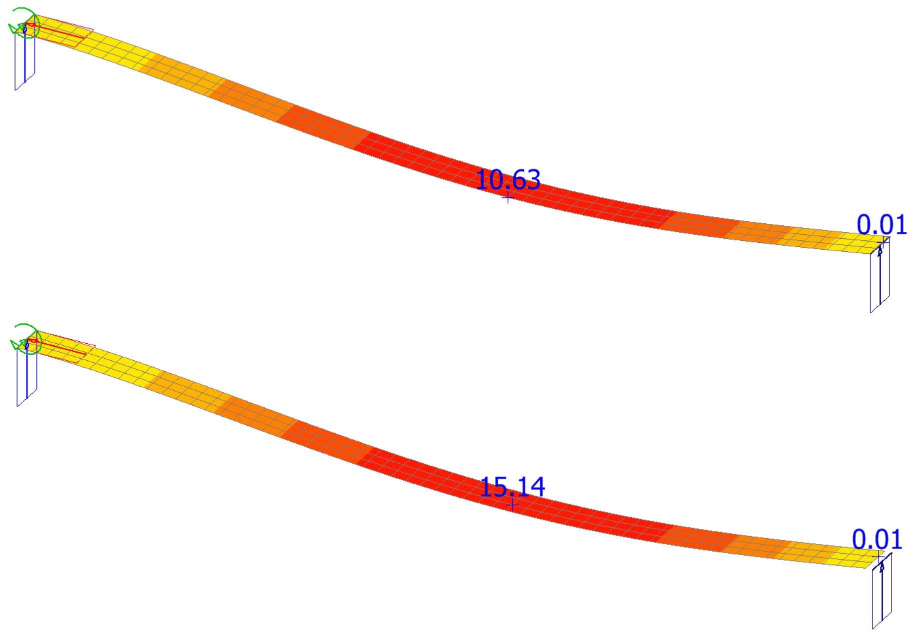


Figure 7.2.2.5 – The deflection [mm] *of the RC shell model in FEM-Design with cracked section analysis (without shrinkage [above], with shrinkage [below])*

Fig. 7.2.2.5 shows the deflection and the finite element mesh after the cracked section analysis without and with considering shrinkage. The deflection of the shell model in FEM-Design:

Cracked section analysis without shrinkage: $w_{kFEM} = 10.63\,\text{mm}$

Cracked section analysis with shrinkage: $w_{k.csFEM} = 15.14\,\text{mm}$

The difference between the hand and FEM-Design calculations is around 10%, but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result and by a statically indeterminate structure it causes greater difference.

Fig. 7.2.2.6 shows the effect of tension stiffening (without shrinkage effects) in FEM-Design at the relevant SLS load interval. We indicated the load level where the first crack occurred at the fixed end and at the mid-span.

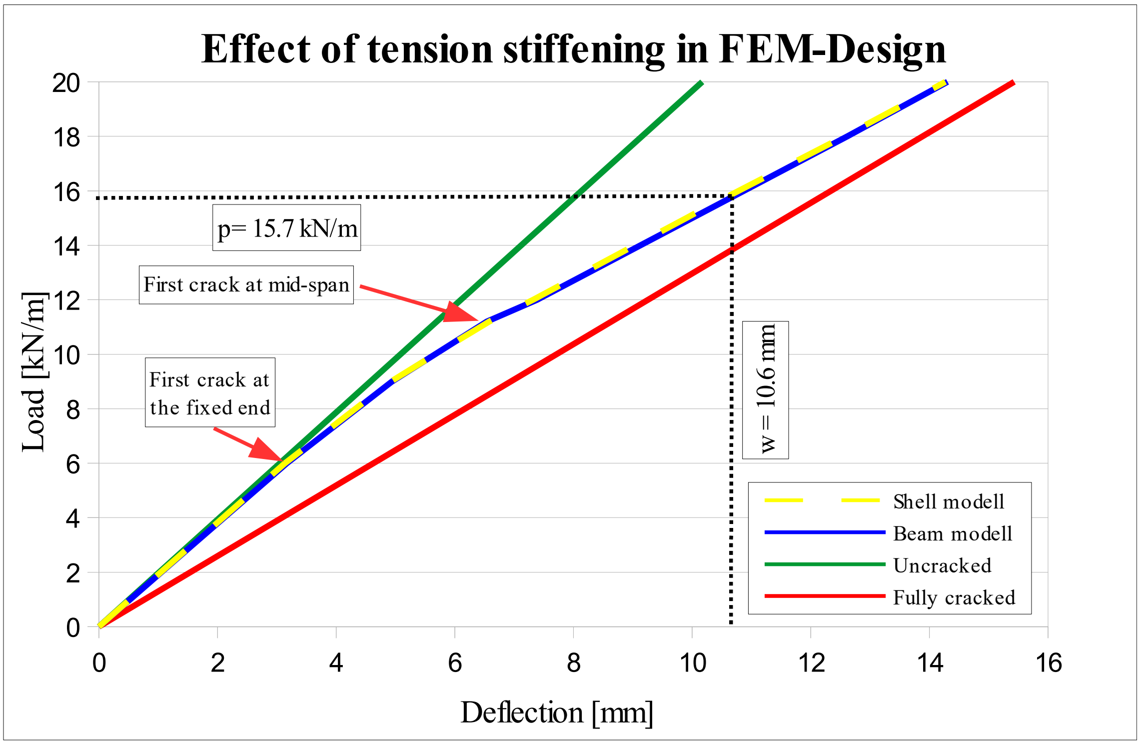


Figure 7.2.2.6 – The effect of tension stiffening by a statically indeterminate structure

Download link to the example files:

Beam model:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.2 Cracked deflection of a statically indeterminate beam.beam.str

Shell model:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.2 Cracked deflection of a statically indeterminate beam.shell.str

7.2.3 Cracked deflection of a cantilever beam

Inputs:

Span length	$L_{eff} = 4$ m
The cross section	Rectangle: b = 300 mm; h = 450 mm
The elastic modulus of concrete	$E_{cm} = 31.476$ GPa, C25/30
The creep factor	$\varphi_{28} = 2.35$
Effective elastic modulus of concrete	$E_{ceff} = E_{cm}/(1+\varphi_{28}) = 9.396$ GPa
Mean tensile strength	$f_{ctm} = 2.565$ MPa
Elastic modulus of steel bars	$E_s = 200$ GPa
Characteristic value of dead load	$g_k = 8.5$ kN/m
Characteristic value of live load	$q_k = 12.0$ kN/m
Live load combination factor	$\psi_2 = 0.6$
Diameter of the longitudinal reinforcement	$\phi_l = 18$ mm
Diameter of the stirrup reinforcement	$\phi_s = 8$ mm
Area of longitudinal reinforcement	$A_l = 5x18^2\pi/4 = 1272.3\ m^2$
Nominal concrete cover	$c_{nom} = 20$ mm
Effective height	$d = h - c_{nom} - \phi_s - \phi_l/2 = 413$ mm
Shrinkage	$\varepsilon_{cs} = 0.4$ ‰

The cross sectional properties without calculation details:

I. stress stadium second moment of inertia	$I_I = 3.075x10^9\ mm^4$
II. stress stadium second moment of inertia	$I_{II} = 2.028x10^9\ mm^4$
I. stress stadium position of neutral axis	$x_I = 256.4$ mm
II. stress stadium position of neutral axis	$x_{II} = 197.3$ mm

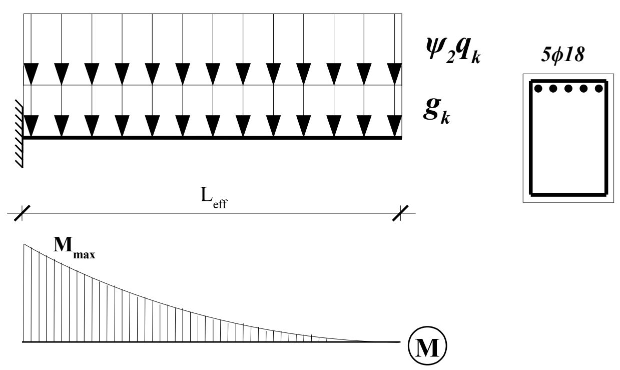


Figure 7.2.3.1 – The cantilever RC beam

The calculation of deflection according to EN 1992-1-1:

The load value for the quasi-permanent load combination:

$$p_{qp}=g_k+\psi_2 q_k=8.5+0.6\cdot 12=15.7\,\frac{\mathrm{kN}}{\mathrm{m}}$$

The maximum deflection with cross sectional properties in Stadium I. (uncracked):

$$w_{k.I}=\frac{1}{8}\frac{p_{qp}L_{eff}^{\;4}}{E_{ceff}I_I}=\frac{1}{8}\frac{15.7\cdot 4^4}{9396000\cdot 0.003075}=0.01739\,\mathrm{m}=17.39\,\mathrm{mm}$$

The maximum deflection with cross sectional properties in Stadium II. (cracked):

$$w_{k.II}=\frac{1}{8}\frac{p_{qp}L_{eff}^{\;4}}{E_{ceff}I_{II}}=\frac{1}{8}\frac{15.7\cdot 4^4}{9396000\cdot 0.002028}=0.02637\,\mathrm{m}=26.37\,\mathrm{mm}$$

The maximum bending moment under the quasi-permanent load:

$$M_{max}=\frac{1}{2}p_{qp}L_{eff}^{\;2}=\frac{1}{2}15.7\cdot 4^2=125.6\,\mathrm{kNm}$$

The cracking moment with the mean tensile strength:

$$M_{cr}=f_{ctm}\frac{I_I}{h-x_I}=2565\frac{0.003075}{0.45-0.2564}=40.74\,\mathrm{kNm}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta=\max\left[1-0.5\left(\frac{M_{cr}}{M_{max}}\right)^2,0\right]=\max\left[1-0.5\left(\frac{40.74}{125.6}\right)^2,0\right]=0.9474$$

This value is almost 1.0, it means that the final deflection will be closer to the cracked deflection than to the uncracked one.

The final deflection with the aim of interpolation factor:

$$w_k=(1-\zeta)w_{k.I}+\zeta\,w_{k.II}=(1-0.9474)\,17.39+0.9474\cdot 26.37=25.90\,\mathrm{mm}$$

This deflection is even greater if we are considering the effect of shrinkage.

The curvatures in uncracked and cracked states due to shrinkage:

$$\kappa_{I,cs}=\varepsilon_{cs}\frac{E_s}{E_{c,eff}}\frac{S_{s,I}}{I_I}=\frac{0.4}{1000}\frac{200}{9.396}\frac{1272.3\cdot(413-256.4)}{3.075\cdot 10^9}=5.517\cdot 10^{-4}\frac{1}{\mathrm{m}}$$

$$\kappa_{II,cs}=\varepsilon_{cs}\frac{E_s}{E_{c,eff}}\frac{S_{s,II}}{I_{II}}=\frac{0.4}{1000}\frac{200}{9.396}\frac{1272.3\cdot(413-197.3)}{2.028\cdot 10^9}=1.152\cdot 10^{-3}\frac{1}{\mathrm{m}}$$

The additional deflection due to shrinkage:

$$w_{k.I.cs}=\frac{1}{2}L_{eff}^{\ 2}\kappa_{I,cs}=\frac{1}{2}4^2\cdot 5.517\cdot 10^{-4}=0.004414\,\mathrm{m}=4.414\,\mathrm{mm}$$

$$w_{k.II.cs}=\frac{1}{2}L_{eff}^{\ 2}\kappa_{II,cs}=\frac{1}{2}4^2\cdot 1.152\cdot 10^{-3}=0.009216\,\mathrm{m}=9.216\,\mathrm{mm}$$

The total deflection considering cracking and the effect of shrinkage:

$$w_{k.cs}=(1-\zeta)(w_{k.I}+w_{k.I.cs})+\zeta(w_{k.II}+w_{k.II.cs})=$$
$$=(1-0.9474)(17.39+4.414)+0.9474(26.37+9.216)=34.86\,\mathrm{mm}$$

We modelled the beam with beam finite elements. In FEM-Design we increased the division number of the beam finite elements to five to get the more accurate results.

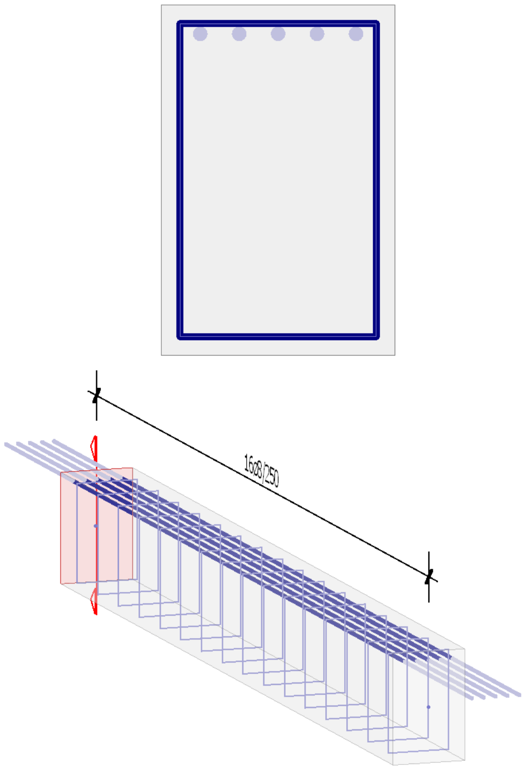


Figure 7.2.3.2 – The cross section of the RC cantilever in FEM-Design

Fig. 7.2.3.2 shows the applied cross section and reinforcement with the defined input parameters.

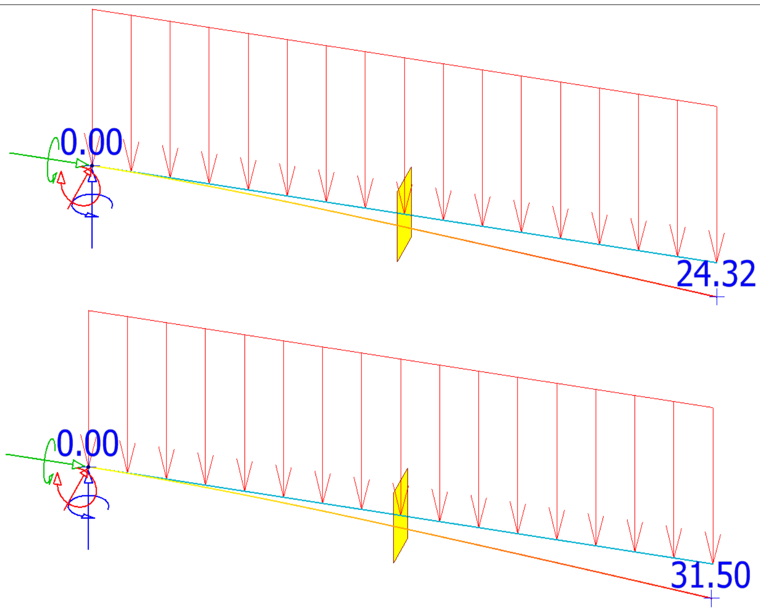


Figure 7.2.2.3 – The deflection [mm] *of the RC bar model in FEM-Design with cracked section analysis (without shrinkage [above], with shrinkage [below])*

Fig. 7.2.2.3 shows the deflection after the cracked section analysis. The deflection of the beam model in FEM-Design:

Cracked section analysis without shrinkage: $w_{kFEM} = 24.32\,\text{mm}$

Cracked section analysis with shrinkage: $w_{k.csFEM} = 31.50\,\text{mm}$

The difference between the hand and FEM-Design calculations is around 9%, but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.3 Cracked deflection of a cantilever beam.str

7.2.4 Cracked deflection of a cantilever beam with compressed reinforcement bars

Inputs:

Span length	$L_{eff} = 3$ m
The cross section	b = 200 mm; h = 400 mm
The elastic modulus of concrete	$E_{cm} = 31.5$ GPa, C25/30
The creep factor	$\varphi_{28} = 2.0$
Effective elastic modulus of concrete	$E_{ceff} = E_{cm}/(1+\varphi_{28}) = 10.5$ GPa
Mean tensile strength	$f_{ctm} = 2.6$ MPa
Elastic modulus of steel bars	$E_s = 200$ GPa
Characteristic value of the point moment at the end	$M_k = 40$ kNm
Diameter of the longitudinal reinforcement	$\phi_l = 12$ mm
Diameter of the stirrup reinforcement	$\phi_s = 10$ mm
Area of longitudinal reinforcement (tension)	$A_l = 4x12^2\pi/4 = 452.4$ m^2
Area of longitudinal reinforcement (compression)	$A_l' = 2x12^2\pi/4 = 226.2$ m^2
Nominal concrete cover	$c_{nom} = 20$ mm
Effective height (tension)	$d = h - c_{nom} - \phi_s - \phi_l/2 = 364$ mm
Effective height (compression)	$d' = c_{nom} + \phi_s + \phi_l/2 = 36$ mm
Shrinkage	$\varepsilon_{cs} = 0.4$ ‰

The cross sectional properties without calculation details:

I. stress stadium second moment of inertia	$I_I = 1.409x10^9$ mm^4
II. stress stadium second moment of inertia	$I_{II} = 6.563x10^8$ mm^4
I. stress stadium position of neutral axis	$x_I = 207.6$ mm
II. stress stadium position of neutral axis	$x_{II} = 128.0$ mm

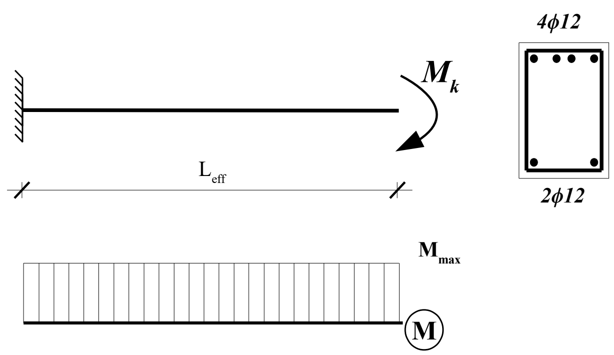


Figure 7.2.4.1 – The cantilever RC beam with compressed RC bars

The calculation of deflection according to EN 1992-1-1:

The load value for the quasi-permanent load combination (see Fig. 7.2.4.1):

$$M_k = 40\,\text{kNm}$$

The maximum deflection with cross sectional properties in Stadium I. (uncracked):

$$w_{k.I} = \frac{1}{2}\frac{M_k L_{eff}^2}{E_{ceff} I_I} = \frac{1}{2}\frac{40\cdot 3^2}{10500000\cdot 0.001409} = 0.01217\,\text{m} = 12.17\,\text{mm}$$

The maximum deflection with cross sectional properties in Stadium II. (cracked):

$$w_{k.II} = \frac{1}{2}\frac{M_k L_{eff}^2}{E_{ceff} I_{II}} = \frac{1}{2}\frac{40\cdot 3^2}{10500000\cdot 0.0006563} = 0.02612\,\text{m} = 26.12\,\text{mm}$$

The maximum bending moment under the quasi-permanent load (see Fig. 7.2.4.1):

$$M_{max} = M_k = 40\,\text{kNm}$$

The cracking moment with the mean tensile strength:

$$M_{cr} = f_{ctm}\frac{I_I}{h - x_I} = 2600\frac{0.001409}{0.40 - 0.2076} = 19.04\,\text{kNm}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta = \max\left[1 - 0.5\left(\frac{M_{cr}}{M_{max}}\right)^2, 0\right] = \max\left[1 - 0.5\left(\frac{19.04}{40}\right)^2, 0\right] = 0.8867$$

The final deflection with the aim of interpolation factor:

$$w_k = (1-\zeta)\,w_{k.I} + \zeta\, w_{k.II} = (1 - 0.8867)\,12.17 + 0.8867\cdot 26.12 = 24.54\,\text{mm}$$

This deflection is even greater if we are considering the effect of shrinkage.

The curvatures in uncracked and cracked states due to shrinkage:

$$\kappa_{I,cs} = \varepsilon_{cs}\frac{E_s}{E_{c,eff}}\frac{S_{s,I}}{I_I} = \frac{0.4}{1000}\frac{200}{10.5}\frac{452.4\cdot(364 - 207.6) - 226.2\,(207.6 - 36)}{1.409\cdot 10^9} = 1.727\cdot 10^{-4}\,\frac{1}{\text{m}}$$

$$\kappa_{II,cs} = \varepsilon_{cs}\frac{E_s}{E_{c,eff}}\frac{S_{s,II}}{I_{II}} = \frac{0.4}{1000}\frac{200}{10.5}\frac{452.4\cdot(364 - 128.0) - 226.2\,(128.0 - 36)}{6.563\cdot 10^8} = 9.979\cdot 10^{-4}\,\frac{1}{\text{m}}$$

The additional deflection due to shrinkage:

$$w_{k.I.cs}=\frac{1}{2}L_{eff}^{2}\kappa_{I,cs}=\frac{1}{2}3^{2}\cdot 1.727\cdot 10^{-4}=0.0007772\,\mathrm{m}=0.7772\,\mathrm{mm}$$

$$w_{k.II.cs}=\frac{1}{2}L_{eff}^{2}\kappa_{II,cs}=\frac{1}{2}3^{2}\cdot 9.979\cdot 10^{-4}=0.004491\,\mathrm{m}=4.491\,\mathrm{mm}$$

The total deflection considering cracking and the effect of shrinkage:

$$w_{k.cs}=(1-\zeta)(w_{k.I}+w_{k.I.cs})+\zeta(w_{k.II}+w_{k.II.cs})=$$
$$=(1-0.8867)(12.17+0.7772)+0.8867(26.12+4.491)=28.61\,\mathrm{mm}$$

We modelled the beam with beam finite elements. In FEM-Design we increased the division number of the beam finite elements to five to get the more accurate results.

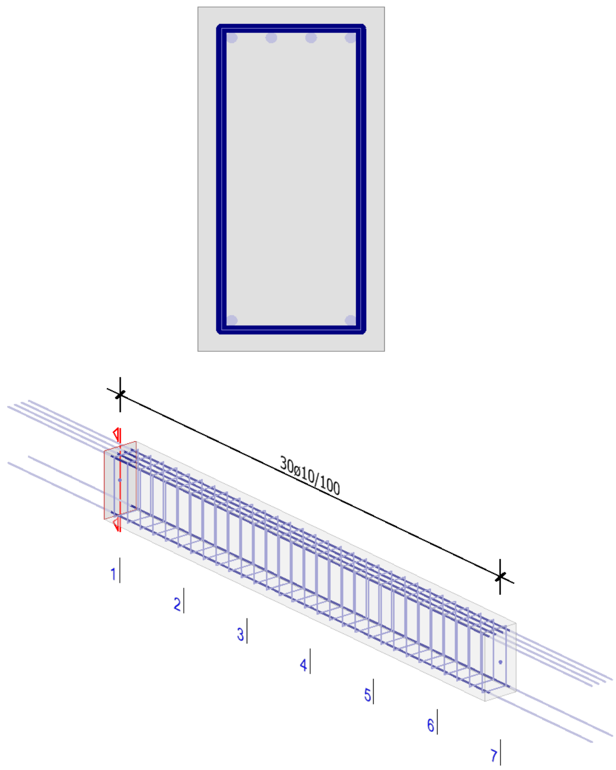


Figure 7.2.4.2 – The cross section of the RC cantilever in FEM-Design

Fig. 7.2.4.2 shows the applied cross section and reinforcement with the defined input parameters.

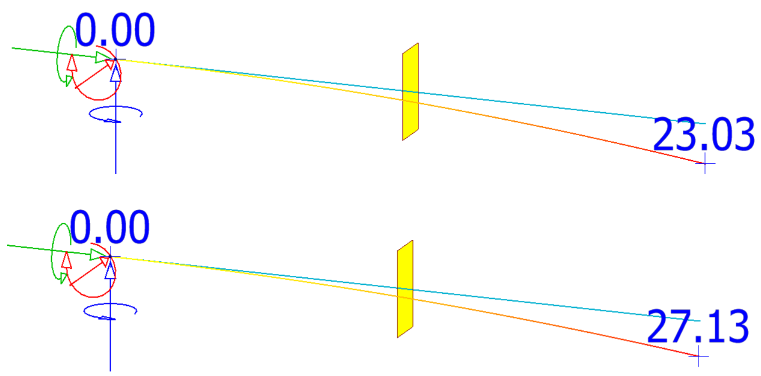


Figure 7.2.4.3 – The deflection [mm] *of the RC bar model in FEM-Design with cracked section analysis (without shrinkage [above], with shrinkage [below])*

Fig. 7.2.4.3 shows the deflection after the cracked section analysis. The deflection of the beam model in FEM-Design:

Cracked section analysis without shrinkage: $w_{kFEM} = 23.03\,\text{mm}$

Cracked section analysis with shrinkage: $w_{k.csFEM} = 27.13\,\text{mm}$

The difference between the hand and FEM-Design calculations is around 7%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.4 Cracked deflection of a cantilever beam with compressed reinforcement bars

7.2.5 Cracked deflection of a cantilever with bending moment and normal forces

Inputs:

Span length	$L_{eff} = 2$ m
The cross section	Rectangle: b = 200 mm; h = 400 mm
The elastic modulus of concrete	$E_{cm} = 31.5$ GPa
The creep factor	$\varphi_{28} = 2.0$
Effective elastic modulus of concrete	$E_{ceff} = E_{cm}/(1+\varphi_{28}) = 10.5$ GPa
Ratio between the moduli	$\alpha = E_s/E_{ceff} = 19.05$
Mean tensile strength	$f_{ctm} = 2.2$ MPa
Elastic modulus of steel bars	$E_s = 200$ GPa
Diameter of the longitudinal reinforcement	$\phi_l = 12$ mm
Diameter of the stirrup reinforcement	$\phi_s = 10$ mm
Area of longitudinal reinforcement (top)	$A_l = 4x12^2\pi/4 = 452.4$ m^2
Area of longitudinal reinforcement (bottom)	$A_l' = 4x12^2\pi/4 = 452.4$ m^2
Nominal concrete cover	$c_{nom} = 20$ mm
Effective height (bottom)	$d = h - c_{nom} - \phi_s - \phi_l/2 = 364$ mm
Effective height (top)	$d' = c_{nom} + \phi_s + \phi_l/2 = 36$ mm

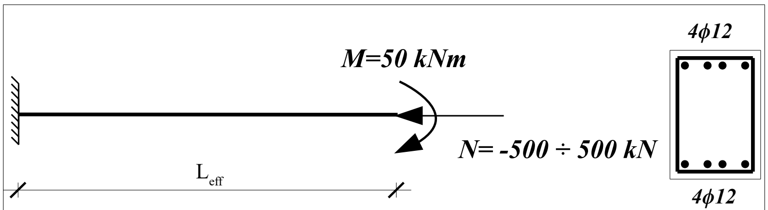


Figure 7.2.5.1 – A RC cantilever with the applied loads

In this chapter we will calculate the deflection (vertical translation) of the end of a cantilever (see Fig. 7.2.5.1). According to the behaviour of the reinforced concrete material this deflections will depend on the amount of the applied normal force. At the free end of the cantilever we applied a constant concentrated bending moment (M=50kNm) and we changed the intensity of the applied concentrated normal force at the end from -500 kN compression to +500 kN tension.

The force is acting on the centroid of the uncracked RC section. Now it is in the middle.

During the hand calculation (and by the FEM-Design calculation as well) we considered eccentricity caused by cracking in cracked section analysis.

We are going to calculate the compressed concrete zone by hand at five notable compressed zone condition and based on these values the inhomogeneous inertias and areas are also can be calculated. The deflections depend on these inertias. After these we calculate the interpolation factors to get the accurate deflection considering the tension stiffening effect. At the end of this chapter we compare the results with FEM-Design solutions.

Case a)

M = 50 kNm; N= - 500 kN (compression):

In this case the complete section is uncracked, the total concrete zone is active.

Therefore we only need to calculate an inhomogeneous cross-section.

$$A_a = b \cdot h + \alpha (A_l + A_l') = 200 \cdot 400 + 19.05 \cdot (452.4 + 452.4) = 0.09724\,\mathrm{m}^2$$

$$I_a = \frac{b \cdot h^3}{12} + \alpha\, A_l \left(d - \frac{h}{2} \right)^2 + \alpha\, A_l' \left(\frac{h}{2} - d' \right)^2 =$$

$$= \frac{200 \cdot 400^3}{12} + 19.05 \cdot 452.4 \left(364 - \frac{400}{2} \right)^2 + 19.05 \cdot 452.4 \left(\frac{400}{2} - 36 \right)^2 = 0.0015303\,\mathrm{m}^4$$

The stresses at the extreme fibres are as follows:

$$\sigma_{top} = \frac{N}{A_a} + \frac{M}{I_a}\frac{h}{2} = \frac{-500}{0.09724} + \frac{50}{0.0015303}\frac{0.4}{2} = 1.392\,\mathrm{MPa} < f_{ctm} = 2.2\,\mathrm{MPa} \quad \text{(tension)}$$

$$\sigma_{bottom} = \frac{N}{A_a} - \frac{M}{I_a}\frac{h}{2} = \frac{-500}{0.09724} - \frac{50}{0.0015303}\frac{0.4}{2} = -7.049\,\mathrm{MPa} \quad \text{(compression)}$$

Based on these equation it is trivial that the first crack occur at the following normal force:

$$N_{crack} = \left(f_{ctm} - \frac{M}{I_a}\frac{h}{2} \right) A_a = \left(2200 - \frac{50}{0.0015303}\frac{0.4}{2} \right) 0.09724 = -421.5\,\mathrm{kN} \quad \text{(compression)}$$

The deflection (vertical translation) of the free end of the cantilever:

$$w_a = \frac{M\, L_{eff}^2}{2\, E_{ceff}\, I_a} = \frac{50 \cdot 2^2}{2 \cdot 10500000 \cdot 0.0015303} = 6.223\,\mathrm{mm}$$

Case b)

M = 50 kNm; N= - 200 kN (compression):

The compressed zone (measured from the compressed side) in this case comes from the solution of a third order polynomial:

$$\frac{M}{N} - \frac{\alpha\, A_l (d - x_b)(d - h/2) + \alpha\, A_l' (x_b - d')(h/2 - d') + \frac{b\, x_b^2}{2}(h/2 - \frac{1}{3}x_b)}{\alpha\, A_l (d - x_b) - \alpha\, A_l' (x_b - d') - \frac{b\, x_b^2}{2}} = 0$$

The relevant solution of this equation:

$$x_b = 199.7\,\mathrm{mm}$$

Based on this, the position of the centroid measured from the compressed side:

$$x_{Sb}=\frac{b\frac{x_b^2}{2}+\alpha A_l d+\alpha A_l' d'}{b x_b+\alpha(A_l+A_l')}=130.0\,\text{mm}$$

According to this the cracked section area and inertia:

$$A_b=b x_b+\alpha(A_l+A_l')=200\cdot 199.7+19.05\cdot(452.4+452.4)=0.05718\,\text{m}^2$$

$$I_b=\frac{b\cdot x_b^3}{12}+b x_b(x_{Sb}-x_b/2)^2+\alpha A_l\left(d-x_{Sb}\right)^2+\alpha A_l'\left(x_{Sb}-d'\right)^2=0.0007171\,\text{m}^4$$

The deflection (vertical translation) of the free end of the cantilever with this cracked condition <u>considering the decreased moment at the cracked centroid</u> from the additional eccentricities (because the original loads are acting on the centroid of the uncracked RC section):

$$w_{bII}=\frac{(M+N(h/2-x_{Sb}))L_{eff}^2}{2E_{ceff}I_b}=\frac{(50-200(0.2-0.130))\cdot 2^2}{2\cdot 10500000\cdot 0.0007171}=9.562\,\text{mm}$$

But this is not the final deflection, because we need to consider the tension stiffening to get a more comparable result.

The stress in the tension reinforcement calculated on the basis of a cracked section and the additional moment due to cracked eccentricity:

$$\sigma_{sb}=\alpha\left[\frac{N}{A_b}+\frac{M+N(h/2-x_{Sb})}{I_b}(d-x_{Sb})\right]=$$

$$=19.05\left[\frac{-200}{0.05718}+\frac{50-200(0.2-0.130)}{0.0007171}(0.364-0.130)\right]=157.2\,\text{MPa}$$

The load conditions causing first crack:

$$\eta_b\left[\frac{N}{A_a}+\frac{M}{I_a}(h/2)\right]=\eta_b\left[\frac{-200}{0.09724}+\frac{50}{0.0015303}(0.4/2)\right]=f_{ctm}=2.2\,\text{MPa}\quad;\text{thus}\quad \eta_b=0.4913$$

Thus the normal force:

$$\eta_b N=0.4913\cdot(-200)=-98.26\,\text{kN}\quad(\text{compression})$$

And the bending moment:

$$\eta_b M=0.4913\cdot 50=24.57\,\text{kNm}$$

The stress in the tension reinforcement calculated on the basis of a cracked section under the loading conditions causing first cracking considering the additional moment due to cracked eccentricity:

$$\sigma_{srb}=\alpha\left[\frac{\eta_b N}{A_b}+\frac{\eta_b M+\eta_b N(h/2-x_{Sb})}{I_b}(d-x_{Sb})\right]=$$

$$=19.05\left[\frac{-98.26}{0.05718}+\frac{24.57-98.26(0.2-0.130)}{0.0007171}(0.364-0.130)\right]=77.24\,\text{MPa}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta_b=\max\left[1-0.5\left(\frac{\sigma_{srb}}{\sigma_{sb}}\right)^2,0\right]=\max\left[1-0.5\left(\frac{77.24}{157.2}\right)^2,0\right]=0.8792$$

The final deflection with the aim of interpolation factor:

$$w_b=(1-\zeta_b)w_a+\zeta_b w_{bII}=(1-0.8792)6.223+0.8792\cdot 9.562=9.159\,\text{mm}$$

Case c)

M = 50 kNm; N= 0 kN (pure bending):

In this case the cross section is under pure bending. In this situation the following equation gives the position of the compressed zone (cracking occur on the tension side).

$$\frac{1}{2}x_c^2 b+\alpha(x_c-d')A_l'+\alpha(x_c-d)A_l=0$$

The relevant solution of this equation:

$$x_c=118.5\,\text{mm}$$

Based on this value the cracked cross-sectional area and inertia:

$$A_c=b\cdot x_c+\alpha(A_l+A_l')=200\cdot 118.51+19.05\cdot(452.4+452.4)=0.04094\,\text{m}^2$$

$$I_c=\frac{b\cdot x_c^3}{3}+\alpha A_l(d-x_c)^2+\alpha A_l'(x_c-d')^2=$$

$$=\frac{200\cdot 118.5^3}{3}+19.05\cdot 452.4(364-118.5)^2+19.05\cdot 452.4(118.5-36)^2=0.000689\,\text{m}^4$$

The deflection (vertical translation) of the free end of the cantilever with this cracked condition:

$$w_{cII}=\frac{M\,L_{eff}^2}{2\,E_{ceff}\,I_c}=\frac{50\cdot 2^2}{2\cdot 10500000\cdot 0.000689}=13.82\,\text{mm}$$

But this is not the final deflection, we need to consider the tension stiffening to get a more comparable solutions. The interpolation factor based on the pure bending condition:

The cracking moment with the mean tensile strength:

$$M_{cr}=f_{ctm}\frac{I_a}{h/2}=2200\frac{0.0015303}{0.4/2}=16.83\,\mathrm{kNm}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta_c=\max\left[1-0.5\left(\frac{M_{cr}}{M}\right)^2,0\right]=\max\left[1-0.5\left(\frac{16.83}{50}\right)^2,0\right]=0.9434$$

The final deflection with the aim of interpolation factor:

$$w_c=(1-\zeta_c)w_a+\zeta_c w_{cII}=(1-0.9434)6.223+0.9434\cdot 13.82=13.39\,\mathrm{mm}$$

Case d)

M = 50 kNm; N= 200 kN (tension):

The compressed zone (measured from the compressed side) in this case comes from the solution of a third order polynomial:

$$\frac{M}{N}-\frac{\alpha A_l(d-x_d)(d-h/2)+\alpha A_l'(x_d-d')(h/2-d')+\frac{b x_d^2}{2}(h/2-\frac{1}{3}x_d)}{\alpha A_l(d-x_d)-\alpha A_l'(x_d-d')-\frac{b x_d^2}{2}}=0$$

The relevant solution of this equation:

$$x_d=58.38\,\mathrm{mm}$$

Based on this the position of the centroid measured from the compressed side:

$$x_{Sd}=\frac{b\frac{x_d^2}{2}+\alpha A_l d+\alpha A_l' d'}{b x_d+\alpha(A_l+A_l')}=131.0\,\mathrm{mm}$$

According to this the cracked section area and inertia:

$$A_d=b x_d+\alpha(A_l+A_l')=200\cdot 58.38+19.05\cdot(452.4+452.4)=0.02891\,\mathrm{m}^2$$

$$I_d=\frac{b\cdot x_d^3}{12}+b x_d(x_{Sd}-x_d/2)^2+\alpha A_l\left(d-x_{Sd}\right)^2+\alpha A_l'\left(x_{Sd}-d'\right)^2=0.0006700\,\mathrm{m}^4$$

The deflection (vertical translation) of the free end of the cantilever with this cracked condition considering the increased moment at the cracked centroid from the additional eccentricities (because the original loads are acting on the centroid of the uncracked RC section):

$$w_{dII}=\frac{(M+N(h/2-x_{Sd}))L_{eff}^2}{2E_{ceff}I_d}=\frac{(50+200(0.2-0.131))\cdot 2^2}{2\cdot 10500000\cdot 0.00067}=18.14\,\text{mm}$$

But this is not the final deflection, we need to consider the tension stiffening to get a more comparable solution.

The stress in the tension reinforcement calculated on the basis of a cracked section and the additional moment due to cracked eccentricity:

$$\sigma_{sd}=\alpha\left[\frac{N}{A_d}+\frac{M+N(h/2-x_{Sd})}{I_d}(d-x_{Sd})\right]=$$

$$=19.05\left[\frac{200}{0.02891}+\frac{50+200(0.2-0.131)}{0.00067}(0.364-0.131)\right]=554.5\,\text{MPa}$$

The load conditions causing first crack:

$$\eta_d\left[\frac{N}{A_a}+\frac{M}{I_a}(h/2)\right]=\eta_d\left[\frac{200}{0.09724}+\frac{50}{0.0015303}(0.4/2)\right]=f_{ctm}=2.2\,\text{MPa}\quad;\text{ thus}\quad \eta_d=0.2560$$

Thus the normal force:

$$\eta_d N=0.2560\cdot 200=51.2\,\text{kN kNm}$$

And the bending moment:

$$\eta_d M=0.2560\cdot 50=12.80\,\text{kNm}$$

The stress in the tension reinforcement calculated on the basis of a cracked section under the loading conditions causing first cracking considering the additional moment due to cracked eccentricity:

$$\sigma_{srd}=\alpha\left[\frac{\eta_d N}{A_d}+\frac{\eta_d M+\eta_d N(h/2-x_{Sd})}{I_d}(d-x_{Sd})\right]=$$

$$=19.05\left[\frac{51.2}{0.02891}+\frac{12.8+51.2(0.2-0.131)}{0.00067}(0.364-0.131)\right]=141.9\,\text{MPa}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta_d=\max\left[1-0.5\left(\frac{\sigma_{srd}}{\sigma_{sd}}\right)^2,0\right]=\max\left[1-0.5\left(\frac{141.9}{554.5}\right)^2,0\right]=0.9673$$

The final deflection with the aim of interpolation factor:

$$w_d=(1-\zeta_d)w_a+\zeta_d w_{dII}=(1-0.9673)6.223+0.9673\cdot 18.14=17.75\,\mathrm{mm}$$

Case e)

M = 50 kNm; N= 500 kN (tension):

In this case the cross section is fully cracked (therefore this is only a theoretical solution). The whole concrete zone is cracked. Practically in means that only the reinforcement bars are active in the section, thus the cross-sectional properties:

$$A_e=\alpha(A_l+A_l')=19.05\cdot(452.4+452.4)=0.01724\,\mathrm{m}^2$$

$$I_e=\alpha A_l(d-h/2)^2+\alpha A_l'(h/2-d')^2=$$

$$=19.05\cdot 452.4(364-200)^2+19.05\cdot 452.4(200-36)^2=0.0004636\,\mathrm{m}^4$$

The deflection (vertical translation) of the free end of the cantilever with this fully cracked condition (only the RC bars are working):

$$w_{eII}=\frac{M L_{eff}^2}{2 E_{ceff} I_e}=\frac{50\cdot 2^2}{2\cdot 10500000\cdot 0.0004636}=20.54\,\mathrm{mm}$$

But this is not the final deflection, we need to consider the tension stiffening to get a more comparable solution.

The stress in the tension reinforcement calculated on the basis of a cracked section:

$$\sigma_{se}=\alpha\left[\frac{N}{A_e}+\frac{M}{I_e}(d-h/2)\right]=19.05\left[\frac{500}{0.01724}+\frac{50}{0.0004636}(0.364-0.2)\right]=889.4\,\mathrm{MPa}$$

The load conditions causing first crack:

$$\eta_e\left[\frac{N}{A_a}+\frac{M}{I_a}(h/2)\right]=\eta_e\left[\frac{500}{0.09724}+\frac{50}{0.0015303}(0.4/2)\right]=f_{ctm}=2.2\,\mathrm{MPa}\quad ;\text{thus}\quad \eta_e=0.4279$$

Thus the normal force:

$$\eta_e N=0.4279\cdot 500=214.0\,\mathrm{kN}$$

And the bending moment:

$$\eta_e M=0.4279\cdot 50=21.40\,\mathrm{kNm}$$

The stress in the tension reinforcement calculated on the basis of a cracked section under the loading conditions causing first cracking:

$$\sigma_{sre}=\alpha\left[\frac{\eta N}{A_e}+\frac{\eta M}{I_e}(d-h/2)\right]=19.05\left[\frac{214.0}{0.01724}+\frac{21.40}{0.0004636}(0.364-0.2)\right]=380.7\,\text{MPa}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta_e=\max\left[1-0.5\left(\frac{\sigma_{sre}}{\sigma_{se}}\right)^2,0\right]=\max\left[1-0.5\left(\frac{380.7}{889.4}\right)^2,0\right]=0.9084$$

The final deflection with the aim of interpolation factor:

$$w_e=(1-\zeta_e)w_a+\zeta_e w_{eII}=(1-0.9084)6.223+0.9084\cdot 20.54=19.23\,\text{mm}$$

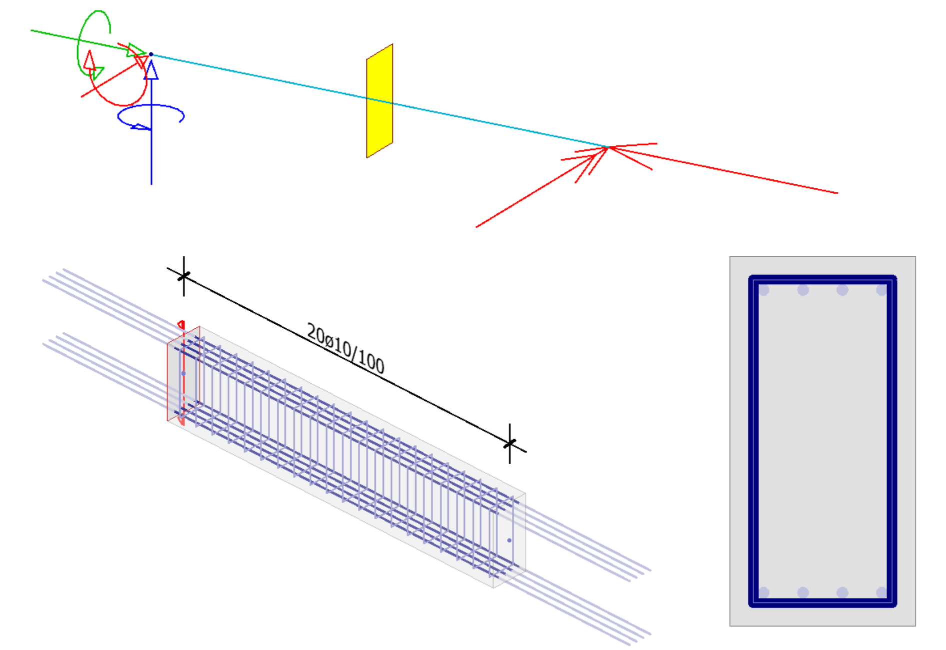


Figure 7.2.5.2 – A RC cantilever with the applied loads in FEM-Design

Fig. 7.2.5.2 shows the FEM-Design model with bars. Fig. 7.2.5.3 shows the deflections (vertical translations) of the free end of cantilever under the constant bending moment regarding different normal forces (compression and tension as well). In Fig. 7.2.5.3 in addition to the hand calculation we indicated the FEM-Design results with beam model and with shell model also.

The differences between the hand and FEM-Design calculations are less than 5%, thus we can say that the results are identical to each other.

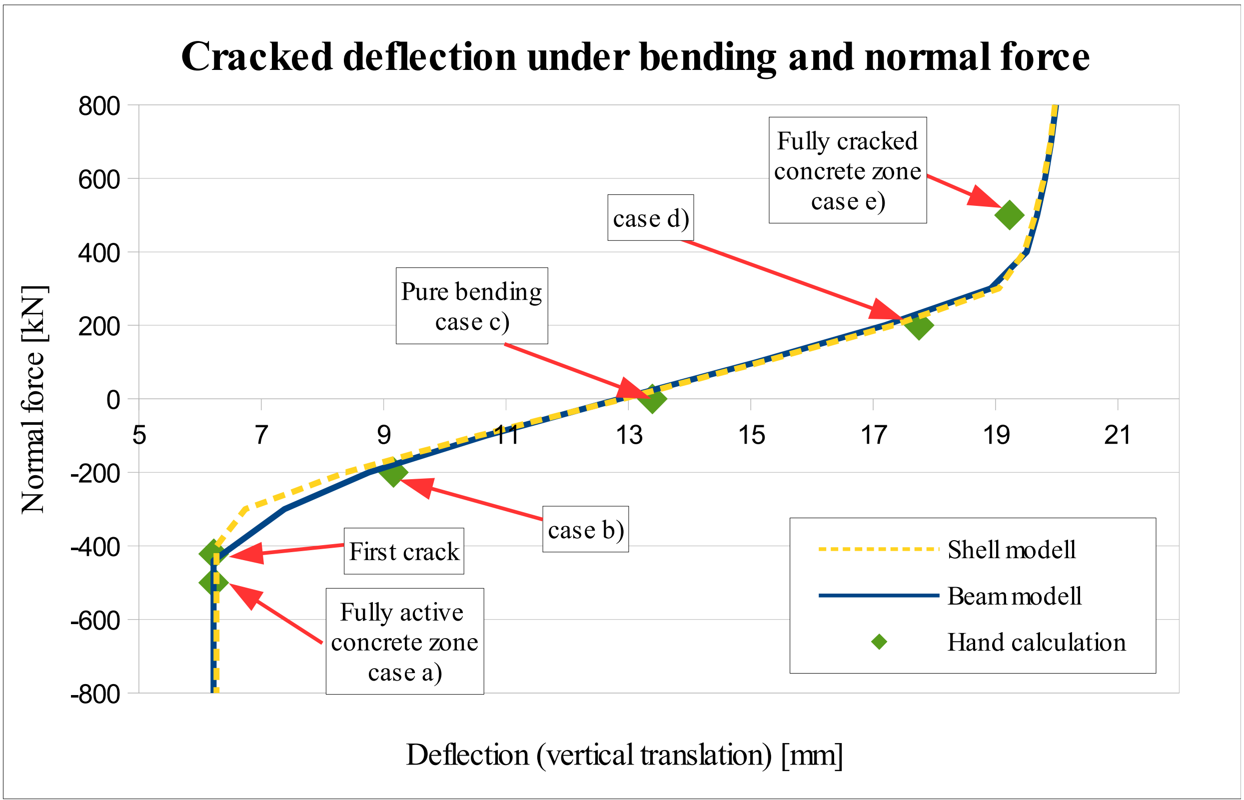


Figure 7.2.5.3 – A results with FEM-Design (beam and shell also) compared to the hand calculation

NOTE: The cracked section analysis in FEM-Design is based on a non-linear calculation but only accurate by SLS combinations because the used material model for the reinforcement is linear (both compression and tension) and for the concrete is a non-tension material (only compression, assumed to be linear) if the extreme fibres reached the mean tensile strength in the cross-section.

Download link to the example files:

Beam model:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.5 Cracked deflection of a cantilever with bending moment and different normal forces.beam.str

Shell model:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.5 Cracked deflection of a cantilever with bending moment and different normal forces.shell.str

7.2.6 Cracked deflection of a simply supported square slab

Inputs:

Span length	L_{eff} = 6 m (Fig. 6.2.3.1)
The thickness	h = 200 mm
The elastic modulus of concrete	E_{cm} = 30 GPa, C20/25
The Poisson's ratio of concrete	$\nu = 0.2$
The creep factor	$\varphi_{28} = 2.35$
Effective elastic modulus of concrete	$E_{ceff} = E_{cm}/(1+\varphi_{28}) = 8.96$ GPa
Mean tensile strength	f_{ctm} = 2.2 MPa
Elastic modulus of steel bars	E_s = 200 GPa
Characteristic value of dead load	g_k = 6 kN/m
Characteristic value of live load	q_k = 10 kN/m
Live load combination factor	$\psi_2 = 0.6$
Diameter of the longitudinal reinforcement	ϕ_l = 10 mm/200 mm
The specific reinforcement	A_s = 0.393 mm^2/mm
Nominal concrete cover	c_x = 30 mm; c_y = 40 mm
Average effective height	d = 160 mm
The ratio of the elastic modulus	$\alpha_s = E_s/E_{ceff} = 22.32$

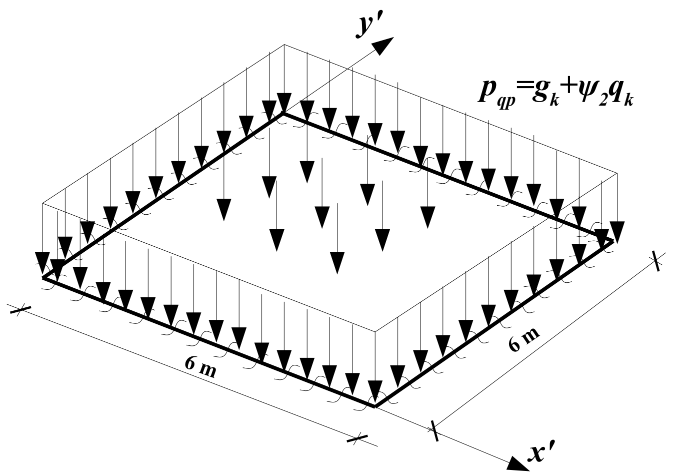


Figure 7.2.6.1 – Simply supported RC slab with constant total distributed load

The load value for the quasi-permanent load combination:

$$p_{qp} = g_k + \psi_2 q_k = 6 + 0.6 \cdot 10 = 12 \frac{\text{kN}}{\text{m}^2}$$

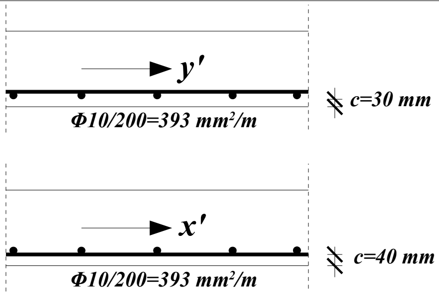


Figure 7.2.6.2 – The reinforcement in the slab

The deflection of an isotropic simly supported square slab under uniform load (see Chapter 1.3):

$$w_{max}=0.00416\frac{p\,a^4}{\left(\frac{E\,h^3}{12(1-\nu^2)}\right)}$$

The uncracked unreinforced specific inertia (only with the concrete):

$$I_c=\frac{h^3}{12}=\frac{200^3}{12}=6.667\cdot10^5\frac{\mathrm{mm}^4}{\mathrm{mm}}=0.0006667\frac{\mathrm{m}^4}{\mathrm{m}}$$

The deflection based on this inertia:

$$w_{max,c}=0.00416\frac{12\cdot6^4}{\left(\frac{8960000\cdot0.0006667}{1-0.2^2}\right)}=0.01040\,\mathrm{m}=10.4\,\mathrm{mm}$$

The uncracked reinforced specific inertia (Stadium I.):

$$x_I=\frac{\frac{h^2}{2}+\alpha_s a_s d}{h+\alpha_s a_s}=\frac{\frac{200^2}{2}+22.32\cdot0.393\cdot160}{200+22.32\cdot0.393}=102.52\,\mathrm{mm}$$

$$I_I=\frac{x_I^{\,3}}{3}+\frac{(h-x_I)^3}{3}+\alpha_s a_s(d-x_I)^2=\frac{102.52^3}{3}+\frac{(200-102.52)^3}{3}+22.32\cdot0.393(160-102.52)^2$$

$$I_I=6.969\cdot10^5\frac{\mathrm{mm}^4}{\mathrm{mm}}=0.0006969\frac{\mathrm{m}^4}{\mathrm{m}}$$

The deflection based on this inertia:

$$w_{max,I}=0.00416\frac{12\cdot6^4}{\left(\frac{8960000\cdot0.0006969}{1-0.2^2}\right)}=0.009947\,\mathrm{m}=9.947\,\mathrm{mm}$$

The cracked reinforced specific inertia (Stadium II.):

$$x_{II} = \frac{\frac{x_{II}^2}{2} + \alpha_s a_s d}{x_{II} + \alpha_s a_s} = \frac{\frac{x_{II}^2}{2} + 22.32 \cdot 0.393 \cdot 160}{x_{II} + 22.32 \cdot 0.393} \qquad x_{II} = 44.93\,\mathrm{mm}$$

$$I_{II} = \frac{x_{II}^3}{3} + \alpha_s a_s (d - x_{II})^2 = \frac{44.93^3}{3} + 22.32 \cdot 0.393 (160 - 44.93)^2 = 1.4638 \cdot 10^5 \,\frac{\mathrm{mm}^4}{\mathrm{mm}}$$

The deflection based on this inertia:

$$w_{max,II} = 0.00416 \frac{12 \cdot 6^4}{\left(\frac{8960000 \cdot 0.00014638}{1 - 0.2^2}\right)} = 0.04735\,\mathrm{m} = 47.35\,\mathrm{mm}$$

The cracking moment:

$$m_{cr} = f_{ctm} \frac{I_I}{h - x_I} = 2.2 \frac{6.969 \cdot 10^5}{200 - 102.52} = 15730 \frac{\mathrm{Nmm}}{\mathrm{mm}} = 15.73 \frac{\mathrm{kNm}}{\mathrm{m}}$$

$$m_{max} = 0.0469\, p a^2 = 0.0469 \cdot 12 \cdot 6^2 = 20.26 \frac{\mathrm{kNm}}{\mathrm{m}}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta = \max\left[1 - 0.5\left(\frac{m_{cr}}{m_{max}}\right)^2, 0\right] = \max\left[1 - 0.5\left(\frac{15.73}{20.26}\right)^2, 0\right] = 0.6986$$

The deflection based on this interpolation factor:

$$w_{max,cr} = (1 - \zeta) w_{max,I} + \zeta\, w_{max,II} = (1 - 0.6986) 9.947 + 0.6986 \cdot 47.35 = 36.08\,\mathrm{mm}$$

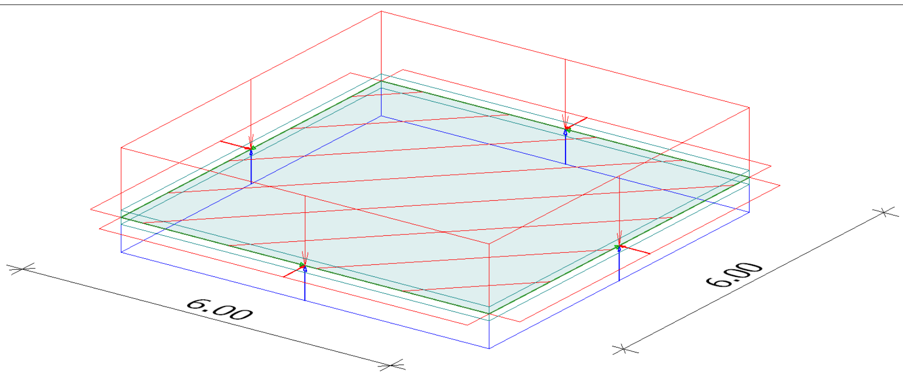


Figure 7.2.6.3 – The FE model

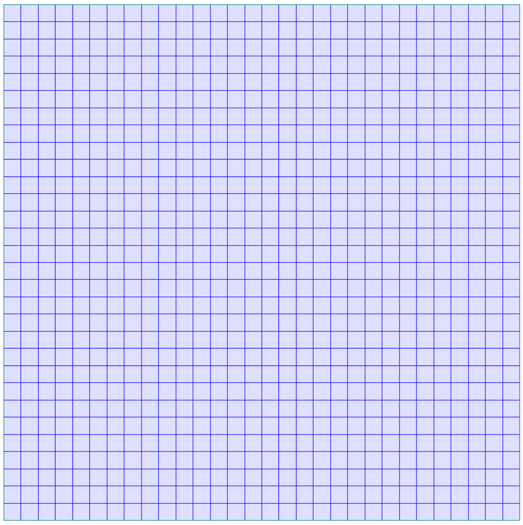


Figure 7.2.6.4 – The finite element mesh [average element size: 0.2 m]

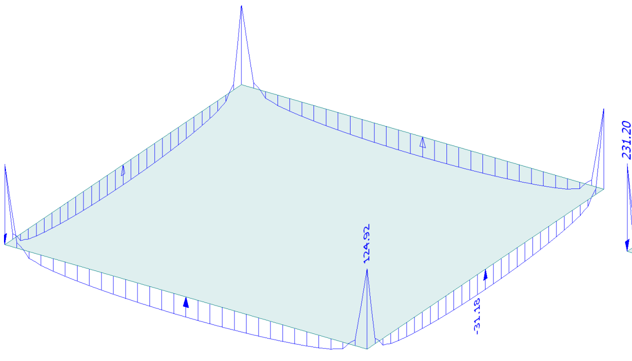


Figure 7.2.6.5 – The reactions [kN/m] *without reinforcement and with reinforced cracked section analysis*

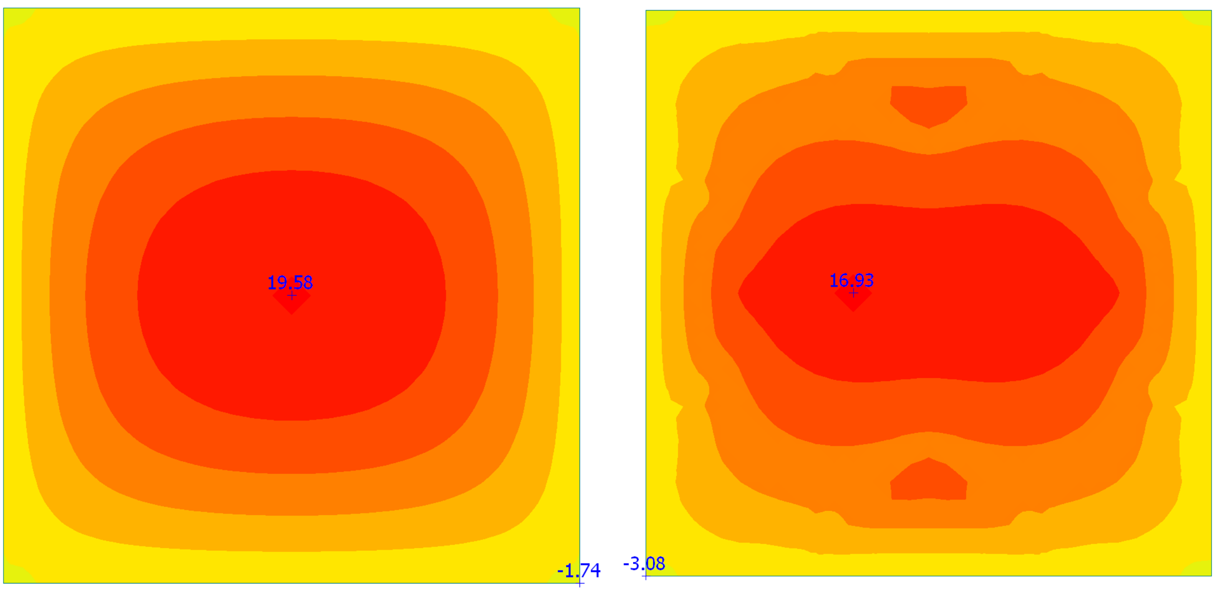


Figure 7.2.6.6 – The m_x [kNm/m] *without reinforcement and with reinforced cracked section analysis*

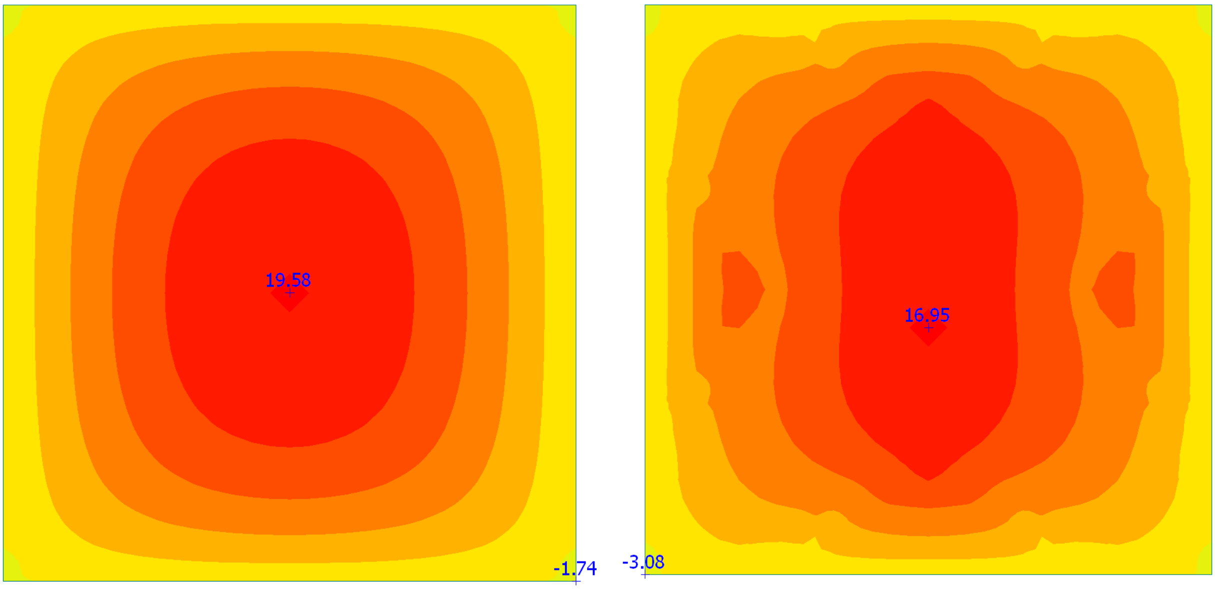


Figure 7.2.6.7 – The m_y [kNm/m] without reinforcement and with reinforced cracked section analysis

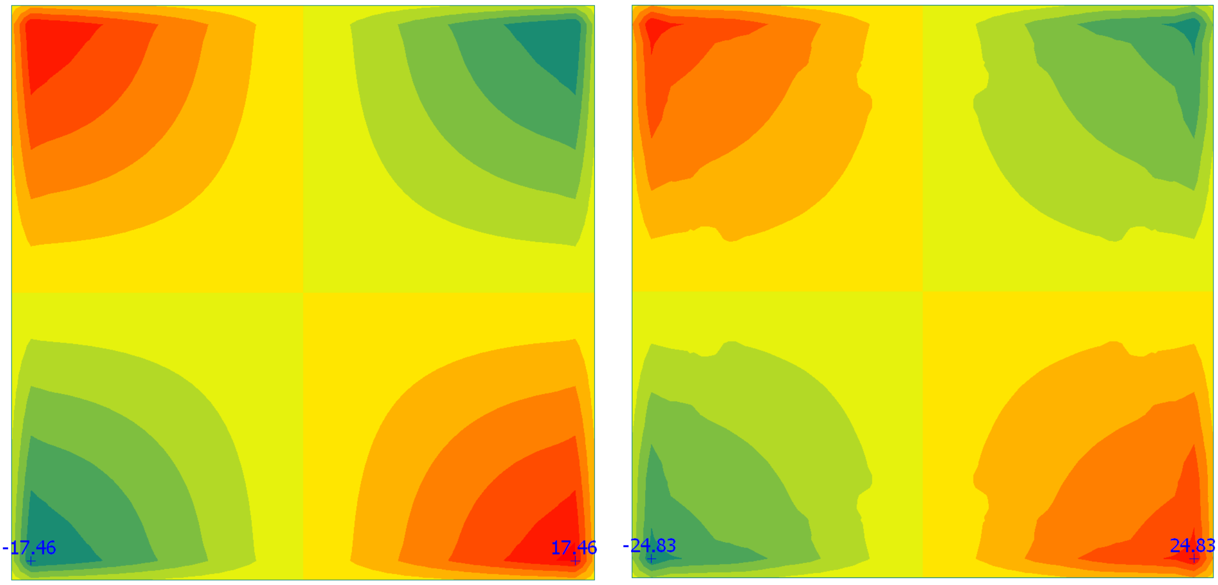


Figure 7.2.6.8 – The m_{xy} [kNm/m] without reinforcement and with reinforced cracked section analysis

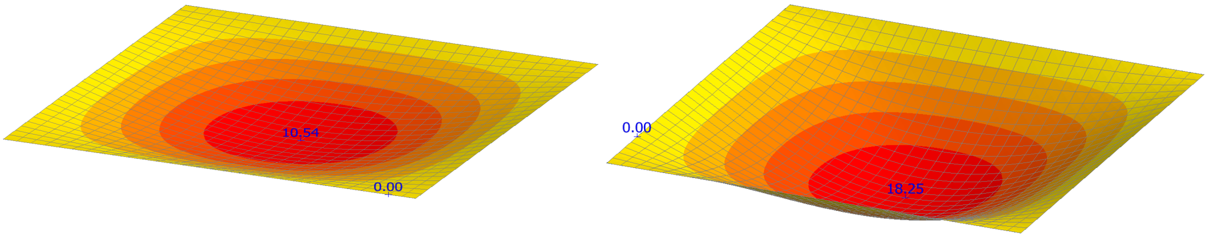


Figure 7.2.6.9 – The deflection [mm] *without reinforcement and with reinforced cracked section analysis*

This hand calculation method is very **very conservative**, thus **do not considering the realistic crack pattern** and the **torsional effects**. In addition the shear deformations in the slab is also neglected.

Fig. 7.2.6.9 shows the deflection based on FEM-Design. Here the difference between the deflections is quite large but this comes from the mentioned very very conservative hand calculation method.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.3 Cracked deflection of a simply supported square slab.str

7.3 Nonlinear soil calculation

This chapter goes beyond the scope of this document, therefore additional informations are located in:

FEM-Design – Geotechnical modul in 3D, Theoretical background and verification and validation handbook

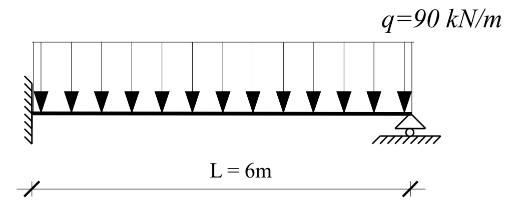
http://download.strusoft.com/FEM-Design/inst170x/documents//3dsoilmanual.pdf

7.4 Elasto-plastic calculations

7.4.1 Elasto-plastic point support in a beam

Inputs:

Span length	$L = 6$ m
Distributed load	$q = 90$ kN/m
Structural steel	S235, $f_y = 235$ MPa
Young's modulus of steel	$E = 210$ GPa
Shear modulus	$G = 80.77$ GPa
Cross section	IPE 400
Cross-sectional area	$A = 8446$ mm^2
The first principal inertia	$I_1 = 231283781$ mm^4
The shear correction factor in the relevant direction	$\rho_2 = 0.4$
The elastic cross sectional modulus	$W_{el} = 1156419$ mm^3
The first moment of the half of the cross sectional area about the centroid	$S_0 = 653575$ mm^3
Plastic load-bearing moment capacity of the section	$M_{pl} = 2S_0f_y = 307.2$ kNm



7.4.1.1 – The beam with a fixed support and a roller

The problem is a beam with fixed end on the left side and a roller on the right side (see Fig. 7.4.1.1). The input parameters are in the table above. First of all we calculate the deflection and the internal forces according to the external total distributed load based on linear elastic theory (Case 1). For the second calculation we assume that the fixed support can only bear maximum M_{pl} bending moment (Case 2). Thus we assume a plastic hinge after a certain amount of load level when the beam reaches this plastic limit moment in the fixed support.

Based on the plastic hinge the distribution of the internal forces and the deflection will be different from the linear elastic calculation.

At the hand calculation we neglect the shear deformation (Euler-Bernoulli beam theory) to get a simple solution for this problem. In FEM-Design the applied beam theory considers the shear deformation (Timoshenko beam theory). It means that when we compare the solution of the hand calculation to the finite element solution the deflection will be a bit larger in case of the

FEM calculation. To avoid this difference in the FEM analysis of the beam then the cross-sectional area must be rewritten with a large number (e.g.: A = 10^6 m^2). Practically it means that the shear stiffness is almost infinite. In this case the solutions will be the same than the results of the hand calculations. At the end of this example we will show two finite element results. One of them will neglect and the other will consider the shear deformation.

Case 1:

The characteristic shear forces, bending moments, rotations and translations are indicated in Fig. 7.4.1.3 left side according to linear elastic calculation. Here are some hand calculation results without further details based on the classical theory of elasticity:

$$V_{max}=\frac{5}{8}q\,L=\frac{5}{8}90\cdot 6=337.5\,\text{kN} \quad ; \quad M^-_{max}=\frac{q\,L^2}{8}=\frac{90\cdot 6^2}{8}=405\,\text{kNm}$$

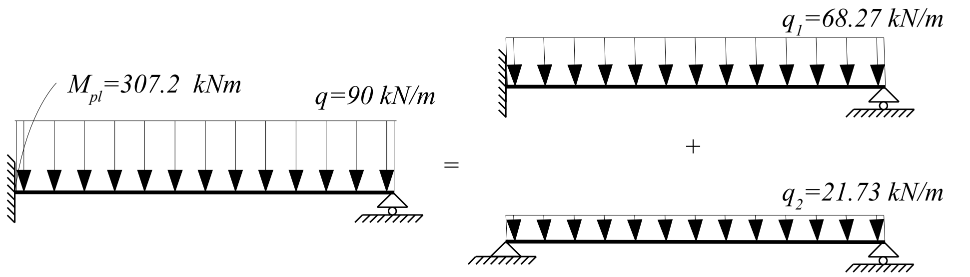
$$\phi_{roller}=\frac{q\,L^3}{48\,E\,I_1}=\frac{90\cdot 6^3}{48\cdot 210\cdot 10^6\cdot 0.000231283781}=0.008339\,\text{rad}$$

$$e_{midspan}=\frac{2}{384}\frac{q\,L^4}{E\,I_1}=\frac{2}{384}\frac{90\cdot 6^4}{210\cdot 10^6\cdot 0.000231283781}=12.51\,\text{mm}$$

Case 2:

To easily consider the bending moment limit (plastic analysis) at the fixed support the following simplification need to be declared at the hand calculation. Until the certain amount of load level (q_1) which causes M_{pl} bending moment at the fixed support the behaviour is linear elastic:

$$M^-_1=M_{pl}=\frac{q_1\,L^2}{8}=307.2\,\text{kNm} \quad ==> \quad q_1=\frac{8\,M_{pl}}{L^2}=68.27\,\text{kN/m}$$



7.4.1.2 – The beam behaviour with the plastic bending moment limit at the fixed end

After this load level the fixed end will be a plastic hinge therefore the statical system differs from the original one (see Fig. 7.4.1.2). The rest of the load level which will act on a simply supported beam (according to the plastic hinge):

$$q_2=q-q_1=90-68.27=21.73\,\text{kN/m}$$

With this assumption we can superpose the two different results based on the two different statical systems. The specific shear forces, bending moments, rotations and translations are as follows without furher details (Fig. 7.4.1.3 right side):

$$V_{max}=\frac{5}{8}q_1 L+\frac{1}{2}q_2 L=\frac{5}{8}68.27\cdot 6+\frac{1}{2}21.73\cdot 6=321.2\,\mathrm{kN}$$

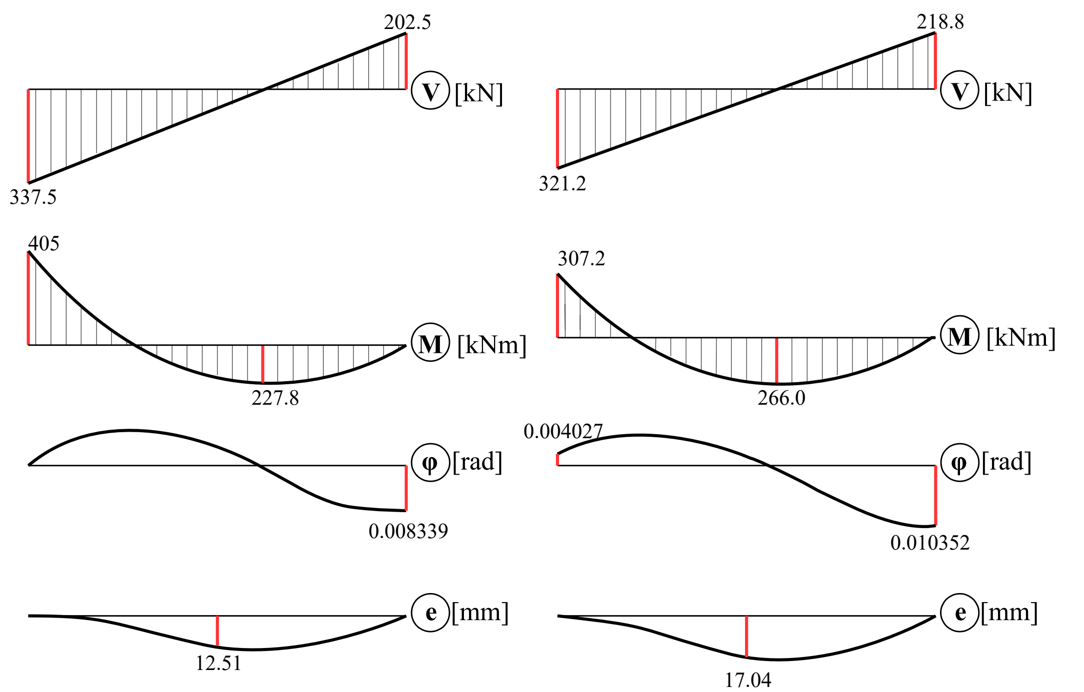
$$M^{-}_{max}=M_{pl}=\frac{q_1 L^2}{8}=\frac{68.27\cdot 6^2}{8}=307.2\,\mathrm{kNm}$$

$$\phi_{roller}=\frac{q_1 L^3}{48 E I_1}+\frac{q_2 L^3}{24 E I_1}=\left(\frac{68.27}{48}+\frac{21.73}{24}\right)\left[\frac{6^3}{210\cdot 10^6\cdot 0.000231283781}\right]=0.010352\,\mathrm{rad}$$

$$\phi_{pl\,hinge}=\frac{q_2 L^3}{24 E I_1}=\left(\frac{21.73}{24}\right)\left[\frac{6^3}{210\cdot 10^6\cdot 0.000231283781}\right]=0.004027\,\mathrm{rad}$$

$$e_{midspan}=\frac{2}{384}\frac{q_1 L^4}{E I_1}+\frac{5}{384}\frac{q_2 L^4}{E I_1}=\left(\frac{2\cdot 68.27}{384}+\frac{5\cdot 21.73}{384}\right)\left[\frac{6^4}{210\cdot 10^6\cdot 0.000231283781}\right]=17.04\,\mathrm{mm}$$

Figure 7.4.1.3 left side shows the linear elastic and right side shows the plastic results.

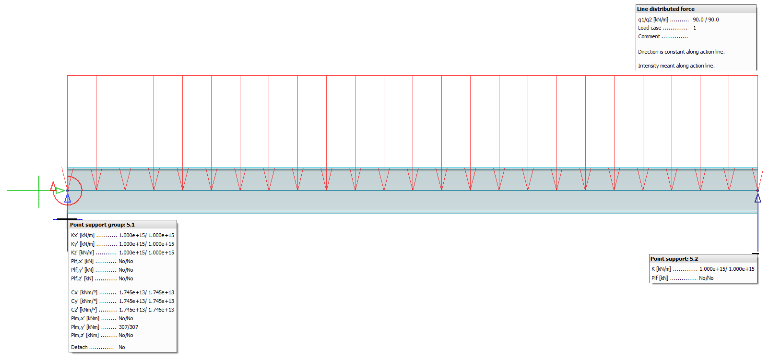


7.4.1.3 – The internal forces, the rotations and translations in the beam according to linear elastic (left) and plastic calculation (right) by hand

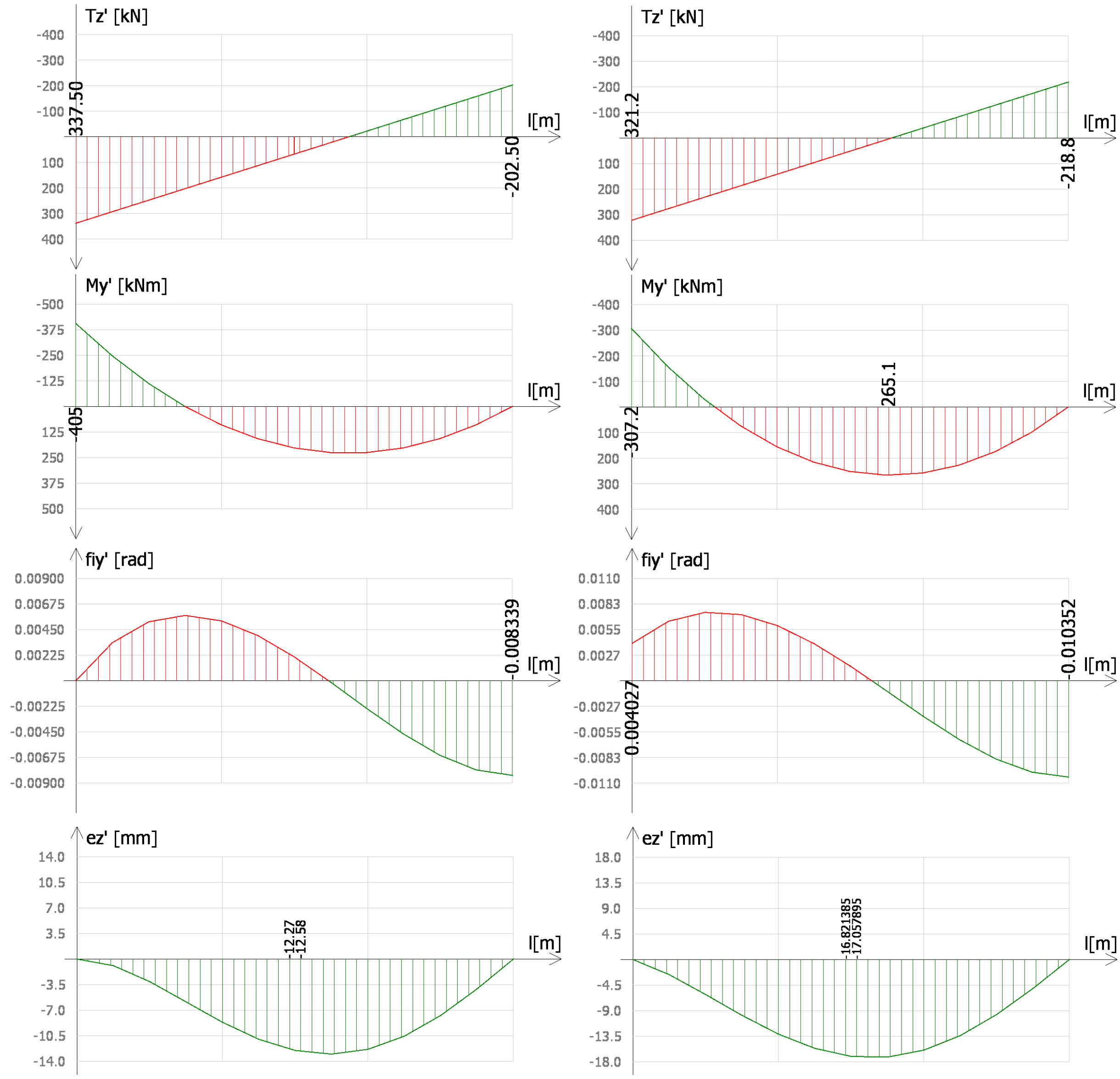
Figure 7.4.1.4 shows the FEM-Design model with the input data.

Figure 7.4.1.5 shows the FEM-Design results without shear deformation ($A = 10^6\,mm^2$). The left side shows the linear elastic calculation and the right side shows the plastic analysis.

We can say that the results are identical with the hand calculations.



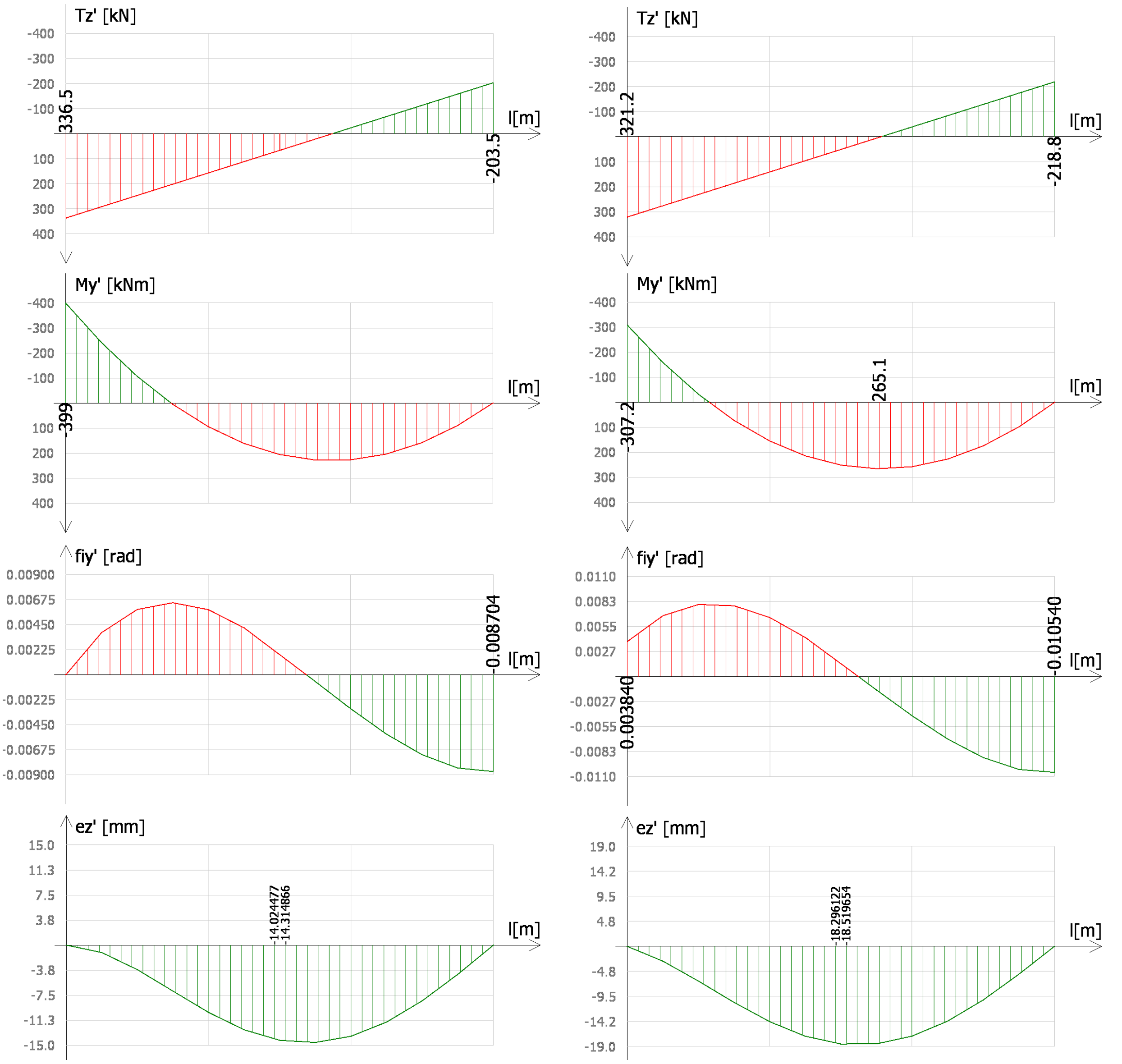
7.4.1.4 – The model with the input data in FEM-Design



7.4.1.4 – The internal forces, the rotations and translations in the beam according to linear elastic (left) and plastic limit calculation (right) with FEM-Design without shear deformation ($A=10^6$ mm^2)

Figure 7.4.1.5 shows the FEM-Design results with shear deformation (A = 8446 mm^2). The left side shows the linear elastic calculation and the right side shows the plastic analysis.

In this case the result differs a bit from the hand calculations according to the shear deformation (Timoshenko beam theory).



7.4.1.5 – The internal forces, the rotations and translations in the beam according to linear elastic (left) and plastic limit calculation (right) with FEM-Design with shear deformation (A=8446 mm^2)

Download links to the example files:

Without shear deformation:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.1 Elasto-plastic point supports in a beam_without_shear def.str

With shear deformation:

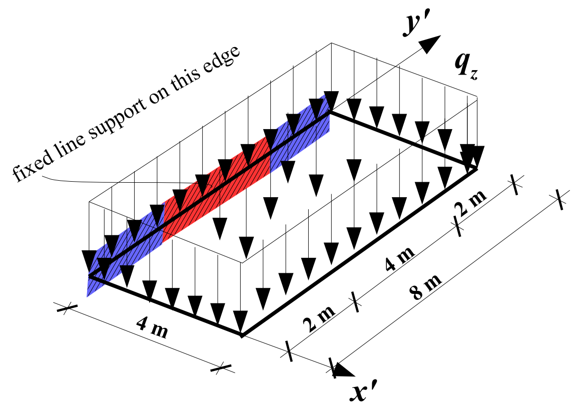
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.1 Elasto-plastic point supports in a beam_with_shear def.str

7.4.2 Elasto-plastic line support in a plate

Inputs:

Dimensions of the slab	$L_x = 4$ m ; $L_y = 8$ m
Distributed load	$q_z = 10$ kN/m^2
Concrete slab	C 30/37
Young's modulus of concrete	E = 33 GPa
Poisson's ratio	ν = 0.2
Thickness	t = 200 mm
Line support	$m_{pl} = m_{x'} = 30$ kNm/m (around line)

The problem is a concrete slab with a fixed support on its edge and with a total distributed load (see Fig. 7.4.2.1).



7.4.2.1 – A concrete slab with a fixed support on its edge

<u>Case 1:</u>

In this case we assume that the line support has a **plastic limit moment capacity** (m_{pl}, around the edge) along the whole support (blue and red line equally, see Fig. 7.4.2.1). The slab behaves similarly as a cantilever beam. If the plastic limit moment capacity is valid along the whole line support the load bearing capacity can be assumed with the help of the following equation:

$$m_{pl} = \frac{q_1 L_x^2}{2}$$ This is the approximated specific moment value at the fixed end.

It means that the maximum of the total distributed load which the structure can bear is:

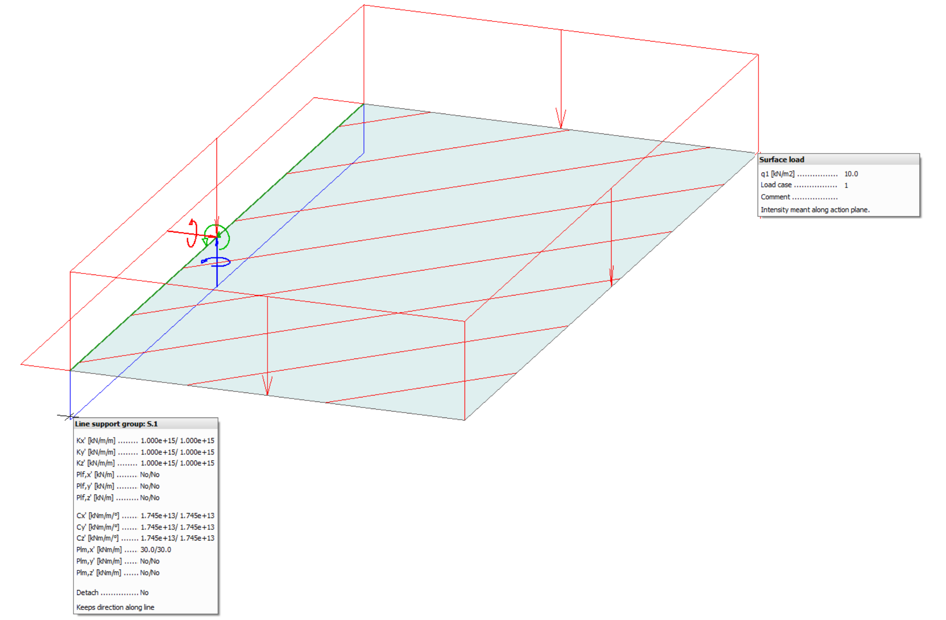
$$q_1 = \frac{2 m_{pl}}{L_x^2} = \frac{2 \cdot 30}{4^2} = 3.75 \,\mathrm{kN/m^2}$$

Thus if we apply q_z = 10 kN/m^2 load on the slab there is no equilibrium on this final load level. The last converged load level must be around:

$$\eta = \frac{q_1}{q_z} = \frac{3.75}{10} = 0.375 = 37.5\%$$

We modelled this case in FEM-Design. Let's see the results based on the finite element analysis.

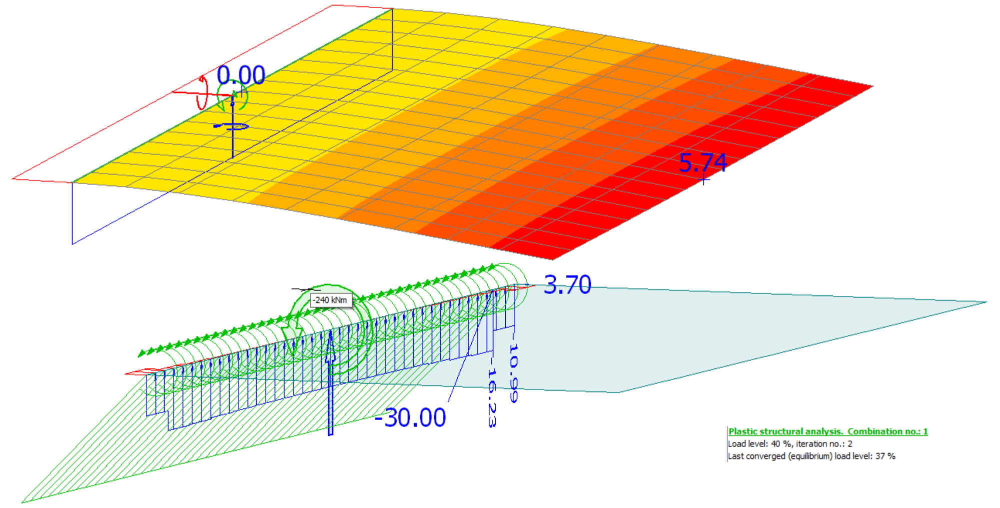
Fig. 7.4.2.2 shows the model with the adjusted parameters.



7.4.2.2 – The concrete slab with a fixed support on its edge in FEM-Design (Case 1)

After the elasto-plastic analysis the last converged load level based on the program is **37%** (see Fig. 7.4.2.3). Fig. 7.4.2.3 also shows the reactions. We can see that the reaction moment around the line support is a constant **30.00 kNm/m** which is the expected value according to the adjusted limit moment value ($m_{pl} = m_{x'} = 30$ kNm/m).

Thus we can say that the program gives the same solution as the hand calculation.



7.4.2.3 – Last converged load level 37%, the deflection [mm] *and the reactions by this load level*

Case 2:

In this case we assume that the line support has a **plastic limit moment capacity** (m_{pl}, around the edge) along the **red part** (see Fig. 7.4.2.1). The **blue parts behave in a linear elastic** way. In this case the slab behaves also as a cantilever beam. The slab behaves linearly until load level q_1=3.75 kN/m^2. Above this load level there will be a plastic hinge line along the red part of the fixed support (see Fig. 7.4.2.1). This remaining load is:

$$q_2 = q_z - q_1 = 10 - 3.75 = 6.25\,\mathrm{kN/m^2}$$

Since the sum of the length of the blue parts is equal to the length of the red one (see Fig. 7.4.2.1) we can estimate that (as a conservative assumtion) the remaining load level is redistributing on the blue linear elastic fixed supports. To assume the maximum deflection of the slab first of all we need to calculate the deflection according to q_1=3.75 kN/m^2 (linear elastic behaviour):

The approximated bending stiffness of the slab:

$$D_{11} = \frac{E\,t^3}{12(1-\nu^2)} = \frac{33\cdot 10^6 \cdot 0.2^3}{12(1-0.2^2)} = 22917\,\frac{\mathrm{kN\,m^2}}{\mathrm{m}}$$

Based on this value the deflection (as a cantilever):

$$e_1 \approx \frac{q_1 L_x^4}{8\,D_{11}} = \frac{3.75\cdot 4^4}{8\cdot 22917} = 5.236\,\mathrm{mm}$$

Since the remaining part of the load is redistributing to the blue (linear elastic) parts of the support we can assume that the remaining middle part of the distributed total load acts on the outer part. Therefore the remaining distributed load which has affect on the linear elastic parts of the line support:

$$q_2^* = 2q_2 = 2 \cdot 6.25 = 12.5\,\mathrm{kN/m^2}$$

Because the length of the red part equals to blue ones.

According to this load the deflection of the slab after the linear elastic behaviour can be assumed as:

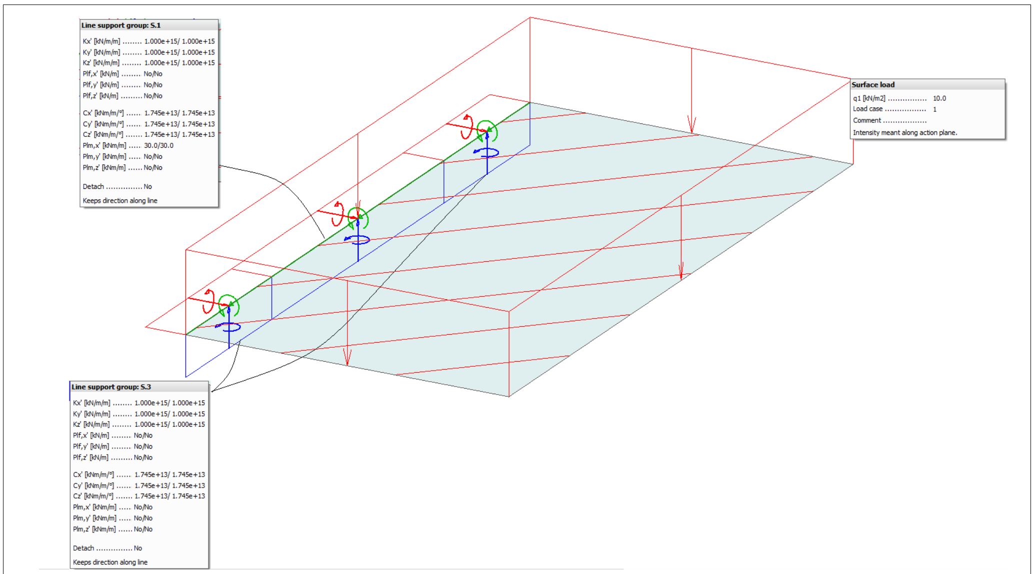
$$e_2 \approx \frac{q_2^* L_x^4}{8D_{11}} = \frac{12.5 \cdot 4^4}{8 \cdot 22917} = 17.454\,\mathrm{mm}$$

Thus the approximation of the total deflection with elasto-plastic calculation is:

$$e_{pl,tot} = e_1 + e_2 = 5.236 + 17.454 = 22.69\,\mathrm{mm}$$ This is a very conservative result.

We modelled this case in FEM-Design. Let's see the results based on the finite element analysis.

Fig. 7.4.2.4 shows the model with the adjusted parameters.



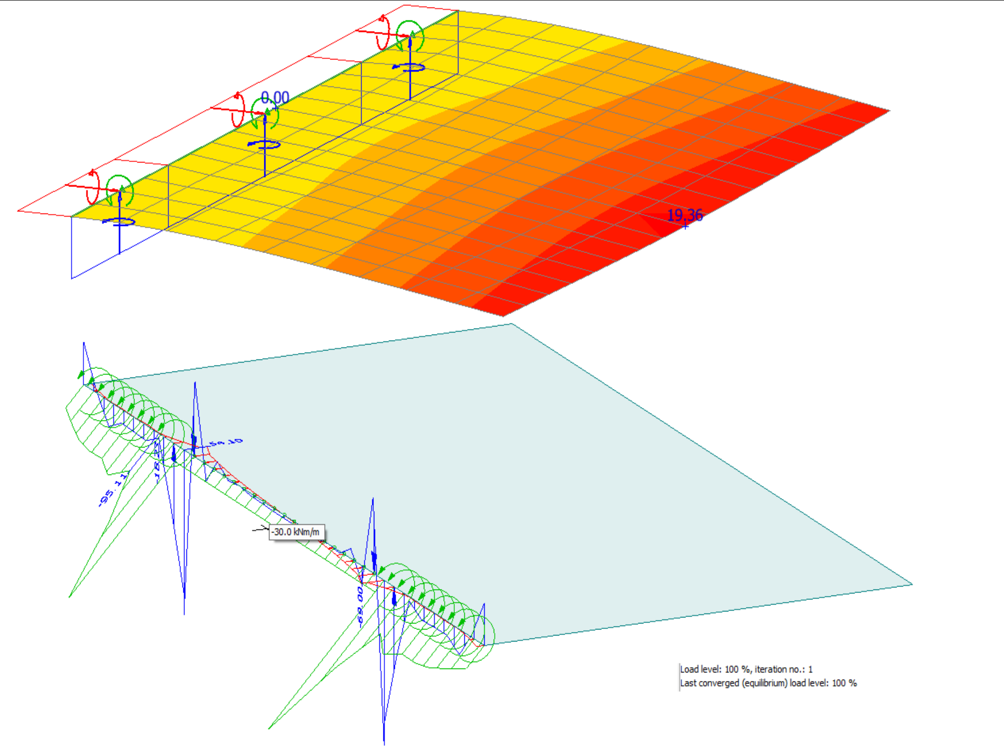
7.4.2.4 – The concrete slab with a fixed support on its edge in FEM-Design (Case 2)

After the elasto-plastic analysis the last converged load level based on the program is **100%** (see Fig. 7.4.2.5). Fig. 7.4.2.5 also shows the reactions. We can see that the reaction moment around the line support of the middle part is a constant **30.00 kNm/m** which is the expected value according to the adjusted limit moment value ($m_{pl} = m_{x'} =$ 30 kNm/m). The remaining part of the line support behaves linearly as expected. At the end of the linear behaviour (load level 37%)

the deflection is 5.74 mm (see Fig. 7.4.2.3). The difference between the FEM-Design and hand calculation is **less than 10% (5.74 mm vs. 5.236 mm**). This difference comes from the very conservative hand calculation formula and from the fact that FEM-Design considers the shear deformations (Mindlin plate theory).

At load level **100%** the deflection is **19.36 mm** (see Fig. 7.4.2.5). The results based on the hand calculation was **22.69 mm**. The difference is around **15%**. This difference comes from the very conservative hand calculation formula and from the fact that FEM-Design considers the shear deformations (Mindlin plate theory).

Thus we can say that the program gives an accurate elasto-plastic solution.



7.4.2.5 – Last converged load level 100%, the deflection [mm] *and the reactions* [kN/m, kNm/m] *by this load level*

Download link to the example files:

Case 1:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.2 Elasto-plastic line supports in a plate case1.str

Case 2:

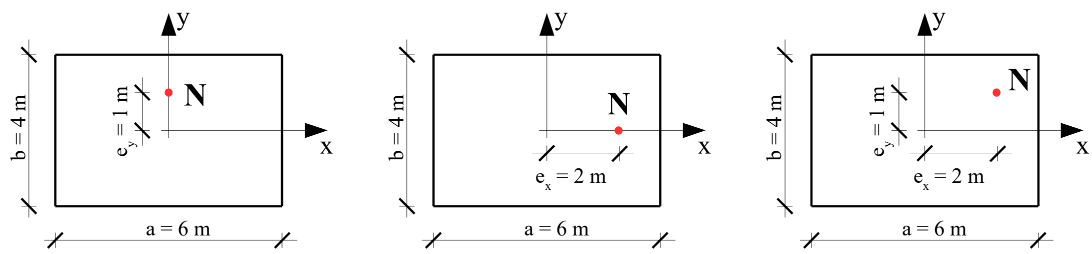
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.2 Elasto-plastic line supports in a plate case2.str

7.4.3 Elasto-plastic surface support with detach in an embedded plate

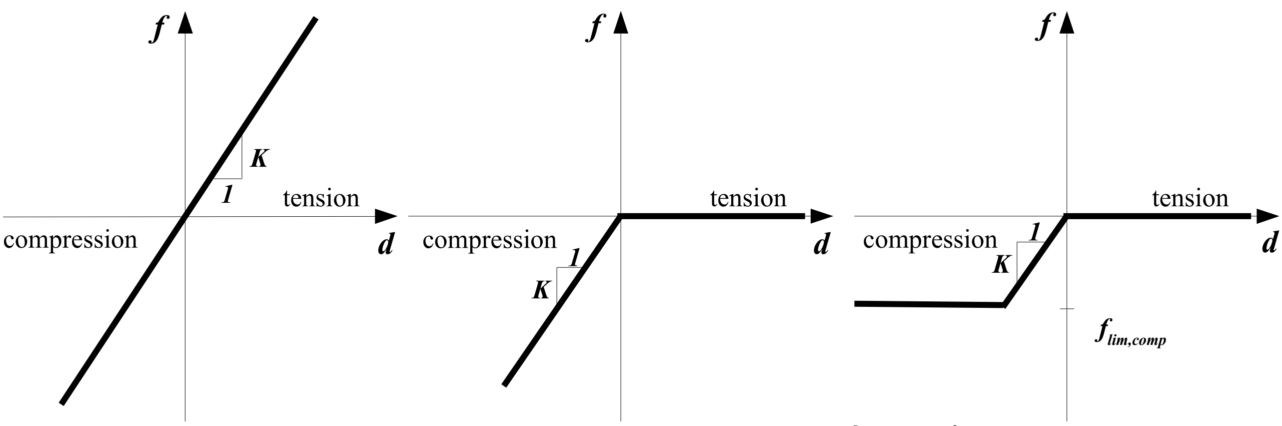
In this example the structure is an embedded rectangle plate (a=6m, b=4m). The plate is assumed to be infinitely rigid. There are three different load positions of compression point load (*N=3000 kN,* see Fig. 7.4.3.1). The behaviour of the bedding of the plate will be considered in three different ways (see Fig. 7.4.3.2).

The table below shows the nine analyzed cases:

No.	**Eccentricity of the compression point load (N = 3000 kN)**	**Behaviour of the bedding**
1	in y direction, e_y=1 m	linear elastic
2	in x direction, e_x=2 m	linear elastic
3	in y direction, e_y=1 m and in x direction, e_x=2 m	linear elastic
4	in y direction, e_y=1 m	linear elastic, non-tension
5	in x direction, e_x=2 m	linear elastic, non-tension
6	in y direction, e_y=1 m and in x direction, e_x=2 m	linear elastic, non-tension
7	in y direction, e_y=1 m	elasto-plastic, non-tension
8	in x direction, e_x=2 m	elasto-plastic, non-tension
9	in y direction, e_y=1 m and in x direction, e_x=2 m	elasto-plastic, non-tension



7.4.3.1 – Dimensions of the embedded plate and the positions of the compression point load on it



7.4.3.2 – The material behaviour of the bedding of the plate (K= 10^5 kN/m/m^2, $f_{lim,comp}$=200 kPa)

Case 1):

In this case the behaviour of the bedding is linear elastic therefore the reaction forces can be calculated according to the superposition of pure compression and uniaxial bending:

$$\sigma_{max}^{(1)} = -\frac{N}{A} + \frac{M_x}{I_x} y_{max} = -\frac{3000}{24} + \frac{3000}{32} 2 = +62.5\,\text{kPa}$$

$$\sigma_{min}^{(1)} = -\frac{N}{A} - \frac{M_x}{I_x} y_{min} = -\frac{3000}{24} - \frac{3000}{32} 2 = -312.5\,\text{kPa}$$

Case 2):

In this case the behaviour of the bedding is linear elastic therefore the reaction forces can be calculated according to the superposition of pure compression and uniaxial bending:

$$\sigma_{max}^{(2)} = -\frac{N}{A} + \frac{M_y}{I_y} x_{max} = -\frac{3000}{24} + \frac{6000}{72} 3 = +125\,\text{kPa}$$

$$\sigma_{min}^{(2)} = -\frac{N}{A} - \frac{M_y}{I_y} x_{min} = -\frac{3000}{24} - \frac{6000}{72} 3 = -375\,\text{kPa}$$

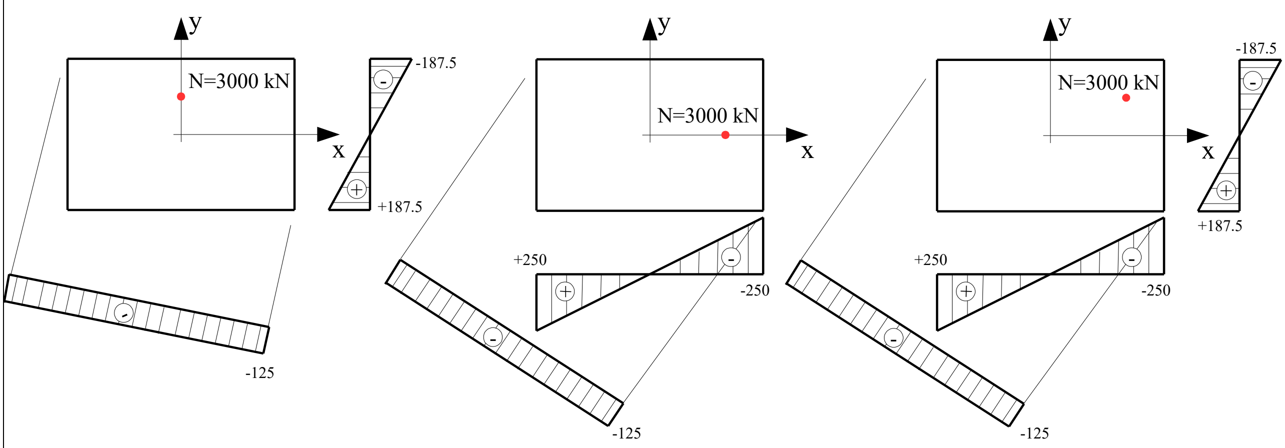
Case 3):

In this case the behaviour of the bedding is linear elastic therefore the reaction forces can be calculated according to the superposition of pure compression and biaxial bending:

$$\sigma_{max}^{(3)} = -\frac{N}{A} + \frac{M_x}{I_x} y_{max} + \frac{M_y}{I_y} x_{max} = -\frac{3000}{24} + \frac{3000}{32} 2 + \frac{6000}{72} 3 = +312.5\,\text{kPa}$$

$$\sigma_{min}^{(3)} = -\frac{N}{A} + \frac{M_x}{I_x} y_{min} + \frac{M_y}{I_y} x_{min} = -\frac{3000}{24} - \frac{3000}{32} 2 - \frac{6000}{72} 3 = -562.5\,\text{kPa}$$

The specific results of the hand calculation can be seen in Fig. 7.4.3.3.



7.4.3.3 – The distribution of the reaction forces [kPa] *of the linear elastic cases*

Case 4):

In this case the behaviour of the bedding is linear elastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the linear stress volume must be equal to the acting point load:

$$N=-\frac{6\cdot 3}{2}\sigma_{min}^{(1)}$$

$$\sigma_{min}^{(1)}=-\frac{2N}{6\cdot 3}=-\frac{2\cdot 3000}{6\cdot 3}=-333.3\,\text{kPa}$$

Case 5):

In this case the behaviour of the bedding is linear elastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the linear stress volume must be equal to the acting point load:

$$N=-\frac{4\cdot 3}{2}\sigma_{min}^{(2)}$$

$$\sigma_{min}^{(2)}=-\frac{2N}{4\cdot 3}=-\frac{2\cdot 3000}{4\cdot 3}=-500\,\text{kPa}$$

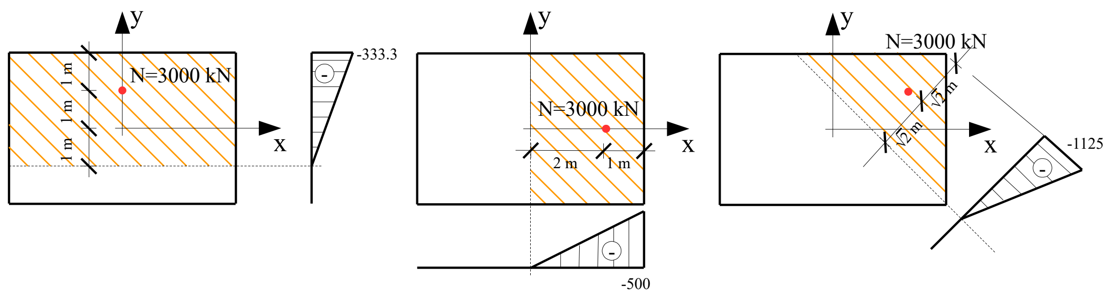
Case 6):

In this case the behaviour of the bedding is linear elastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the linear stress volume must be equal to the acting point load:

$$N=\frac{-\left[2\cdot\sqrt{2}\cdot\sqrt{2}\right]^2}{2}\frac{1}{3}\sigma_{min}^{(3)}$$

$$\sigma_{min}^{(3)}=-\frac{6N}{4^2}=-\frac{6\cdot 3000}{16}=-1125\,\text{kPa}$$

The specific results of the hand calculation can be seen in Fig. 7.4.3.4.



7.4.3.4 – The distribution of the reaction forces [kPa] *of the linear elastic, non-tension cases*

In Case 7-9 the bahaviours are elasto-plastic, non-tension thus it means that according to the given $\boldsymbol{f_{lim,comp}}$***=200 kPa*** limit force (see Fig. 7.4.3.2) there are different values of the load-bearing capacity due to the positions of the point load.

Case 7):

In this case the behaviour of the bedding is elasto-plastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the constant stress volume (with $\boldsymbol{f_{lim,comp}}$***=200 kPa***) must be equal to the acting point load:

$$N_{limit}^{(1)} = 6 \cdot 2 \cdot f_{lim,comp} = 6 \cdot 2 \cdot 200 = 2400\,\mathrm{kN}$$

$$\eta^{(1)} = \frac{N_{limit}^{(1)}}{N} = \frac{2400}{3000} = 80\,\%$$

Case 8):

In this case the behaviour of the bedding is elasto-plastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the constant stress volume (with $\boldsymbol{f_{lim,comp}}$***=200 kPa***) must be equal to the acting point load:

$$N_{limit}^{(2)} = 4 \cdot 2 \cdot f_{lim,comp} = 4 \cdot 2 \cdot 200 = 1600\,\mathrm{kN}$$

$$\eta^{(2)} = \frac{N_{limit}^{(2)}}{N} = \frac{1600}{3000} = 53.3\,\%$$

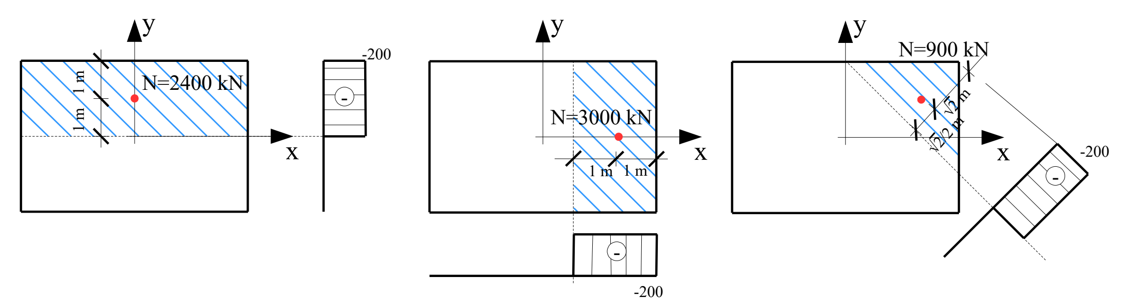
Case 9):

In this case the behaviour of the bedding is elasto-plastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the constant stress volume (with $\boldsymbol{f_{lim,comp}}$***=200 kPa***) must be equal to the acting point load:

$$N_{limit}^{(3)} = \frac{\left[\sqrt{2}\left(\sqrt{2}+\frac{\sqrt{2}}{2}\right)\right]^2}{2} \cdot f_{lim,comp} = \frac{3^2}{2} \cdot 200 = 900\,\mathrm{kN}$$

$$\eta^{(3)} = \frac{N_{limit}^{(3)}}{N} = \frac{900}{3000} = 30\,\%$$

The specific results of the hand calculation can be seen in Fig. 7.4.3.5.

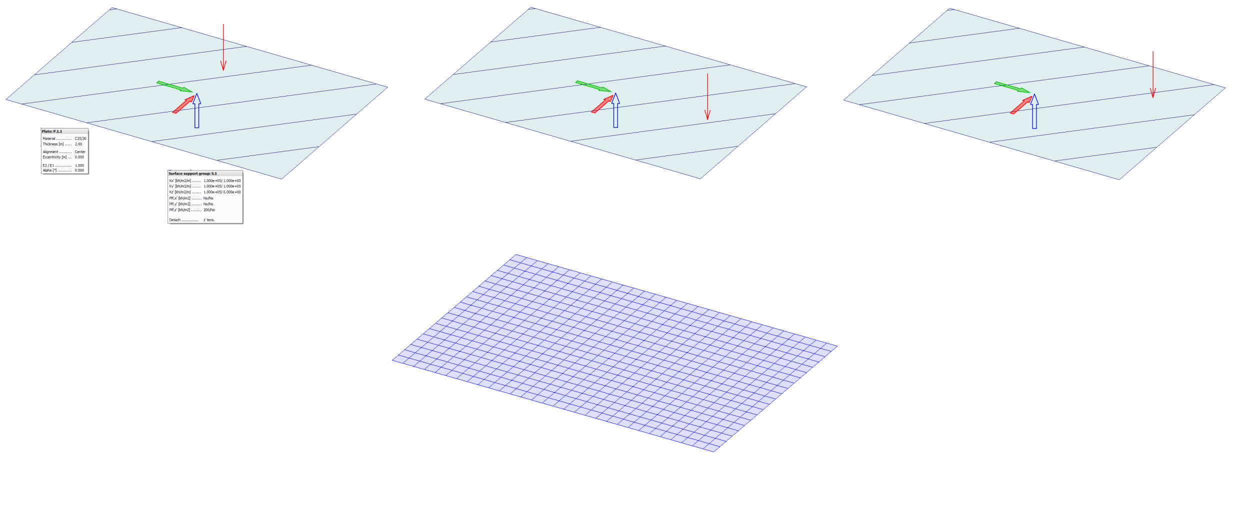


7.4.3.5 – The distribution of the reaction forces [kPa] *of the elasto-plastic, non-tension cases*

In the second part of this chapter we make a FEM-Design model for the problem and compare the results with the hand calculation. Fig. 7.4.3.6 shows the main input properties of the model (concrete slab with 2 m thickness). In FEM-Design the reaction result of a surface support element is the average value. For more precise results the average element (mesh) size was 0.2 m.

By the calculation of the surface support reactions we exrapolated the element average FEM-Design numeric values to the extreme fibre (edge of the plate) to get more comparable results.

You can find below one by one the 9 different cases and their results based on FEM-Design.

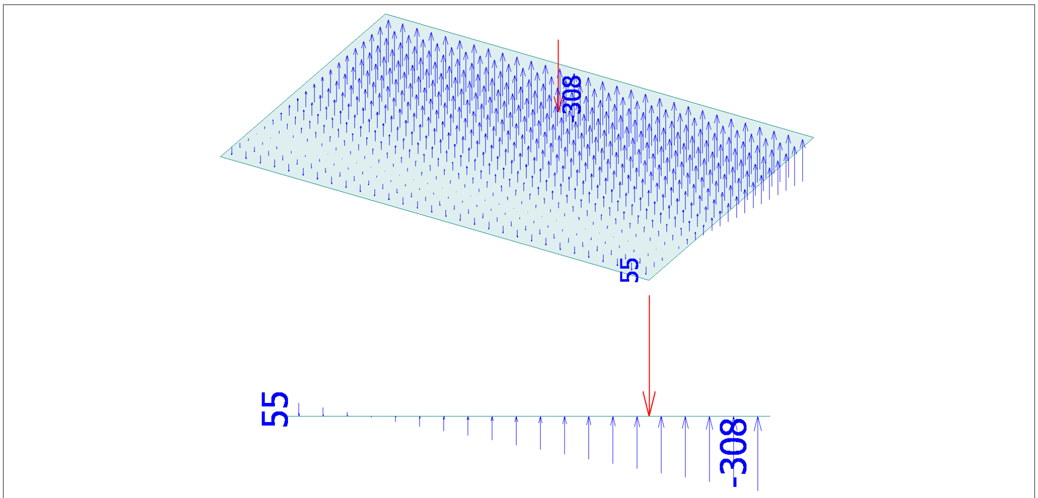


7.4.3.6 – The FEM-Design modell (plate with surface support) with the input data and the different position of the point loads and the finite element mesh

Case 1) FEM (see Fig. 7.4.3.7):

$$\sigma_{max}^{\mathrm{FEM}(1)} = +64.7\,\mathrm{kPa}$$

$$\sigma_{min}^{\mathrm{FEM}(1)} = -317.5\,\mathrm{kPa}$$

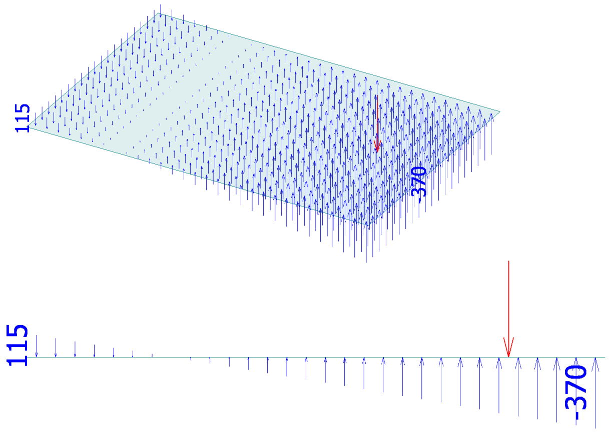


7.4.3.7 – The reaction results in the surface support with the specific numerical values (element average) for Case 1) [kPa]

Case 2) FEM (see Fig. 7.4.3.8):

$$\sigma_{max}^{FEM(2)} = +123.4\,kPa$$

$$\sigma_{min}^{FEM(2)} = -379\,kPa$$

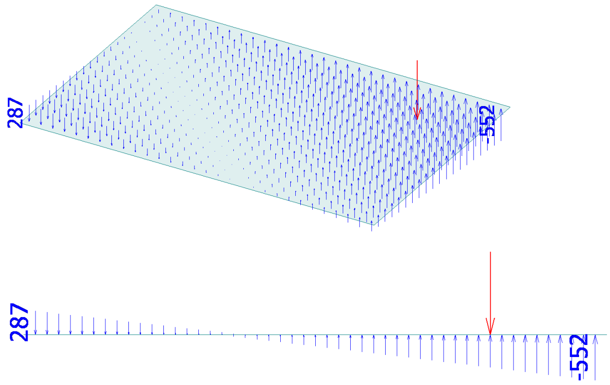


7.4.3.8 – The reaction results in the surface support with the specific numerical values (element average) for Case 2) [kPa]

Case 3) FEM (see Fig. 7.4.3.9):

$$\sigma_{max}^{FEM(3)} = +295.5\,kPa$$

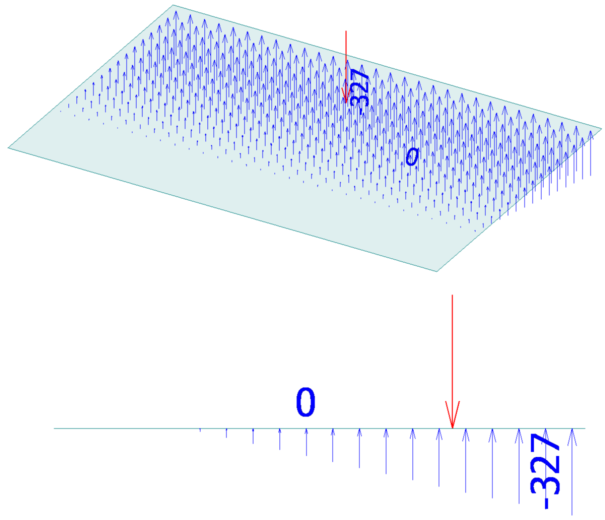
$$\sigma_{min}^{FEM(3)} = -561\,kPa$$



7.4.3.9 – The reaction results in the surface support with the specific numerical values (element average) for Case 3) [kPa]

Case 4) FEM (see Fig. 7.4.3.10):

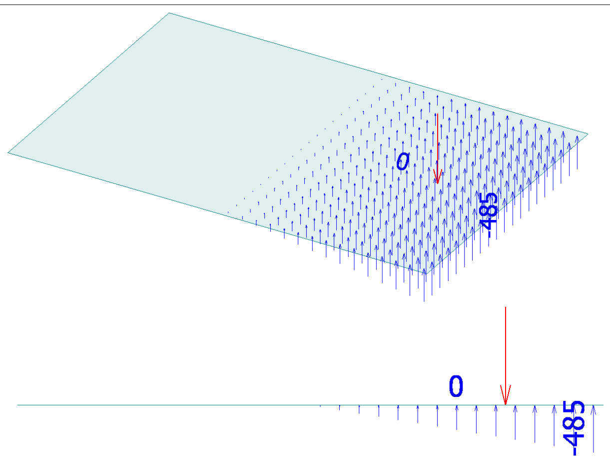
$$\sigma_{min}^{\mathrm{FEM}(1)} = -338\ \mathrm{kPa}$$



7.4.3.10 – The reaction results in the surface support with the specific numerical values (element average) for Case 4) [kPa]

Case 5) FEM (see Fig. 7.4.3.11):

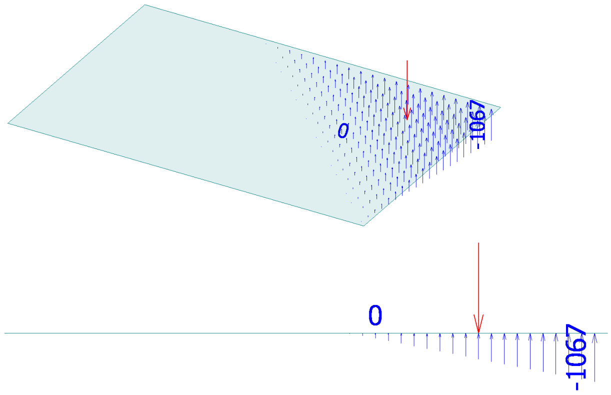
$$\sigma_{min}^{\mathrm{FEM}(2)} = -501.5\ \mathrm{kPa}$$



7.4.3.11 – The reaction results in the surface support with the specific numerical values (element average) for Case 5) [kPa]

Case 6) FEM (see Fig. 7.4.3.12):

$\sigma_{min}^{FEM(3)} = -1095\,\text{kPa}$

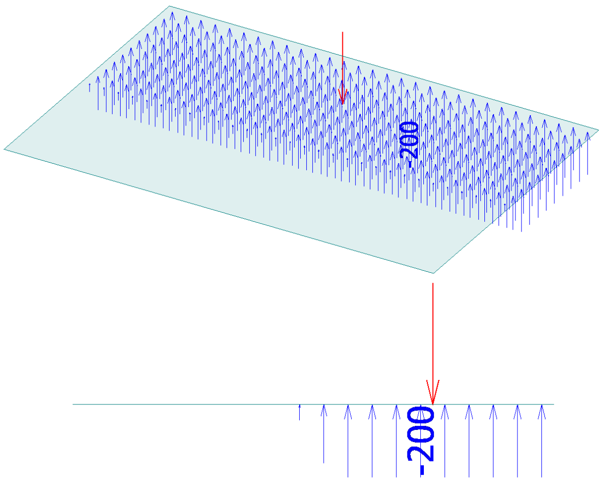


7.4.3.12 – The reaction results in the surface support with the specific numerical values (element average) for Case 6) [kPa]

Case 7) FEM (see Fig. 7.4.3.13):

Last converged load level and the bearing capacity.

$\eta^{FEM(1)} = 80\%$ $N_{limit}^{FEM(1)} = 0.80 \cdot 3000 = 2400\,\text{kN}$

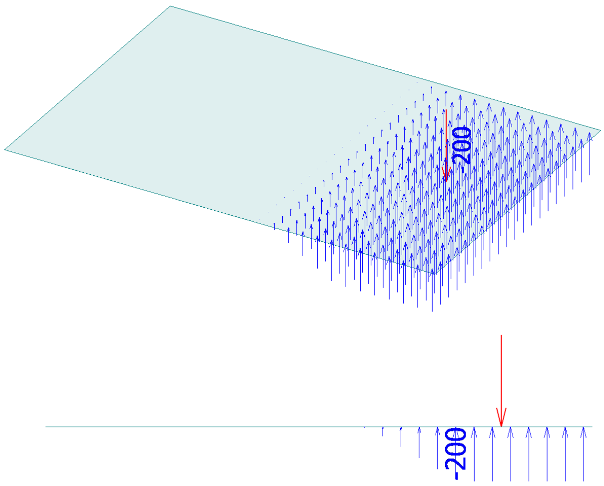


7.4.3.13 – The reaction results in the surface support with the specific numerical values (element average) for Case 7) [kPa]

Case 8) FEM (see Fig. 7.4.3.14):

Last converged load level and the bearing capacity.

$\eta^{FEM(2)} = 52\,\%$ $N_{limit}^{FEM(2)} = 0.52 \cdot 3000 = 1560\,\mathrm{kN}$

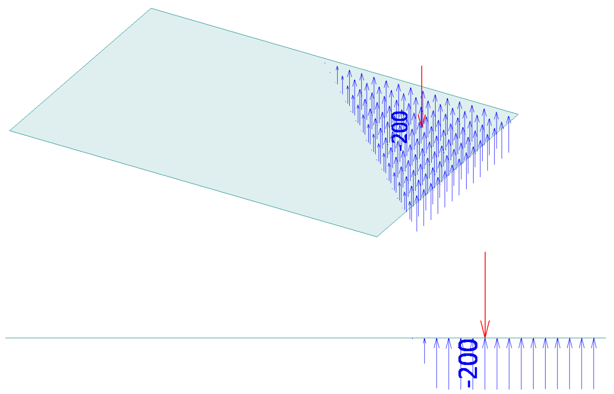


7.4.3.14 – The reaction results in the surface support with the specific numerical values (element average) for Case 8) [kPa]

Case 9) FEM (see Fig. 7.4.3.15):

Last converged load level and the bearing capacity.

$\eta^{FEM(3)} = 30\,\%$ $N_{limit}^{FEM(3)} = 0.30 \cdot 3000 = 900\,\mathrm{kN}$



7.4.3.15 – The reaction results in the surface support with the specific numerical values (element average) for Case 9) [kPa]

The differences between the hand calculations and the FEM-Design results are **less than 5%**.

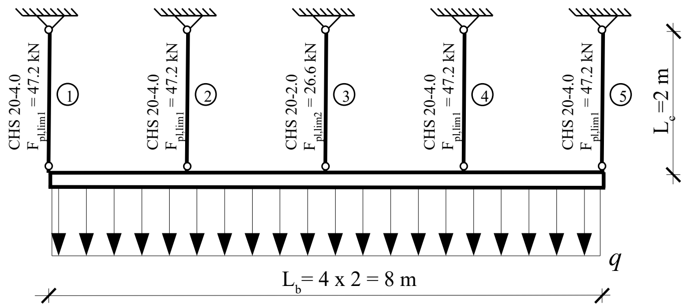
Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.3 Elasto-plastic surface supports with detach in an embedded plate.str

7.4.4 *Elasto-plastic trusses in a multispan continuous beam*

Inputs:

Beam length	$L_b = 8$ m
Truss length	$L_c = 2$ m
Total distributed load	$q = 20$ kN/m
Structural steel (trusses with plastic limit force)	S235, $f_y = 235$ MPa
Young's modulus of steel	$E_s = 210$ GPa
Cross-sectional area (CHS 20-4.0)	$A_1 = 201$ mm^2
Cross-sectional area (CHS 20-2.0)	$A_2 = 113$ mm^2
Plastic limit force (CHS 20-4.0)	$F_{pl,lim1} = f_yA_1 = 47.2$ kN
Plastic limit force (CHS 20-2.0)	$F_{pl,lim2} = f_yA_2 = 26.6$ kN
Concrete (beam, linear elastic material model)	C 25/30
Young's modulus of concrete	$E_c = 31$ GPa
Cross-sectional area (rectangle)	b = 200 mm; h = 300 mm
Inertia	$I_b = 0.00045$ m^4



7.4.4.1 – The trusses with the plastic limit forces support a concrete beam

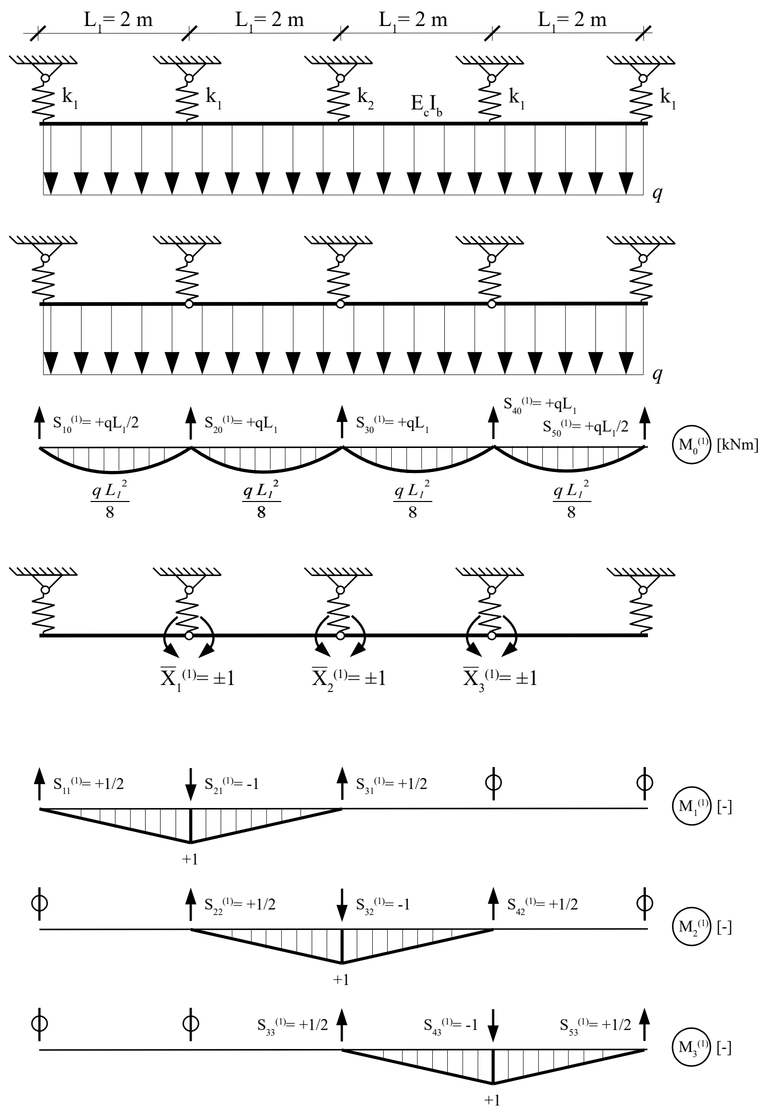
The problem is a concrete beam with five trusses as supports (see Fig. 7.4.4.1). The beam is linear elastic. The trusses are linear elastic, perfectly plastic (with limit force, see the input table above). Truss number 1, 2, 4 and 5 have $F_{pl,lim1} = 47.2$ kN plastic limit force. Truss number 3 has $F_{pl,lim2} = 26.6$ kN plastic limit force. The external load is a total distributed load q = 20 kN/m (see also Fig. 7.4.4.1 for the geometry).

At this problem (loads and supports) the degree of static indeterminacy is three. We will solve the problem with force method. The optimal solution with the force method is when the primary structure is essentially a series of simply supported beams (see Fig. 7.4.4.2). Apply the unit redundant forces $\overline{X}_i$ (i = 1, 2, 3) in the lines of the removed constraints, pairs of opposite moments at the end cross-sections of beams connected to hinges created above the intermediate supports (see the primary structure Fig. 7.4.4.2).

We apply vertical springs to represent the trusses as supports. The flexibility of the trusses (in the elastic region):

$$k_1 = \frac{L_c}{E_s A_1} = \frac{2}{210 \cdot 10^6 \cdot 201 \cdot 10^{-6}} = 4.738 \cdot 10^{-5} \frac{\mathrm{m}}{\mathrm{kN}} \quad \text{for truss number 1, 2, 4 and 5.}$$

$$k_2 = \frac{L_c}{E_s A_2} = \frac{2}{210 \cdot 10^6 \cdot 113 \cdot 10^{-6}} = 8.428 \cdot 10^{-5} \frac{\mathrm{m}}{\mathrm{kN}} \quad \text{for truss number 3.}$$



7.4.4.2 – The primary structure and the redundant forces with the reactions and internal forces for the force method solution (at stage 1)

The bending stiffness of the beam:

$$E_c I_b = 31 \cdot 10^6 \cdot 0.00045 = 13950 \, \mathrm{kNm}^2$$

Fig. 7.4.4.2 shows the statical system of the analyzed problem with the primary structure.

By force method first of all we need to calculate the flexibilty matrix.
Flexible coefficients (see Fig. 7.4.4.2 for the values which provide these coefficients according to the virtual force method):

$$a_{11}=a_{33}=2\frac{L_1}{3E_cI_b}+k_1\frac{1}{2}\frac{1}{2}+k_1 1\cdot 1+k_2\frac{1}{2}\frac{1}{2}=1.7587\cdot 10^{-4}\frac{\text{rad}}{\text{kNm}}$$

$$a_{22}=2\frac{L_1}{3E_cI_b}+2k_1\frac{1}{2}\frac{1}{2}+k_2 1\cdot 1=2.0355\cdot 10^{-4}\frac{\text{rad}}{\text{kNm}}$$

$$a_{13}=a_{31}=k_2\frac{1}{2}\frac{1}{2}=2.107\cdot 10^{-5}\frac{\text{rad}}{\text{kNm}}$$

$$a_{12}=a_{23}=a_{21}=a_{32}=\frac{L_1}{6E_cI_b}-k_1 1\frac{1}{2}-k_2 1\frac{1}{2}=-4.1935\cdot 10^{-5}\frac{\text{rad}}{\text{kNm}}$$

These coefficients are independent from the external loads. The second step is the calculation of the load constants based on the total load intensity q.

Load constants:

$$a_{10}=a_{30}=2\frac{qL_1^3}{24E_cI_b}+k_1\frac{qL_1}{2}\frac{1}{2}-k_1\frac{2qL_1}{2}1+k_2\frac{2qL_1}{2}\frac{1}{2}=1.220\cdot 10^{-3}\,\text{rad}$$

$$a_{20}=2\frac{qL_1^3}{24E_cI_b}+2k_1\frac{2qL_1}{2}\frac{1}{2}-k_2\frac{2qL_1}{2}1=-5.202\cdot 10^{-4}\,\text{rad}$$

The equation system of the force method:

$\boxed{\underline{\underline{A}}\,\underline{X}+\underline{a_0}=0}$ where:

$$\underline{\underline{A}}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}1.7587\cdot 10^{-4} & -4.1935\cdot 10^{-5} & 2.107\cdot 10^{-5}\\ -4.1935\cdot 10^{-5} & 2.0355\cdot 10^{-4} & -4.1935\cdot 10^{-5}\\ 2.107\cdot 10^{-5} & -4.1935\cdot 10^{-5} & 1.7587\cdot 10^{-4}\end{bmatrix}\frac{\text{rad}}{\text{kNm}} \quad \text{and}$$

$$\underline{a_0}=\begin{bmatrix}1.220\cdot 10^{-3}\\ -5.202\cdot 10^{-4}\\ 1.220\cdot 10^{-3}\end{bmatrix}\text{rad}$$

The inverse matrix for the solution:

$$\underline{\underline{A}}^{-1}=\begin{bmatrix} 6013 & 1147 & -447.0 \\ 1147 & 5385 & 1147 \\ -447.0 & 1147 & 6013 \end{bmatrix}\frac{\text{kNm}}{\text{rad}}$$

The solution of the equation system:

$$\underline{X}=\begin{bmatrix} -6.194 \\ 0.003467 \\ -6.194 \end{bmatrix}\text{kNm}$$

The calculation of the truss forces based on the force method (see also Fig. 7.4.4.2):

$$\boxed{S_k=S_{k0}+\sum_{i=1}^{3} S_{ki}X_i}$$

$$S_1'=S_5'=S_{10}+S_{11}X_1+S_{12}X_2+S_{13}X_3=\frac{qL_1}{2}+\frac{1}{2}X_1+0\cdot X_2+0\cdot X_3=16.90\,\text{kN}$$

$$S_2'=S_4'=S_{20}+S_{21}X_1+S_{22}X_2+S_{23}X_3=2\frac{qL_1}{2}-1\cdot X_1+\frac{1}{2}X_2+0\cdot X_3=46.20\,\text{kN}$$

$$S_3'=S_{30}+S_{31}X_1+S_{32}X_2+S_{33}X_3=2\frac{qL_1}{2}+\frac{1}{2}X_1-1\cdot X_2+\frac{1}{2}X_3=33.80\,\text{kN}$$

Since the plastic limit force for truss number 3 is $F_{pl,lim2}$ = 26.6 kN the elastic behaviour with the described statical system ended at load level q_1:

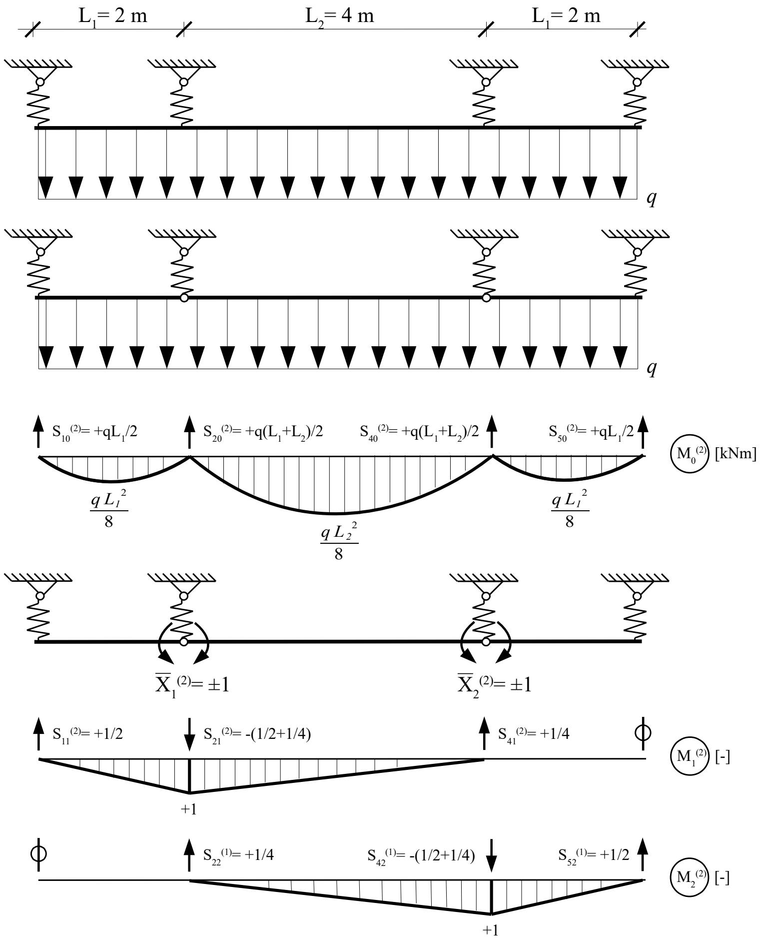
$$\frac{q_1}{q}=\frac{F_{pl,lim2}}{S_3'}=\frac{26.6}{33.80}=0.7870$$

$$q_1=0.7870\,q=15.74\,\text{kN/m}$$

At this load level the normal force in truss number 3:

$$S_3=\frac{q_1}{q}S_3'=26.60\,\text{kN}$$

After this load level the statical system is changing (see Fig. 7.4.4.3). For this system the degree of static indeterminacy is 2. We will solve the problem also with force method. The optimal solution with the force method is when the primary structure is essentially a series of simply supported beams (see Fig. 7.4.4.3). Apply the unit redundant forces $\overline{X}_i$ (i = 1, 2) in the lines of the removed constraints, pairs of opposite moments at the end cross-sections of beams connected to hinges created above the intermediate supports (see the primary structure Fig. 7.4.4.3).



7.4.4.3 – The primary structure and the redundant forces with the reactions and internal forces (at stage 2)

Flexible coefficients (see Fig. 7.4.4.3 for the values which provide these coefficients according to the virtual force method):

$$a_{11}=a_{22}=\frac{L_1}{3E_cI_b}+\frac{L_2}{3E_cI_b}+k_1\frac{1}{2}\frac{1}{2}+k_1\frac{1}{4}\frac{1}{4}+k_1\left(\frac{1}{2}+\frac{1}{4}\right)^2=1.8483\cdot10^{-4}\,\frac{\text{rad}}{\text{kNm}}$$

$$a_{12}=a_{21}=\frac{L_2}{6E_cI_b}-2k_1\frac{1}{4}\left(\frac{1}{2}+\frac{1}{4}\right)=3.0022\cdot10^{-5}\,\frac{\text{rad}}{\text{kNm}}$$

The calculation of the load constants based on the total load intensity q.

Load constants:

$$a_{10}=a_{20}=\frac{qL_1^3}{24E_cI_b}+\frac{qL_2^3}{24E_cI_b}+k_1\frac{qL_1}{2}\frac{1}{2}-k_1\left(\frac{qL_1}{2}+\frac{qL_2}{2}\right)\left(\frac{1}{2}+\frac{1}{4}\right)+k_1\left(\frac{qL_1}{2}+\frac{qL_2}{2}\right)\frac{1}{4}=$$

$$=3.353\cdot 10^{-3}\,\mathrm{rad}$$

The equation system is now:

$\boxed{\underline{\underline{A}}\,\underline{X}+\underline{a_0}=0}$ where:

$$\underline{\underline{A}}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix} 1.8483\cdot 10^{-4} & 3.0022\cdot 10^{-5} \\ 3.0022\cdot 10^{-5} & 1.8483\cdot 10^{-4}\end{bmatrix}\frac{\mathrm{rad}}{\mathrm{kNm}}$$

$$\underline{a_0}=\begin{bmatrix} 3.353\cdot 10^{-5} \\ 3.353\cdot 10^{-5}\end{bmatrix}\mathrm{rad}$$

The inverse matrix for the solution:

$$\underline{\underline{A}}^{-1}=\begin{bmatrix} 5557 & -902.6 \\ -902.6 & 5557\end{bmatrix}\frac{\mathrm{kNm}}{\mathrm{rad}}$$

The solution of the equation system:

$$\underline{X}=\begin{bmatrix} -15.61 \\ -15.61\end{bmatrix}\mathrm{kNm}$$

The calculation of the truss forces:

$$S_1''=S_5''=S_{10}+S_{11}X_1+S_{12}X_2=\frac{qL_1}{2}+\frac{1}{2}X_1+0\cdot X_2=12.20\,\mathrm{kN}$$

$$S_2''=S_4''=S_{20}+S_{21}X_1+S_{22}X_2=\left(\frac{qL_1}{2}+\frac{qL_2}{2}\right)-\left(\frac{1}{2}+\frac{1}{4}\right)X_1+\frac{1}{4}\cdot X_2=67.81\,\mathrm{kN}$$

Since the plastic limit force for truss number 2 and 4 is $F_{pl,lim1}$ = 47.2 kN this elastic behaviour for this second load step with the described statical system ended at load level q_2 (see also Fig. 7.4.4.4):

$$\frac{q_2}{q}=\frac{F_{pl,lim1}-\frac{q_1}{q}S_2'}{S_2''}=\frac{47.2-\frac{15.74}{20}46.2}{67.81}=0.1599$$

$$q_2=0.1599\,q=3.198\,\mathrm{kN/m}$$

On this load level the normal force in truss number 2 and 4:

$$S_2 = S_4 = \frac{q_1}{q} S_2' + \frac{q_2}{q} S_2'' = 47.20\,\text{kN}$$

The remaining load is:

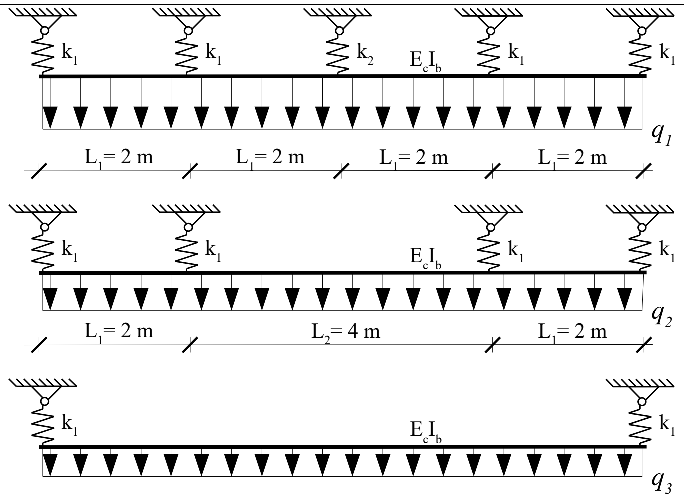
$$q_3 = q - q_1 - q_2 = 1.062\,\text{kN/m}$$

This load is applied now on a new statical system too (see Fig. 7.4.4.4).

$$S_1''' = \frac{q L_b}{2} = 80\,\text{kN}$$

Thus the final normal force in truss number 1:

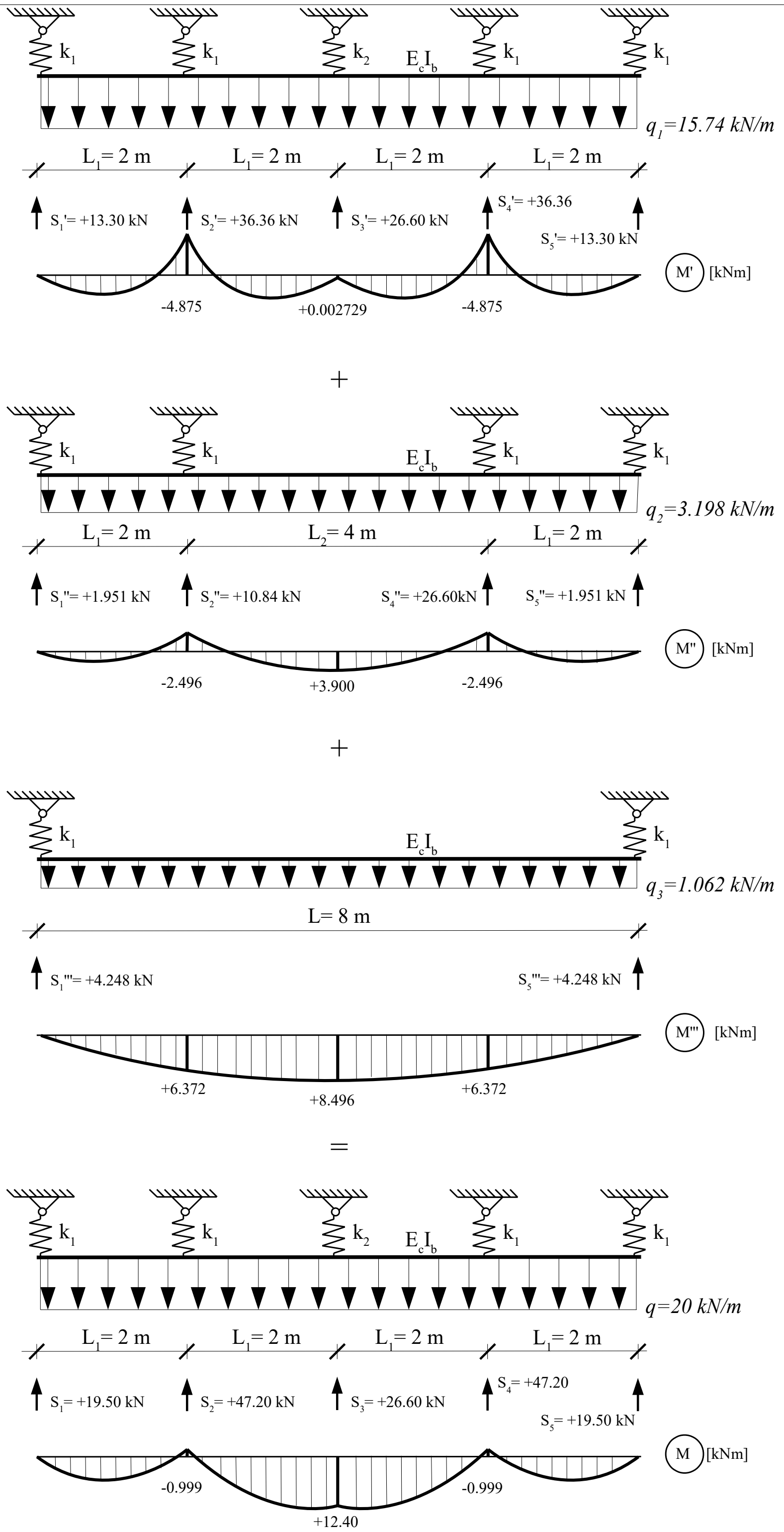
$$S_1 = \frac{q_1}{q} S_1' + \frac{q_2}{q} S_1'' + \frac{q_3}{q} S_1''' = 19.50\,\text{kN}$$



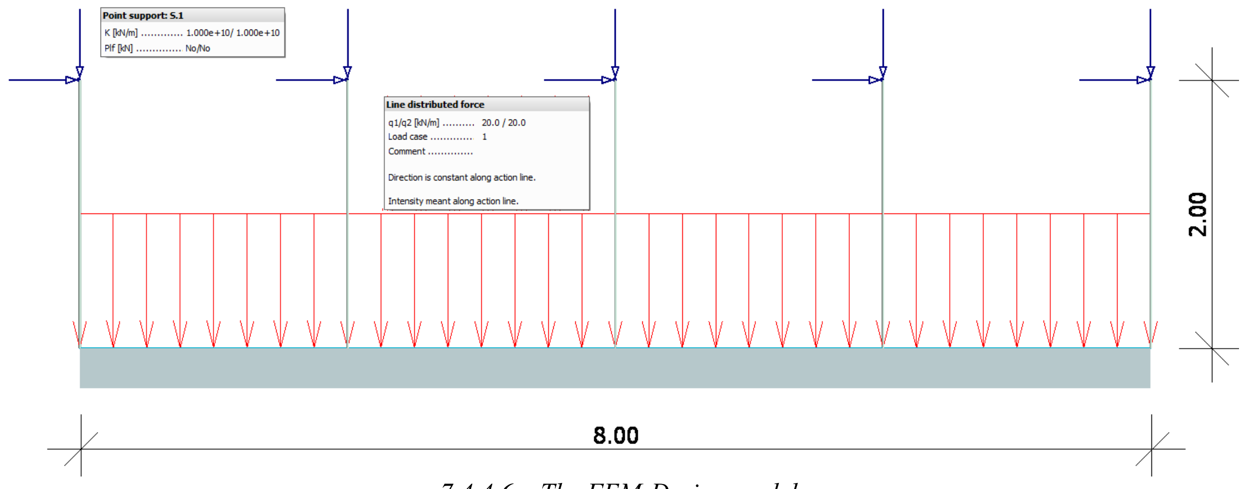
7.4.4.4 – The statical systems of the structure at different load steps (stage 1, stage 2 and stage 3)

Figure 7.4.4.5 shows the truss normal forces and the internal forces for the different load levels with the different statical system. Fig. 7.4.4.5 also shows the final results of the problem (truss normal forces and internal forces).

After the hand calculation we provide the FEM-Design calculation applying these plastic limit forces for the trusses. Fig. 7.4.4.6 shows the FEM-Design model with the geometry and with the main modeling issues.

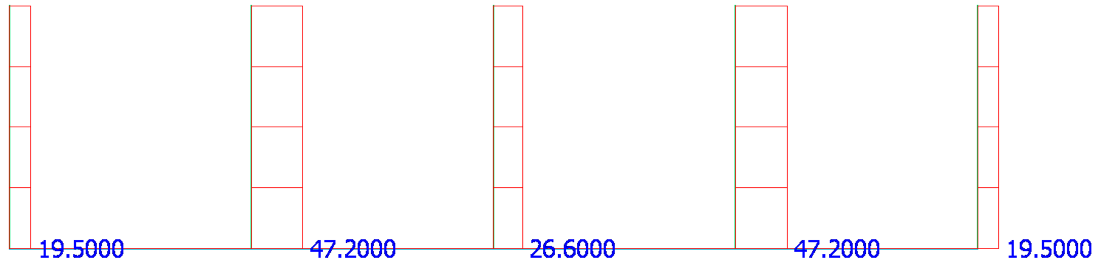


7.4.4.5 – The superposition of the different load level results and the final truss forces and internal forces for the problem

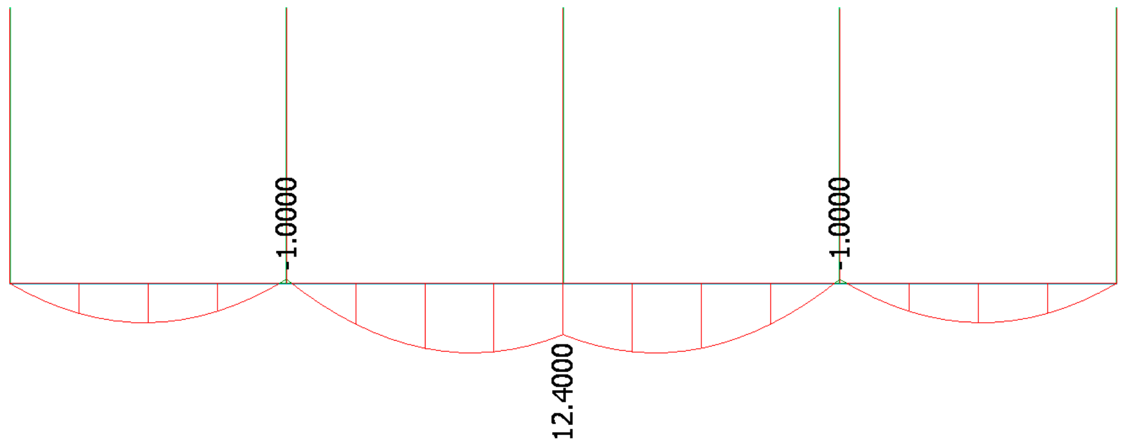


7.4.4.6 – The FEM-Design model

After the nonlinear plastic calculation Fig. 7.4.4.7 shows the normal forces in the trusses. Fig. 7.4.4.8 shows the final moment diagram. The last converged equilibrium load level is at **100%** of the total load, thus the equilibrium exists for the total load level.



7.4.4.7 – The normal forces in the trusses according to the plastic calculation [kN]



7.4.4.8 – The bending moment distribution in the beam according to the plastic calculation [kNm]

We can say that the hand calculation and the FEM calculation results are **identical**!

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.4 Elasto-plastic trusses in a multispan continuous beam.str

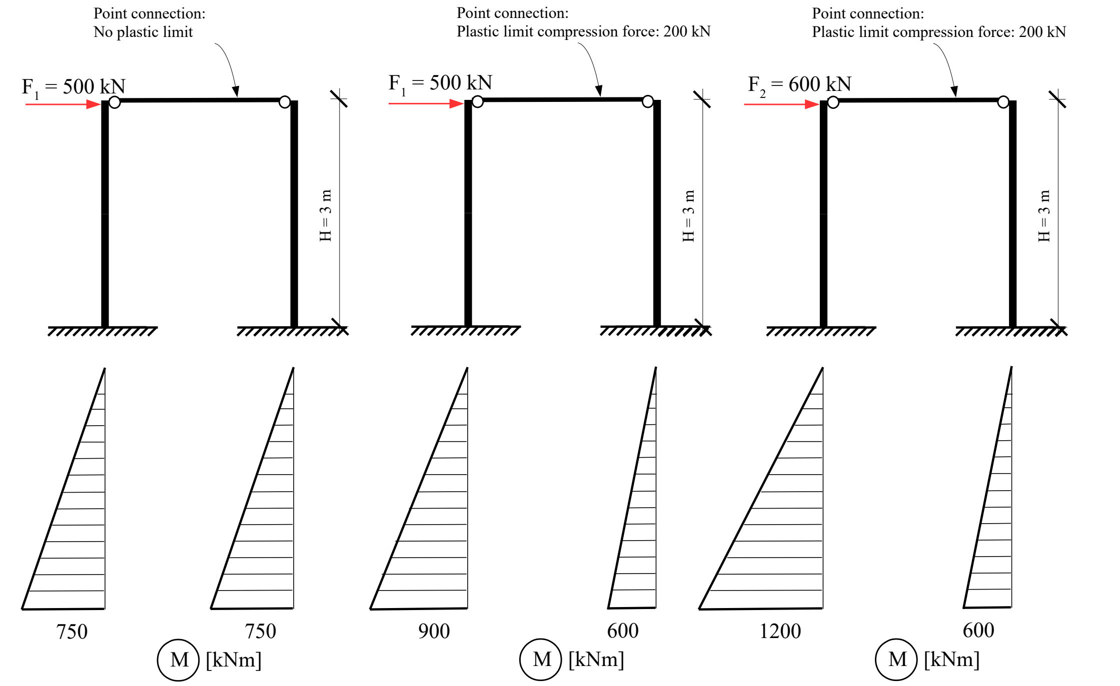
7.4.5 Elasto-plastic point-point connection between cantilevers

Inputs

Column height	H = 3 m
Point load	F_1 = 500 kN or F_2 = 600 kN
Structural steel	S 235
Young's and shear moduli of steel	E_s = 210 GPa, G = 80.77 GPa
Cross-section (both vertical columns)	HEA 600
Cross-sectional area	A_c = 22646 mm^2
Relevant inertia	$I_{c,1}$ = 0.001412 m^4
Relevant shear correction factor	$\rho_{c,2}$ = 0.3314
Point connection with plastic compression limit	N_{pl} = 200 kN

We have two vertical columns (HEA 600, S235) with fixed bottom and hinges at the top ends (see Fig. 7.4.5.1). We applied a horizontal point load (with different intensity, see Fig. 7.4.5.1) at the top of the left column. The tops of the columns are connected to each other with a point-point connection with a plastic compression limit N_{pl} = 200 kN.

The bending moment distributions in the three different cases (see Fig. 7.4.5.1) according to the same bending stiffnesses and the plastic compression limit in the point-point connection are trivial.



7.4.5.1 – The three different problems and the different bending moment distributions in the columns

If there is no limit compression force in the point-point connection the distribution of the point load is equal on both top ends of the columns:

$$M_1^{(1)} = F_1/2 \cdot H = 500/2 \cdot 3 = 750\,\mathrm{kNm}$$

$$M_2^{(1)} = F_1/2 \cdot H = 500/2 \cdot 3 = 750\,\mathrm{kNm}$$

$$M_1^{(2)} = (F_1 - N_{pl})\,H = (500-200)\,3 = 900\,\mathrm{kNm}$$

$$M_2^{(2)} = N_{pl} \cdot H = 200 \cdot 3 = 600\,\mathrm{kNm}$$

$$M_1^{(3)} = (F_2 - N_{pl})\,H = (600-200)\,3 = 1200\,\mathrm{kNm}$$

$$M_2^{(3)} = N_{pl} \cdot H = 200 \cdot 3 = 600\,\mathrm{kNm}$$

The displacements of the top points of the columns:

$$e_1^{(1)} = \frac{(F_1/2)\cdot H^3}{3E_s I_c} + \frac{F_1/2 \cdot H}{\rho_{c,2} G A_c} = \frac{(500/2)\cdot 3^3}{3\cdot 210\cdot 10^6 \cdot 0.001412} + \frac{500/2\cdot 3}{0.3314\cdot 80.77\cdot 10^6\cdot 2.2646\cdot 10^{-2}} = 8.83\,\mathrm{mm}$$

$$e_2^{(1)} = \frac{(F_1/2)\cdot H^3}{3E_s I_c} + \frac{F_1/2 \cdot H}{\rho_{c,2} G A_c} = \frac{(500/2)\cdot 3^3}{3\cdot 210\cdot 10^6 \cdot 0.001412} + \frac{500/2\cdot 3}{0.3314\cdot 80.77\cdot 10^6\cdot 2.2646\cdot 10^{-2}} = 8.83\,\mathrm{mm}$$

$$e_1^{(2)} = \frac{(F_1 - N_{pl})\cdot H^3}{3E_s I_c} + \frac{(F_1 - N_{pl})\cdot H}{\rho_{c,2} G A_c} = \frac{(500-200)\cdot 3^3}{3\cdot 210\cdot 10^6 \cdot 0.001412} + \frac{(500-200)\cdot 3}{0.3314\cdot 80.77\cdot 10^6\cdot 2.2646\cdot 10^{-2}} = \\ = 10.59\,\mathrm{mm}$$

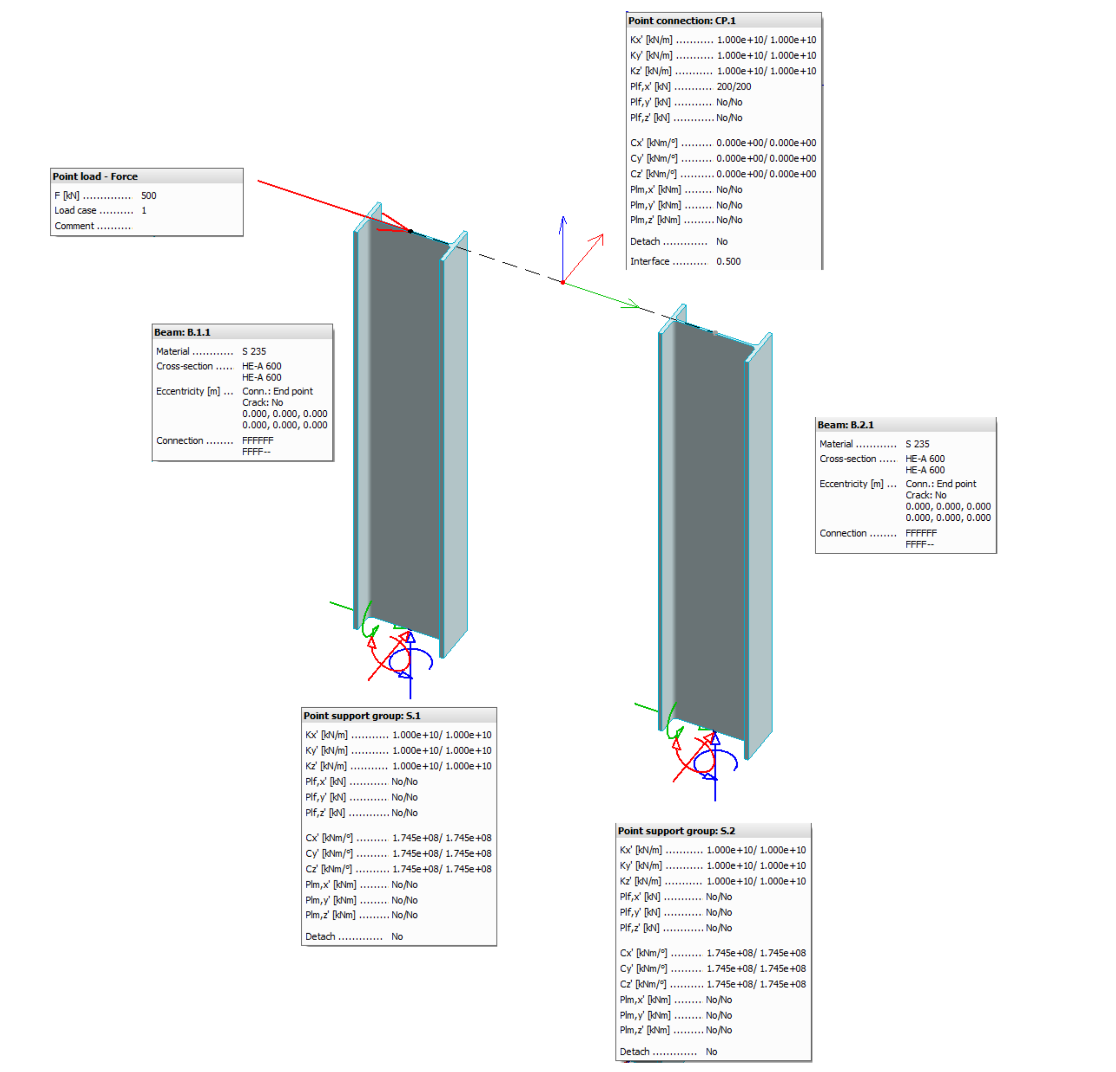
$$e_2^{(2)} = \frac{N_{pl}\cdot H^3}{3E_s I_c} + \frac{N_{pl}\cdot H}{\rho_{c,2} G A_c} = \frac{200\cdot 3^3}{3\cdot 210\cdot 10^6 \cdot 0.001412} + \frac{200\cdot 3}{0.3314\cdot 80.77\cdot 10^6\cdot 2.2646\cdot 10^{-2}} = 7.06\,\mathrm{mm}$$

$$e_1^{(3)} = \frac{(F_2 - N_{pl})\cdot H^3}{3E_s I_c} + \frac{(F_2 - N_{pl})\cdot H}{\rho_{c,2} G A_c} = \frac{(600-200)\cdot 3^3}{3\cdot 210\cdot 10^6 \cdot 0.001412} + \frac{(600-200)\cdot 3}{0.3314\cdot 80.77\cdot 10^6\cdot 2.2646\cdot 10^{-2}} = \\ = 14.12\,\mathrm{mm}$$

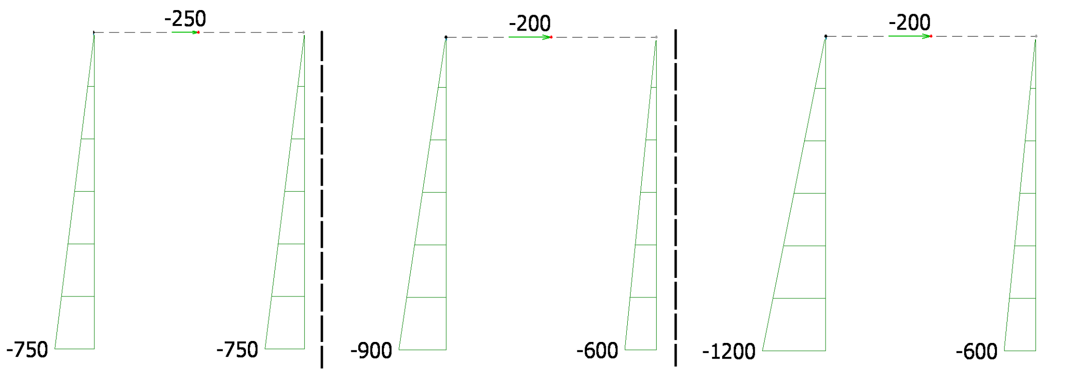
$$e_2^{(3)} = \frac{N_{pl}\cdot H^3}{3E_s I_c} + \frac{N_{pl}\cdot H}{\rho_{c,2} G A_c} = \frac{200\cdot 3^3}{3\cdot 210\cdot 10^6 \cdot 0.001412} + \frac{200\cdot 3}{0.3314\cdot 80.77\cdot 10^6\cdot 2.2646\cdot 10^{-2}} = 7.06\,\mathrm{mm}$$

Fig. 7.4.5.2 shows the FEM-Design model and their input parameters. The results of the FE calculations according to the three different cases are in Fig. 7.4.5.3 and 7.4.5.4 (see also Fig. 7.4.5.1).

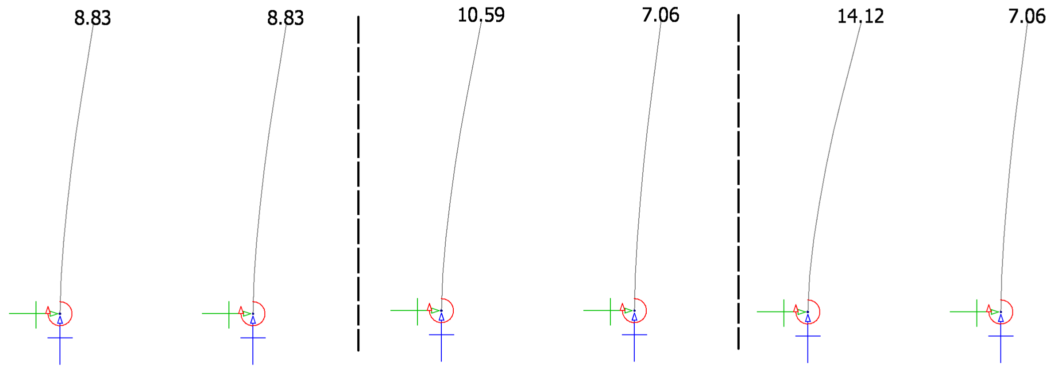
The bending moment diagrams and the displacement values of the top of the columns are identical with the hand calculation.



7.4.5.2 – The FEM-design model of the problem with the inputs



7.4.5.3 – The bending moment diagrams of the FEM solution for the three different cases
bending moments [kNm], *connection force* [kN]



7.4.5.4 – The displacement diagrams of the FEM solution for the three different cases [mm]

Download link to the example file:

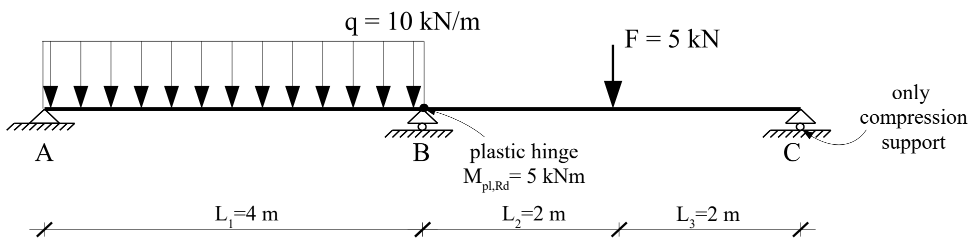
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.5 Elasto-plastic point-point connection between cantilevers.str

7.4.6 Elasto-plastic point-point connection with uplift in a multispan continuous beam

Inputs:

Beam length	$L_1 = 4$ m; $L_2 = L_3 = 2$ m
Partial distributed load	q = 10 kN/m
Point load	F = 5 kN
Concrete (beam, linear elastic material model)	C 25/30
Young's modulus of concrete	$E_c = 31$ GPa
Cross-sectional area (rectangle)	b = 120 mm; h = 150 mm
Inertia	$I_b = 0.00003375$ m^4
Plastic hinge above support *B* (point-point connection)	$M_{pl,Rd} = 5$ kNm
C support only bears compression	*uplift*

A multispan continuous beam, a plastic hinge with limit moment capacity (as a point-point connection) above support B and only compression resistance at support C with different types of loads are given in Fig. 7.4.6.1.



7.4.6.1 – Multispan continuous beam with plastic hinge above support B and uplift at support C

First of all we make a hand calculation and then make a FEM-Design plastic calculation. At the end of this chapter these two results will be compared.

According to the uplift and plastic behaviour we need to analyze and calculate the results in load steps. The first load step q_1, F_1 are at a certain load level where the plastic hinge reaches its limit moment value. On this load level there will be zero reaction force in support C, because this support only can bear compression.

The 50% of the total load level: $q_1 = 5$ kN/m, $F_1 = 2.5$ kN.

At this load level the specific bending moment values (see also Fig. 7.4.6.2):

$M_1^B = F_1 L_2 = 2.5 \cdot 2 = 5\,\text{kNm}$ this is the plastic limit moment capacity above support B.

$$M_1^{ABmid} = -\frac{F_1 L_2}{2} + \frac{q_1 L_1^2}{8} = -\frac{2.5 \cdot 2}{2} + \frac{5 \cdot 4^2}{8} = 7.50\,\text{kNm}$$

$$M_1^{BCmid} = 0\,\text{kNm}$$

At this load level the specific reaction values (see also Fig. 7.4.6.2):

$$A_1=\frac{q_1\frac{L_1^2}{2}-F_1L_2}{L_1}=\frac{5\frac{4^2}{2}-2.5\cdot 2}{4}=8.75\,\text{kN}$$

$$B_1=\frac{q_1\frac{L_1^2}{2}+F_1(L_1+L_2)}{L_1}=13.75\,\text{kN}$$

$$C_1=0\,\text{kN}$$

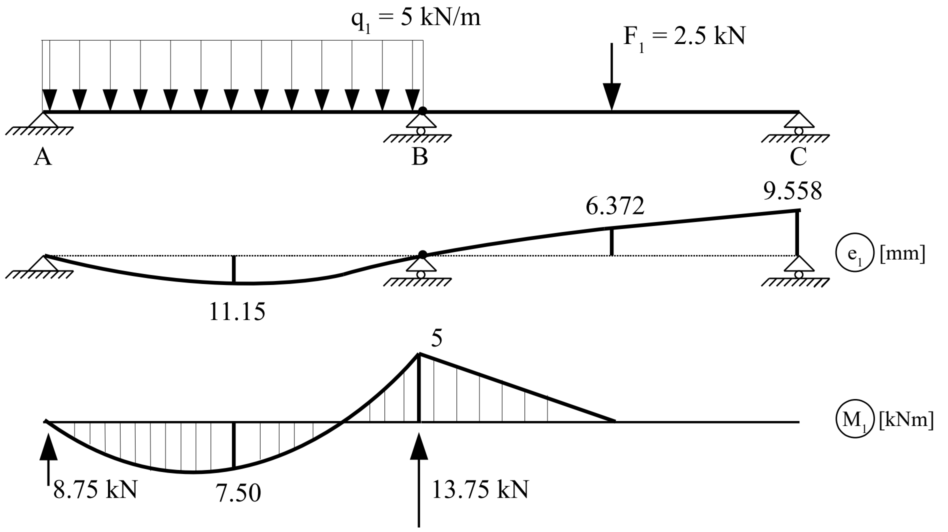
At this load level the specific deflection values according to virtual force method (see also Fig. 7.4.6.2):

$$e_1^{ABmid}=\frac{1}{E_cI_b}\left[-\frac{L_1}{4}\frac{L_1M_1^B}{2\cdot 2}+2\left(\frac{q_1L_1^2}{8}\frac{L_1}{2}\frac{2}{3}\frac{5}{8}\frac{L_1}{4}\right)\right]=11.15\,\text{mm}$$

$$e_1^{BCmid}=\frac{1}{E_cI_b}\left[\frac{L_2}{2}\frac{L_2}{2}\frac{2}{3}M_1^B+\frac{M_1^BL_1}{2}\frac{2}{3}L_2-\frac{q_1L_1^2}{8}L_1\frac{2}{3}\frac{L_2}{2}\right]=-6.372\,\text{mm}$$

$$e_1^C=\frac{1}{E_cI_b}\left[M_1^B\frac{L_2}{2}\left(\frac{L_2+L_3}{2}+\frac{2}{3}\frac{L_2+L_3}{2}\right)+\frac{(L_2+L_3)^2}{2}\frac{2}{3}M_1^B-\frac{q_1L_1^2}{8}L_1\frac{2}{3}\frac{L_2+L_3}{2}\right]=-9.558\,\text{mm}$$

After this load step the remaining loads (q_2, F_2) are acting on a different statical system (see Fig. 7.4.6.3). The second 50% part of the total load level: $q_2 = 5$ kN/m, $F_2 = 2.5$ kN are acting on separate simply supported beams.



7.4.6.2 – The deflection and the bending moment diagram at the first load step (50%) which causes $M_{pl,Rd}$ moment above support B, and uplift at support C

At this load step the specific bending moment values (see also Fig. 7.4.6.3):

$$M_2^{ABmid}=\frac{q_2 L_1^2}{8}=\frac{5\cdot 4^2}{8}=10.0\,\text{kNm}$$

$$M_2^{BCmid}=F_2\frac{L_2+L_3}{4}=2.5\frac{2+2}{4}=2.5\,\text{kNm}$$

At this load step the specific reaction values (see also Fig. 7.4.6.3):

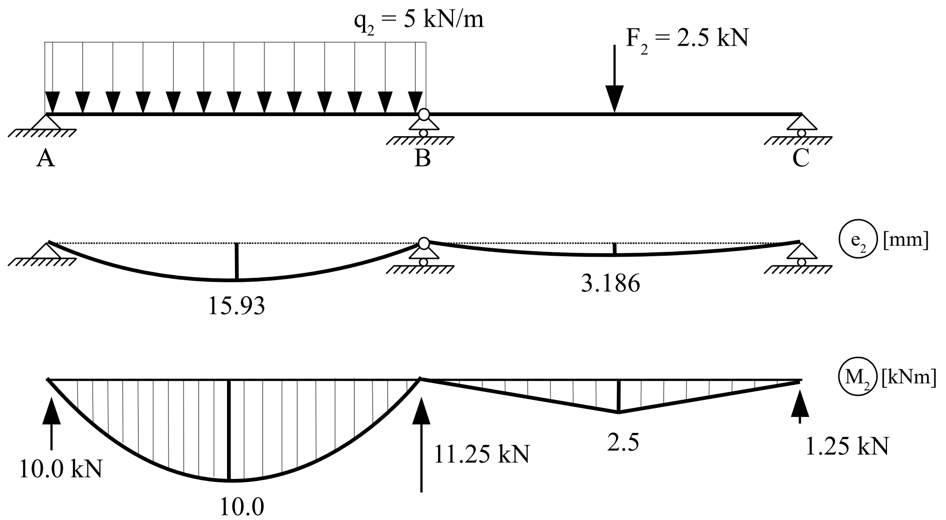
$$A_2=q_2\frac{L_1}{2}=5\frac{4}{2}=10.0\,\text{kN}$$

$$B_2=q_2\frac{L_1}{2}+\frac{F_2}{2}=5\frac{4}{2}+\frac{2.5}{2}=11.25\,\text{kN}$$

$$C_2=\frac{F_2}{2}=\frac{2.5}{2}=1.25\,\text{kN}$$

At this load step the specific deflection values according to virtual force method (see also Fig. 7.4.6.3):

$$e_2^{ABmid}=\frac{5\,q_2 L_1^4}{384\,E_c I_b}=15.93\,\text{mm} \quad ; \quad e_2^{BCmid}=\frac{F_2(L_2+L_3)^3}{48\,E_c I_b}=3.186\,\text{mm}$$



7.4.6.3 – Second load step (50%) which is acting on a different statical system with the plastic hinge above support B and compression in support C

The final results come from the superposition of the two former calculated cases.

The final specific bending moment values (see also Fig. 7.4.6.4):

$$M^B=M_1^B=5\,\text{kNm}$$

$$M^{ABmid}=M_1^{ABmid}+M_2^{ABmid}=7.50+10.0=17.5\,\text{kNm}$$

$$M^{BCmid} = M_1^{BCmid} + M_2^{BCmid} = 0 + 2.5 = 2.5\,\mathrm{kNm}$$

The final specific reaction values (see also Fig. 7.4.6.4):

$$A = A_1 + A_2 = 8.75 + 10.0 = 18.75\,\mathrm{kN}$$

$$B = B_1 + B_2 = 13.75 + 11.25 = 25.0\,\mathrm{kN}$$

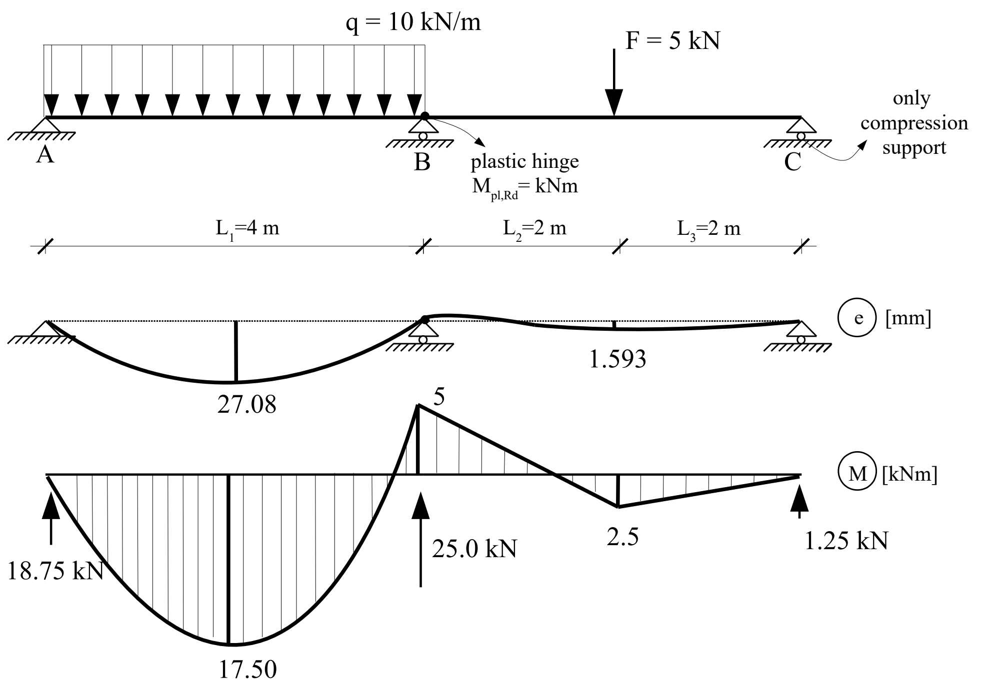
$$C = C_1 + C_2 = 0 + 1.25 = 1.25\,\mathrm{kN}$$

The final specific deflection values (see also Fig. 7.4.6.4):

$$e^{ABmid} = e_1^{ABmid} + e_2^{ABmid} = 11.15 + 15.93 = 27.08\,\mathrm{mm}$$

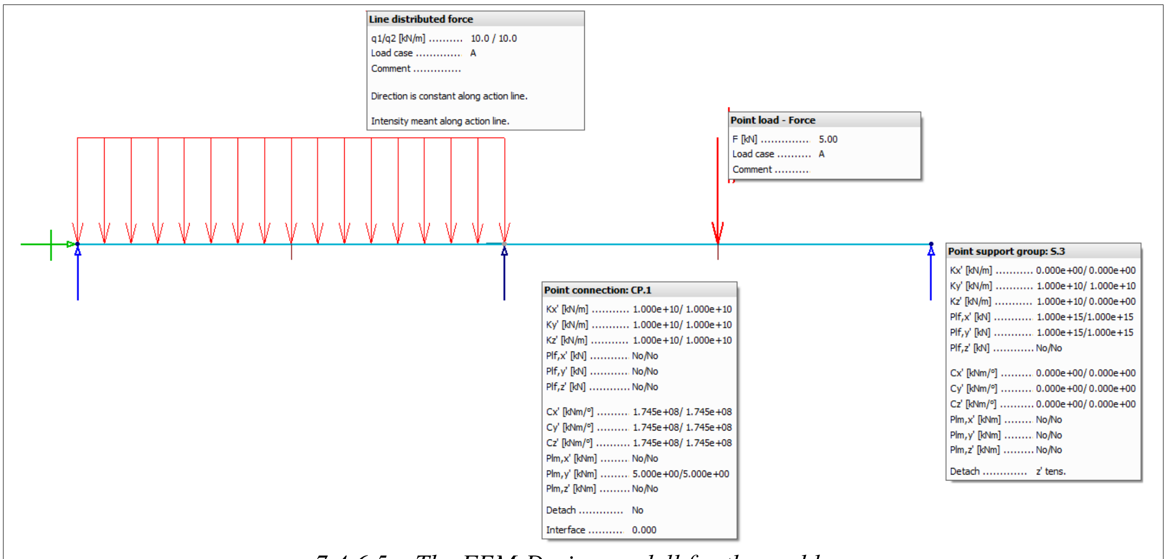
At the final deflection calculation of the middle point of BC span we need to consider the fact that above support B a plastic hinge occurred and hence in support C compression is arising in the second load step.

$$e^{BCmid} = e_1^{BCmid} - \frac{e_1^C}{2} + e_2^{BCmid} = -6.372 - \frac{(-9.558)}{2} + 3.186 = 1.593\,\mathrm{mm}$$



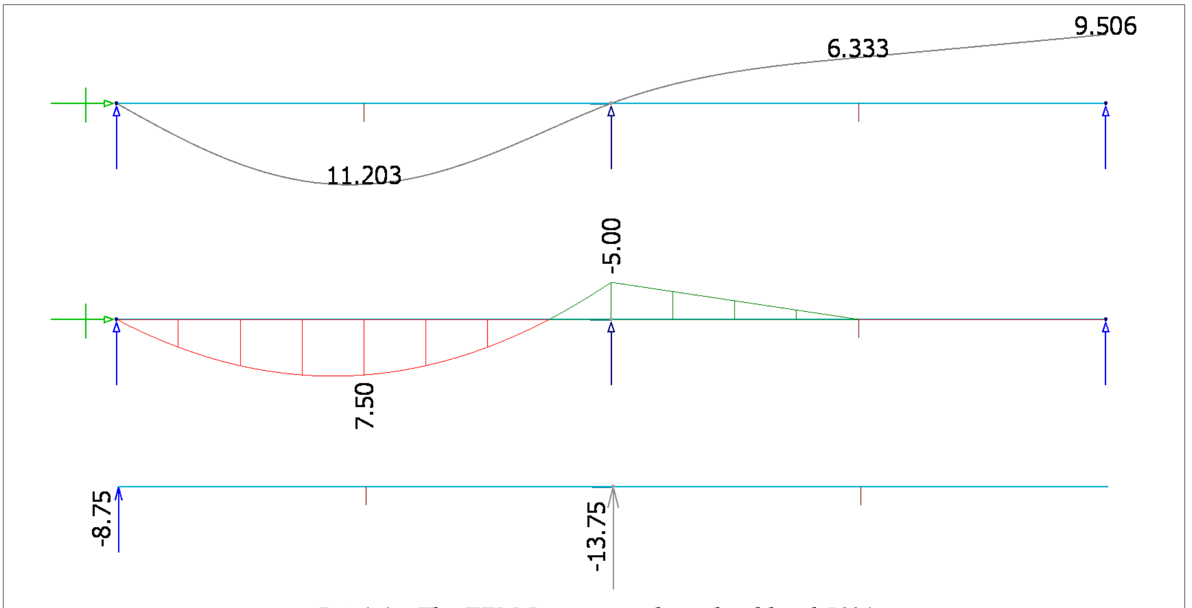
7.4.6.4 – The final deflection and the final bending moment diagram with the reactions

After the hand calculation we modelled this problem in FEM-Design. We adjusted a point-point connection with plastic limit moment value above support B (see Fig. 7.4.6.5) and uplift at support C (see Fig. 7.4.6.5).



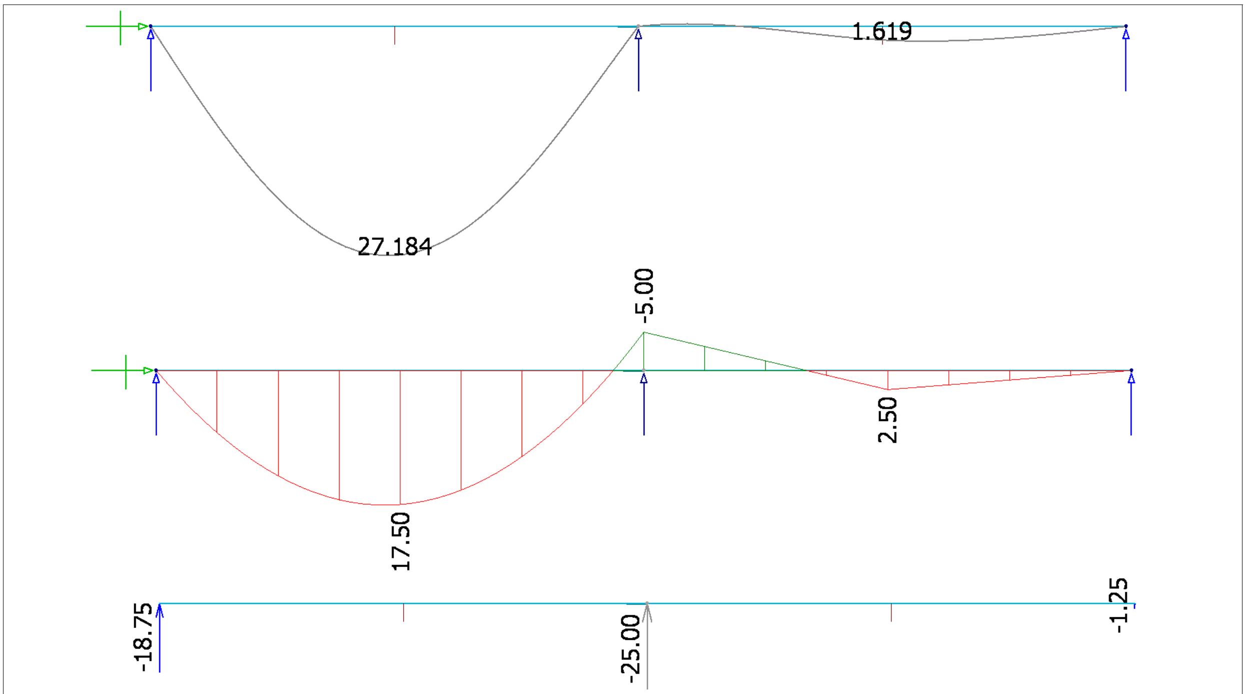
7.4.6.5 – The FEM-Design modell for the problem

Fig. 7.4.6.6 shows the reactions, bending moments and deflection values at load level **50%**. The differences between hand and FEM calculations are **less than 1%**.



7.4.6.6 – The FEM-Design results at load level 50%

Fig. 7.4.6.7 shows the reactions, bending moments and deflection values at load level **100%**. The differences between hand and FEM calculations are **less than 1%**.



7.4.6.7 – The FEM-Design results at load level 100%

Download link to th example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.6 Elasto-plastic point-point connection with uplift in a multispan continuous beam.str

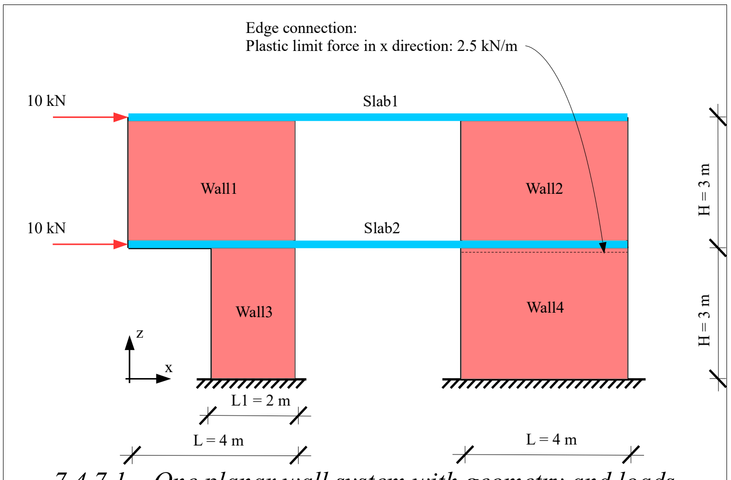
7.4.7 Elasto-plastic edge connections in a building braced by shear walls

Inputs:

Geometry	Fig. 7.4.7.1
Horizontal loads (on one storey, on one wall plane)	V = 10 kN
Concrete (walls and slabs)	C 25/30
Young's modulus of concrete	E_c = 31 GPa
Thickness of the walls/slabs	t = 20 cm
Plastic edge connection shear force limit (above Wall4)	v_{pl} = 2.5 kN/m
Edge connection behaviour	Without detach

The shear wall problem is given in Fig. 7.4.7.1. The given loads are only two concentrated horizontal loads on the two storeys (see Fig. 7.4.7.1). The top of Wall4 only can bear v_{pl} = 2.5 kN/m plastic limit force.

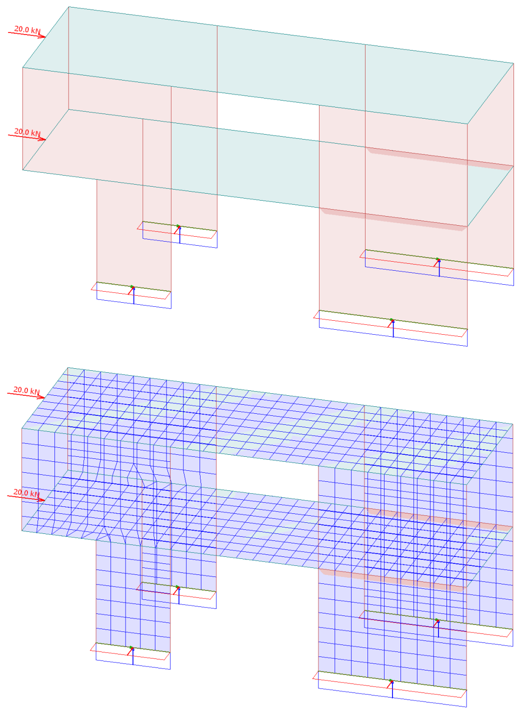
We defined the problem and the model in FEM-Design (7.4.7.2).



7.4.7.1 – One planar wall system with geometry and loads

If all of the elements behaviour are linear elastic then the solution of this problem based on the finite element method (according to the mechanical model of the elements) is “exact”.

Fig. 7.4.7.2 shows the finite element surface mesh of the problem.



7.4.7.2 – The model with the equivalent loads, supports and the plastic edge connections

Fig. 7.4.7.3 shows the resultants of the reactions and the edge connection forces/moments according to the linear elastic calculation.

We can see that the horizontal shear force in the edge connections (above Wall4, see Fig. 7.4.7.1 and 7.4.7.3) is:

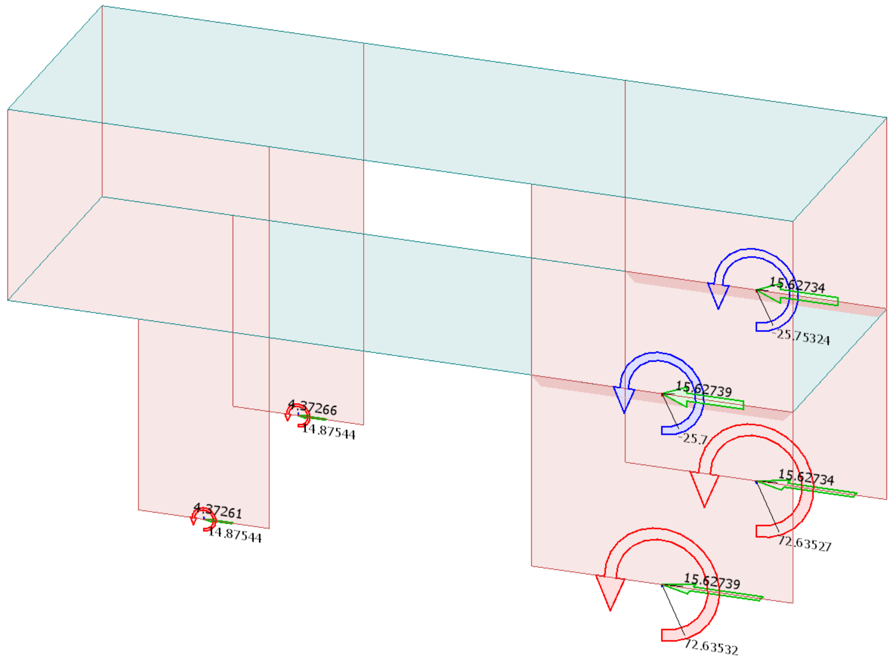
$$V_{el} = 15.6\,\text{kN}$$

The ultimate plastic resultant shear force bearing capacity of the edge connection above Wall4 is:

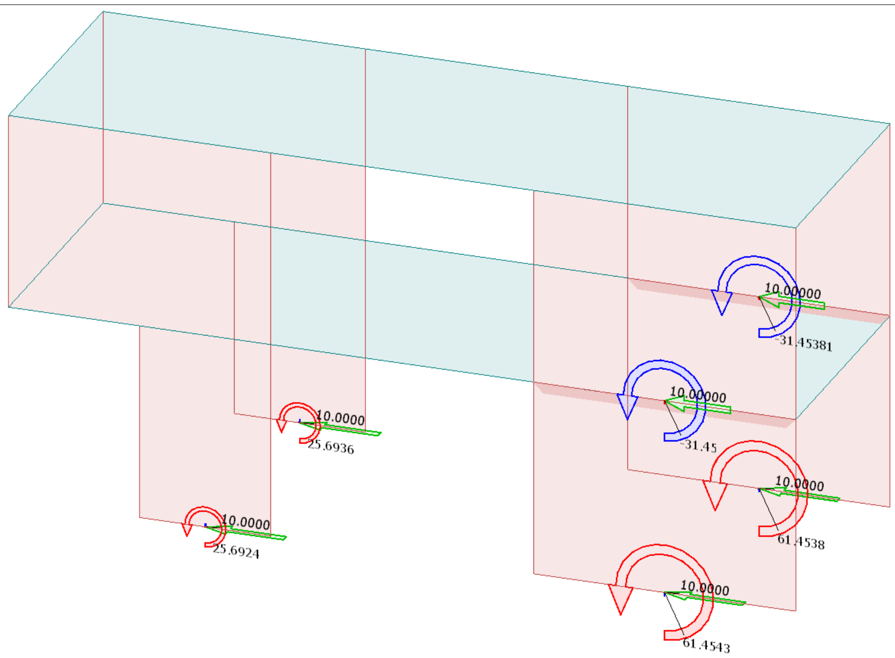
$$V_{ult} = v_{pl} L = 2.5 \cdot 4 = 10.0\,\text{kN}$$

Fig. 7.4.7.4 shows the resultants of the reactions and the edge connection forces/moments according to the elasto-plastic calculation. The edge connection resultant shear force based on the FEM-Design calculation (Fig. 7.4.7.4):

$$V_{ult}^{\text{FEM}} = 10.0\,\text{kN}$$



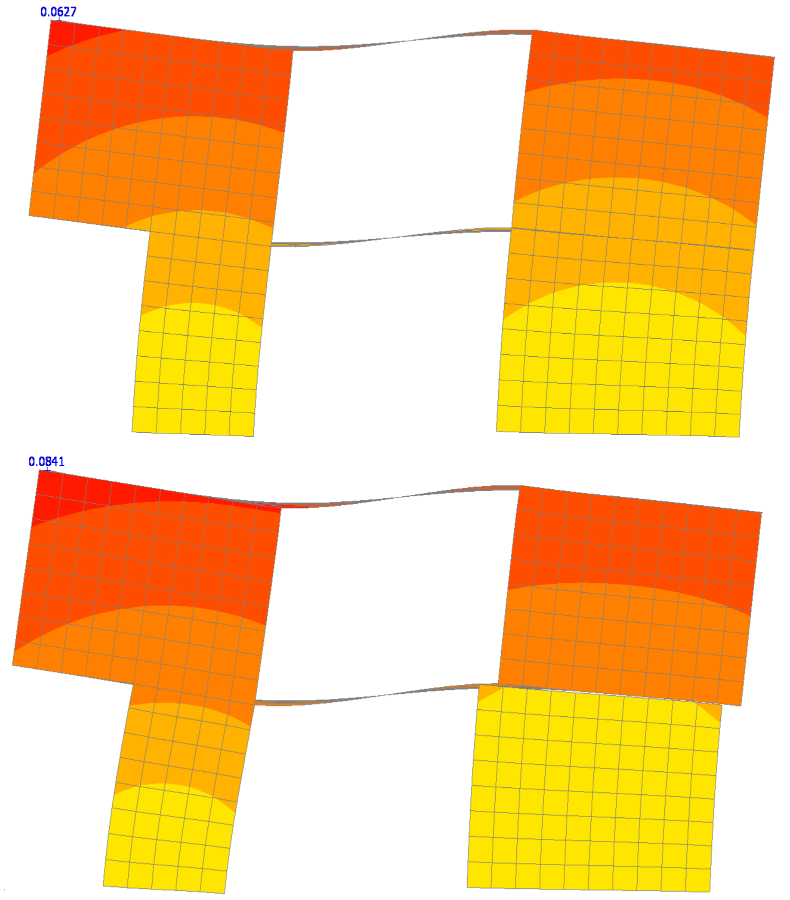
7.4.7.3 – The resultants of the reactions and the edge connections according to linear elastic calculation (forces [kN], *moments* [kNm])



7.4.7.4 – The resultants of the reactions and the edge connections according to elasto-plastic calculation (forces [kN], *moments* [kNm])

It is obvious that according to the elasto-plastic calculation due to the plastic deformations on the edge connections the final displacements need to be greater than the displacements of the linear elastic calculation.

Fig. 7.4.7.5 shows the displacements of the linear elastic (above) and the elasto-plastic (below) calculations.



7.4.7.5 – The displacements according to linear elastic (above) *and elasto-plastic* (below) *calculation* [mm]

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.7 Elasto-plastic edge connections in a building braced by shear walls.str

7.4.8 Elasto-plastic edge connections with detach in a shear wall

Inputs:

Geometry	Fig. 7.4.8.1
Vertical distributed load	q_V = 50 kN/m (N = 200 kN)
Horizontal distributed load intermediate levels	q_{H2} = 15 kN/m (V = 60 kN)
Horizontal distributed load top level	q_{H1} = 7.5 kN/m (V/2 = 30 kN)
Concrete (wall)	C 25/30
Young's modulus of concrete	E_c = 31 GPa
Thickness of the wall	t = 20 cm
Plastic edge connection shear force limit	$v_{pl,1}$ = 20 kN/m
Plastic support shear force limit	$v_{pl,2}$ = 50 kN/m
Edge connection / Fixed support behaviour	Detach in vertical direction

Fig. 7.4.8.1 shows the shear wall problem with the geometry, external loads and the behaviour of the edge connections and the support.

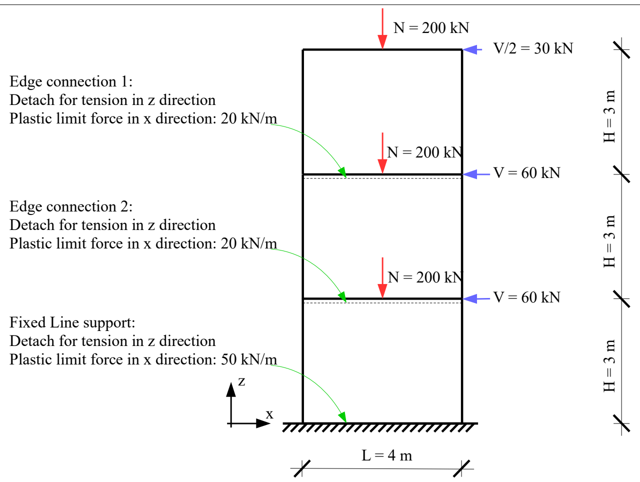


Figure 7.4.8.1 – The properties of the supports and edge connections of a shear wall

According to the loads and the geometry the internal forces and the eccentricities can be calculated (see Fig. 7.4.8.2).

$$e_1=\frac{M_1}{N_1}=\frac{90}{200}=0.45\,\mathrm{m} \quad ; \quad e_2=\frac{M_2}{N_2}=\frac{360}{400}=0.90\,\mathrm{m} \quad ; \quad e_3=\frac{M_3}{N_3}=\frac{810}{600}=1.35\,\mathrm{m}$$

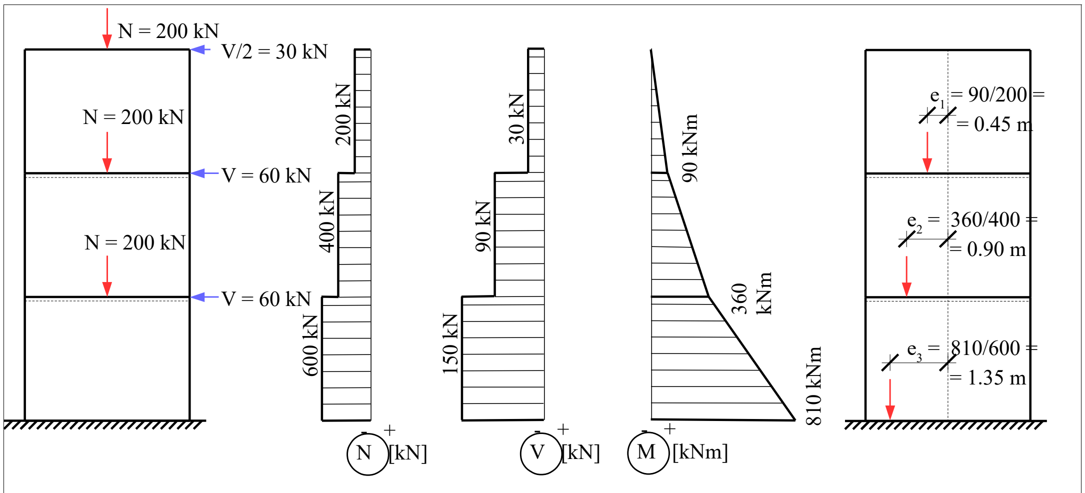


Figure 7.4.8.2 – The internal force distribution and the eccentricities of the normal forces

The eccentric normal force at the top level (see Fig. 7.4.8.2) is inside the Culmann's kernel therefore the specific normal force distribution can be calculated with the following equations and approximations:

$$n_{z1}^{+}=\frac{N_1}{A}t+\frac{M_1}{I}x_{\max}t=\frac{-200}{0.2\cdot 4}\cdot 0.2+\frac{90}{0.2\cdot 4^3/12}2\cdot 0.2=-16.25\,\mathrm{kN/m}$$

$$n_{z1}^{-}=\frac{N_1}{A}t-\frac{M_1}{I}x_{\max}t=\frac{-200}{0.2\cdot 4}\cdot 0.2-\frac{90}{0.2\cdot 4^3/12}2\cdot 0.2=-83.75\,\mathrm{kN/m}$$

The eccentric normal forces at the intermediate levels (see Fig. 7.4.8.2) are outside the Culmann's kernel therefore the specific normal force distribution can be calculated with the following equations and approximations (based on the resultant of the stress volume):

$$3\left(\frac{L}{2}-e_2\right)\frac{1}{2}n_{z2}^{-}=N_2 \qquad n_{z2}^{-}=\frac{N_2}{3\left(\frac{L}{2}-e_2\right)\frac{1}{2}}=\frac{-400}{3\left(\frac{4}{2}-0.9\right)\frac{1}{2}}=-242.4\,\mathrm{kN/m}$$

$$3\left(\frac{L}{2}-e_3\right)\frac{1}{2}n_{z3}^{-}=N_3 \qquad n_{z3}^{-}=\frac{N_3}{3\left(\frac{L}{2}-e_3\right)\frac{1}{2}}=\frac{-600}{3\left(\frac{4}{2}-1.35\right)\frac{1}{2}}=-615.4\,\mathrm{kN/m}$$

The average shear forces at the different levels:

$$n_{xz1,average} = \frac{V_1}{A} t = \frac{30}{0.2 \cdot 4} 0.2 = 7.5 \text{ kN/m}$$

$$n_{xz2,average} = \frac{V_2}{A_{active2}} t = \frac{90}{0.2 \cdot 3 \left(\frac{4}{2} - 0.9 \right)} 0.2 = 27.27 \text{ kN/m}$$

$$n_{xz3,average} = \frac{V_3}{A_{active3}} t = \frac{150}{0.2 \cdot 3 \left(\frac{4}{2} - 1.35 \right)} 0.2 = 76.92 \text{ kN/m}$$

You can see these calculated values in Fig. 7.4.8.3 also.

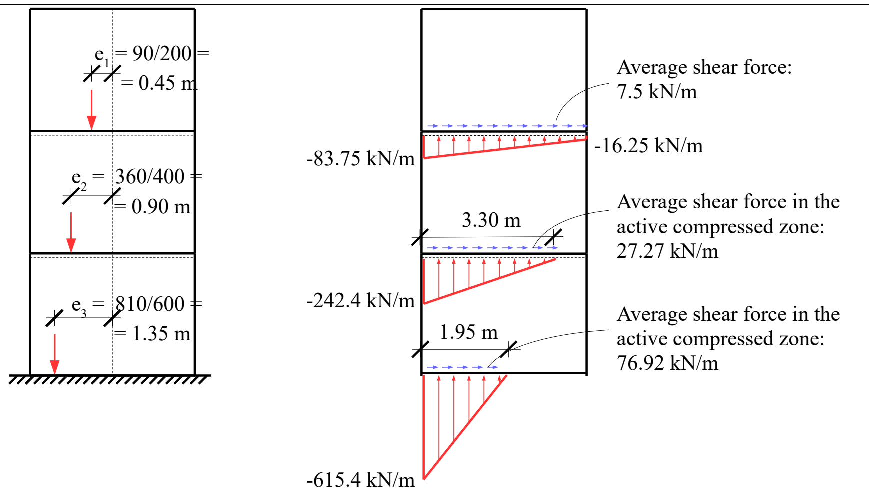


Figure 7.4.8.3 – The eccentricities and the specific normal forces and average shear forces

The plastic limit shear forces differ from each other at the edge connections and the fixed supports hence one-by-one the load-bearing capacities are (see Fig. 7.4.8.1 and 7.4.8.3):

$$\eta_1 = \frac{v_{pl,1}}{n_{xz1,average}} = \frac{20}{7.5} = 266.7\%$$

$$\eta_2 = \frac{v_{pl,1}}{n_{xz2,average}} = \frac{20}{27.27} = 73.34\%$$

$$\eta_3 = \frac{v_{pl,2}}{n_{xz3,average}} = \frac{50}{76.92} = 65.00\%$$

Thus the significant value which gives us the load-bearing capacity of the structure is the last value: **65%** of the external load.

After the hand calculation we have made a finite element calculation with FEM-Design.

The model can be seen with the adjusted parameters in Fig. 7.4.8.4 (see also the input table for the data).

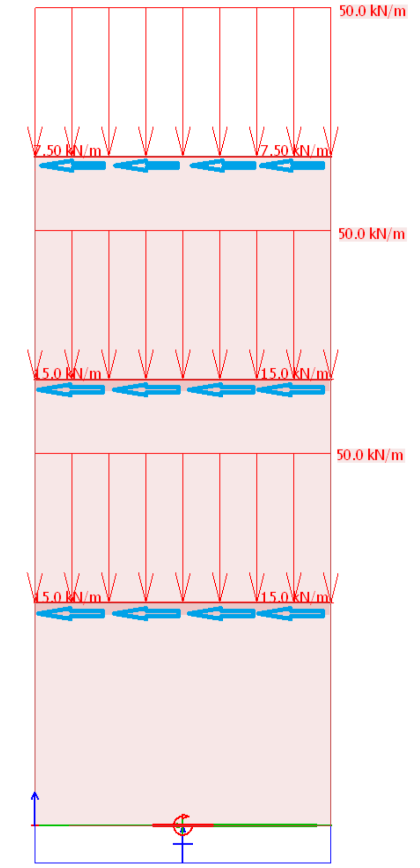


Figure 7.4.8.4 – FEM-Design model with fixed support and the vertical and horizontal forces

The first FEM-Design results can be seen in Fig. 7.4.8.5. In this case the results are based on the uplift (detach) calculation (without plastic calculation).

The results (Fig. 7.4.8.5) are in good agreement with the hand calculation. Do not forget that by the hand calculation we assumed prismatic beam behaviour by the specific normal force calculation but by the FEM calculation the behaviour is more compound.

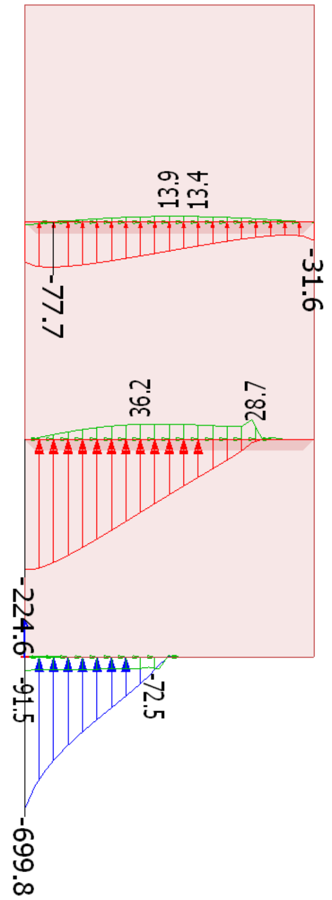


Figure 7.4.8.5 – Distribution of the reactions and connection forces with detach calculation specific normal forces [red and blue, kN/m] *and shear forces* [green lines, kN/m]

In the second calculation we considered plastic analysis with detach (uplift) in FEM-Design. According to the plastic behaviour the last converged (equilibrium) load level was:

$$\eta^{FEM} = 62\,\%$$

Fig. 7.4.8.6 shows the edge connection forces and the reactions for the last converged solution.

The difference between the hand and FEM calculation is 4.6 %.

$$\Delta = \frac{\eta_3^{HAND} - \eta^{FEM}}{\eta_3^{HAND}} = \frac{65\,\% - 62\,\%}{65\,\%} = 4.6\%$$

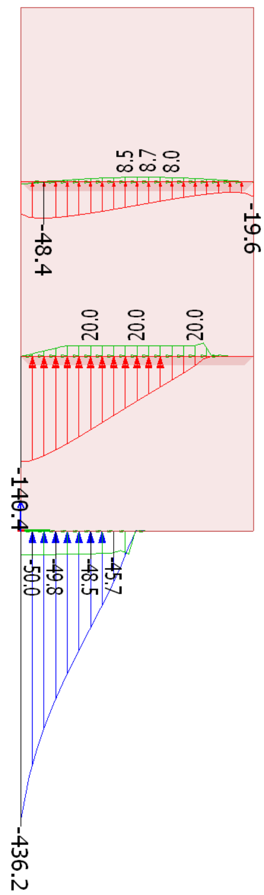


Figure 7.4.8.6 – Distribution of the reactions and connection forces with plastic detach calculation specific normal forces [red and blue, kN/m] *and shear forces* [green lines, kN/m]

Last converged (equilibrium) load level: 62 %

Download link to the example file:

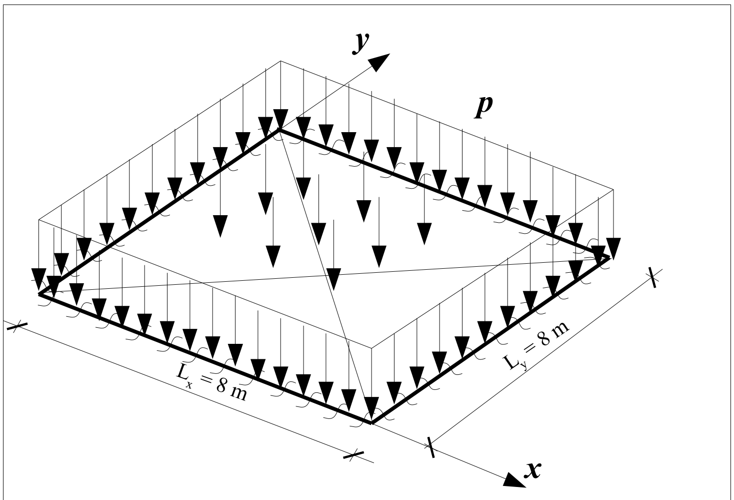
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.8 Elasto-plastic edge connections with detach in a shear wall.str

7.4.9 Elasto-plastic line-line connections in a square plate

Inputs:

Square slab	$L = L_x = L_y = 8$ m
Slab thickness	t = 20 cm
Applied distributed load	$p = 15$ kN/m^2
Concrete	C 25/30
Young's modulus of concrete	$E_c = 31$ GPa
Isotropic positive plastic moment capacity	$m^+ = m_x^+ = m_y^+ = 30$ kNm/m
Line-line connection, with plastic limit specific moment (see the adjusted local coordinate system in Fig. 7.4.9.3)	$m_{y'}^{+pl.limit} = 30$ kNm/m

In this example line-line connections with plastic limit force/moment will be used to calculate the plastic load-bearing capacity of a simly supported square plate with isotropic positive plastic moment capacity. If the slab has constant isotropic positive plastic moment capacity ($m_x^+ = m_y^+$ =30 kNm/m) the yield-line layout is known in this case [11] (see Fig. 7.4.9.1 and 7.4.9.2).



7.4.9.1 – Simply supported square slab with distributed load

If the correct (real) yield-line layout is known the plastic load-bearing capacity can be calculated according to a kinematically admissible virtual displacement field which belong to this layout. The maximum total distributed load which causes the plastic failure of this square slab comes from the equality of the external and the internal virtual work.

$$W_{int} = W_{ext}$$

The external work done by loads:

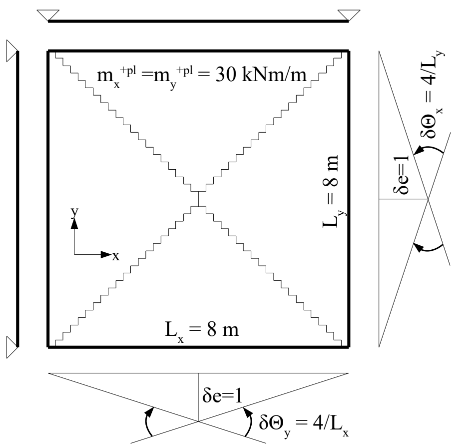
$$W_{ext} = p_{HAND}^{max} \delta V = p_{HAND}^{max} L_x L_y \frac{1}{3} 1 = p_{HAND}^{max} L L \frac{1}{3} 1$$

The internal work done by resisting moments:

$$W_{int} = m_x^{+pl} \delta \Theta_y L_y + m_y^{+pl} \delta \Theta_x L_x = m_x^{+pl} \frac{4}{L_x} L_y + m_y^{+pl} \frac{4}{L_y} L_x = 8 m^+$$

Hence the plastic load-bearing capacity:

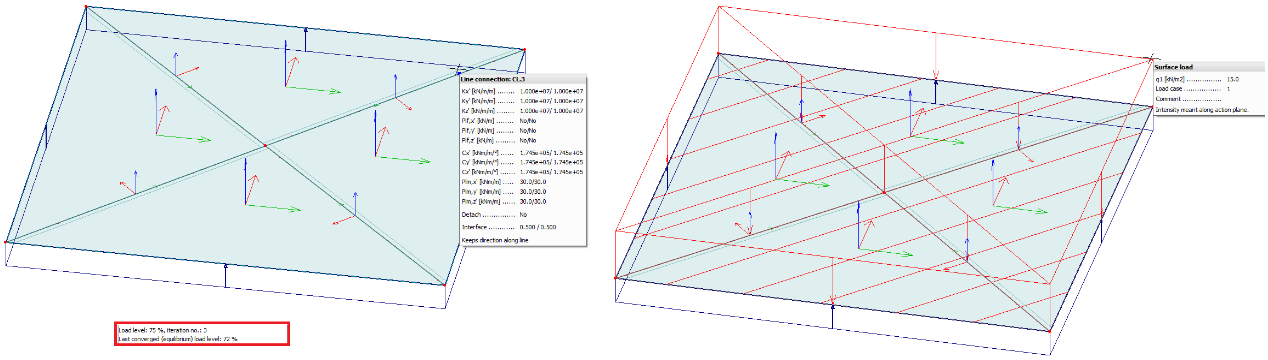
$$p_{HAND}^{max} = \frac{24 m^+}{L^2} = \frac{24 \cdot 30}{8^2} = 11.25 \frac{\text{kN}}{\text{m}^2}$$



7.4.9.2 – Simply supported slab with the virtual displacement system

In FEM-Design along the yield lines line-line connections will be applied with adjusted plastic limit capacity $m_{y'}^{+pl.limit}$= 30 kNm/m. Fig. 7.4.9.3 shows the local co-ordinate system of the line-line connections. We applied these systems because the resisting positive moments were assumed to be isotropic, hence the adjusted plastic limit capacity of the line-line connection was specified along the line (see Fig. 7.4.9.3).

Fig. 7.4.9.3 also shows the adjusted parameters, geometry and the applied p = 15 kN/m^2 uniform distributed load.



7.4.9.3 – FEM-Design modell with plastic line-line connections along the yield-lines

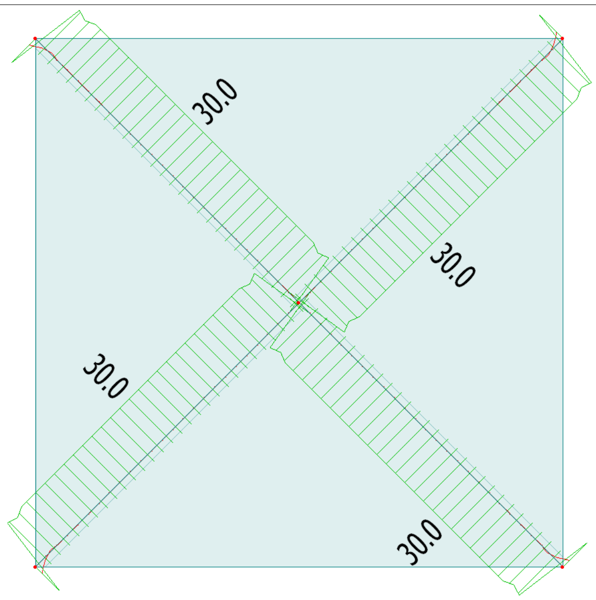
With FEM-Design the last converged equilibrium load level was at **72%**. It means that the load-bearing capacity due to the plastic calculation is:

$$p_{FEM}^{max}=0.72\cdot p=0.72\cdot 15=10.8\,\mathrm{kN/m^2}$$

The difference between the hand calculation and FEM analysis is 4%.

$$\Delta=\frac{p_{HAND}^{max}-p_{FEM}^{max}}{p_{HAND}^{max}}=\frac{11.25-10.80}{11.25}=4\%$$

Fig. 7.4.9.4 shows the plastic connection forces in the adjusted line-line connections (which are the expected values according to the specific limit moment capacity).



7.4.9.4 – The relevant plastic connection moments [kNm/m]

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.9 Elasto-plastic line-line connections in a square plate.str

7.5 Calculation with construction stages

7.5.1 A steel frame building with construction stages calculation

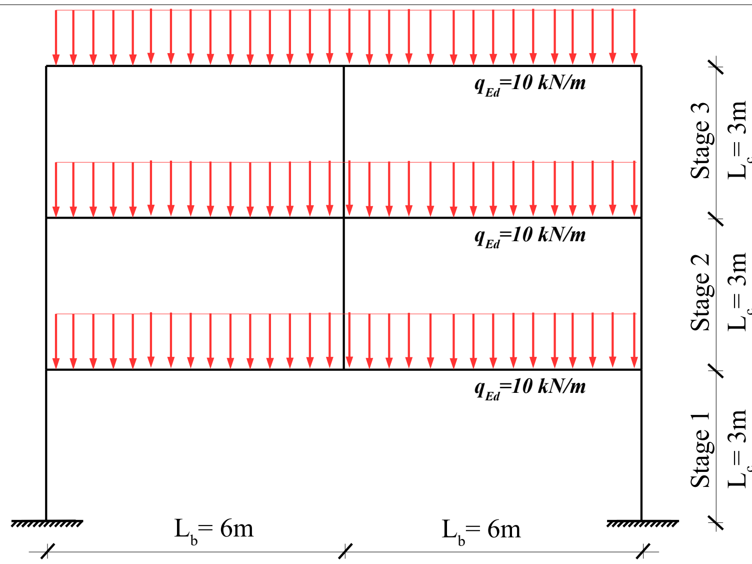
Inputs:

Strength of the steel	S235; E = 210 GPa
Columns	HEA 400
Beams	IPE 600
Line load on each floors	p_{Ed} = 10 kN/m

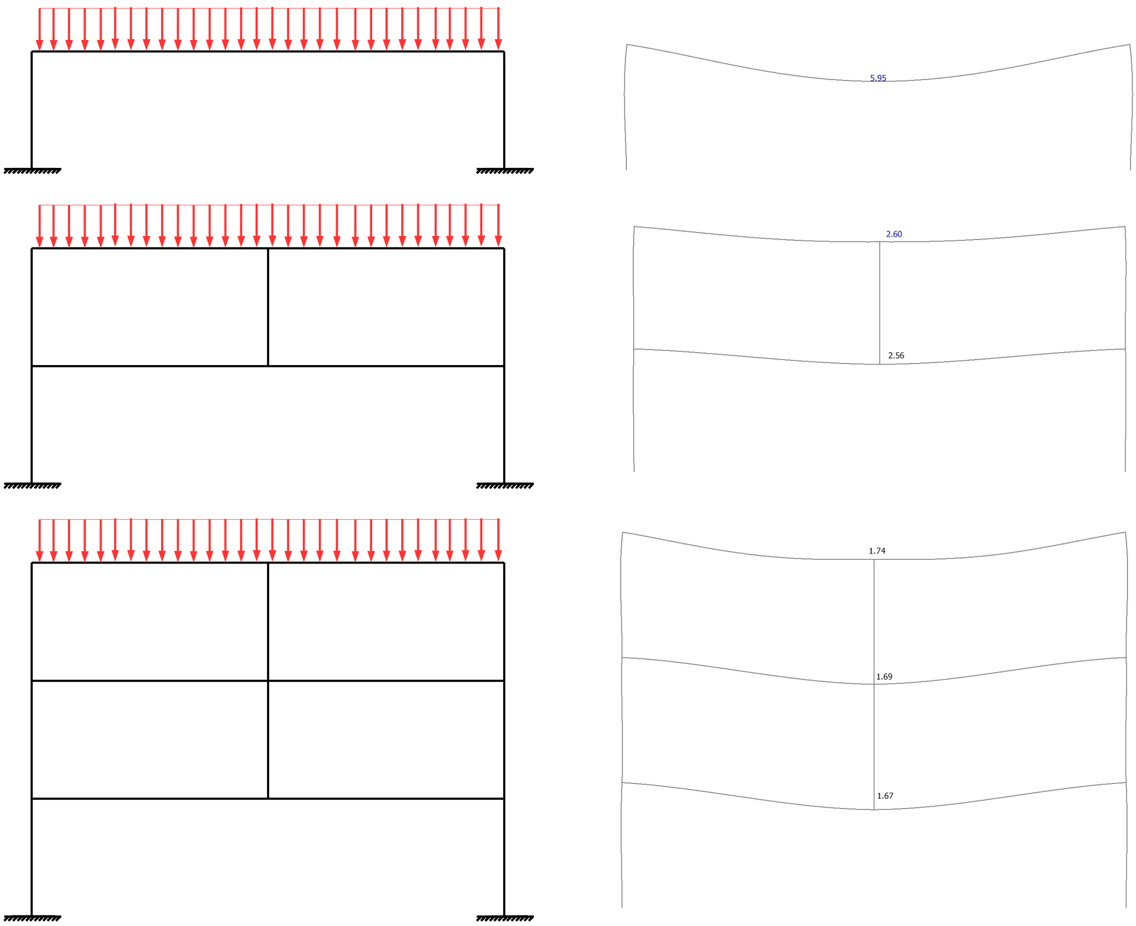
In this chapter we will show a short example about the construction stage calculation through a steel frame building with the "Incremental Tracking Method". Fig. 7.5.1.1 shows the geometry of the frame building. In this figure we also indicated the loads and the three different stages which were considered in this analysis. In this example there were three different stages by the three different storeys. As a simplification by the three different stages there were only a 10 kN/m line load on the beams one-by-one on each stages (see also Fig. 7.5.1.1) for the better comparison.

By the incremetal tracking method of the construction stage calculation FEM-Design handles the different stages as different structures. We apply the given loads on these different structures and apply a superposition by the internal forces and displacements respectively. <u>Thus basically the construction stage calculation is a nonlinear calculation with nonlinear boundary conditions and statical systems through the whole analysis.</u>

First of all we will show the different stage calculations one-by-one with the different boundary conditions and statical systems and superpose the results to verify the FEM-Design construction stage calculation method. In the end we will compare the results and show the differences between the regular and the constuction stage calculation method as well.



7.5.1.1 – The geometry and the considered loads on a steel frame building



7.5.1.2 – *The translational displacements* [mm] *separately according to the three different stages and the different load situations*

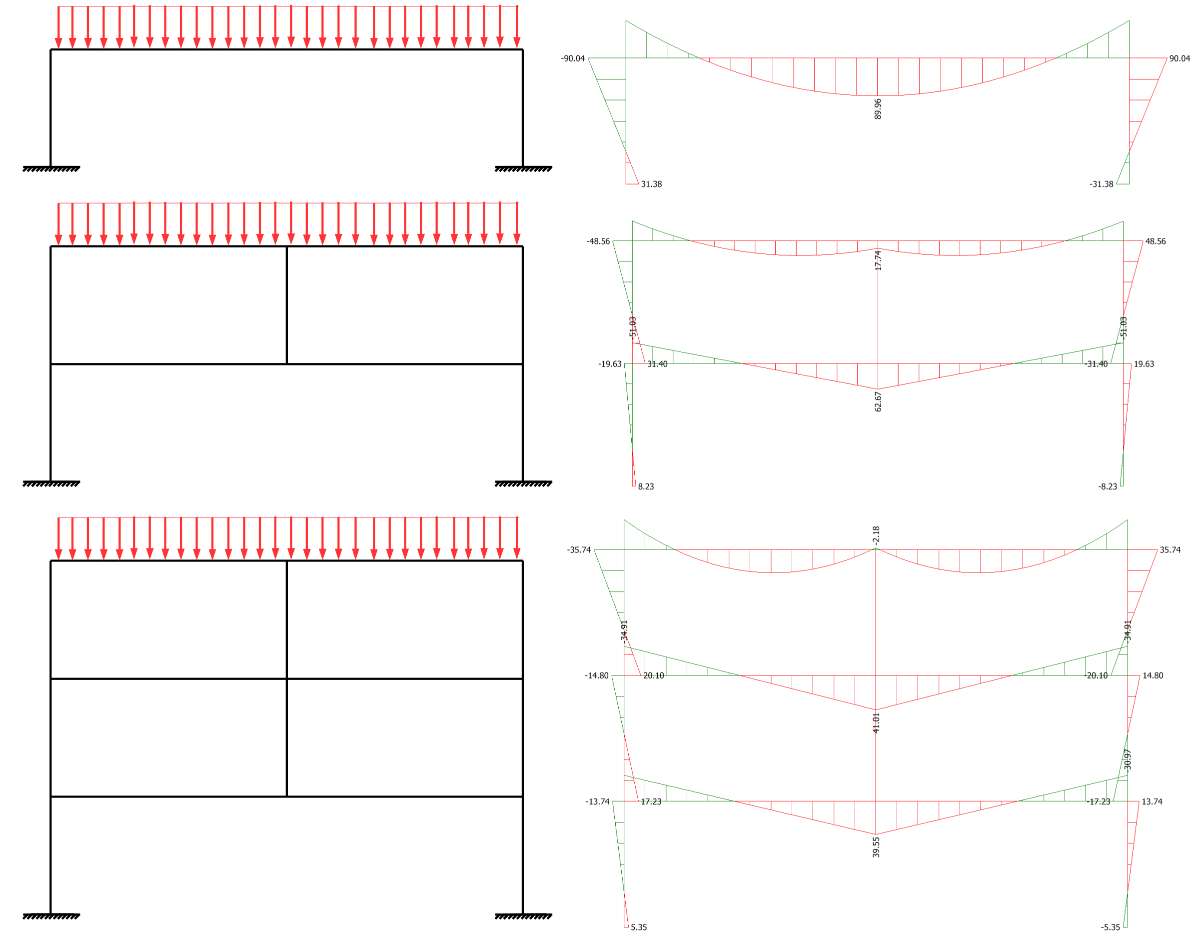
Fig. 7.5.1.2 shows the different stages with the different statical systems and loads. Next to the structural view you can see the translational displacements for each stages one-by-one.

With the construction stage calculation we accumulate these results therefore for the verification we need to superpose these values to get the final stage results. We calculated the displacements at specific points of the beams by the symmetry axis of the structure (see Fig. 7.5.1.2):

$$e_{B1} = e_{ST1B1} + e_{ST2B1} + e_{ST3B1} = 5.95 + 2.56 + 1.67 = 10.18\,\text{mm}$$

$$e_{B2} = e_{ST1B2} + e_{ST2B2} + e_{ST3B2} = 0 + 2.60 + 1.69 = 4.29\,\text{mm}$$

$$e_{B3} = e_{ST1B3} + e_{ST2B3} + e_{ST3B3} = 0 + 0 + 1.74 = 1.74\,\text{mm}$$



7.5.1.3 – The bending moment diagrams [kNm] *separately according to the three different stages and the different load situations*

Fig. 7.5.1.3 shows the different stages with the different statical systems and loads. Next to the structural view you can see the bending moment diagrams for each stages one-by-one.

With the construction stage calculation we accumulate these results therefore for the verification we need to superpose these values to get the final stage results. We calculated the bending moments at specific points of the beams by the symmetry axis of the structure (see Fig. 7.5.1.3):

$$M_{B1}=M_{ST1B1}+M_{ST2B1}+M_{ST3B1}=89.96+62.67+39.55=192.2\,\text{kNm}$$

$$M_{B2}=M_{ST1B2}+M_{ST2B2}+M_{ST3B2}=0+17.74+41.01=58.75\,\text{kNm}$$

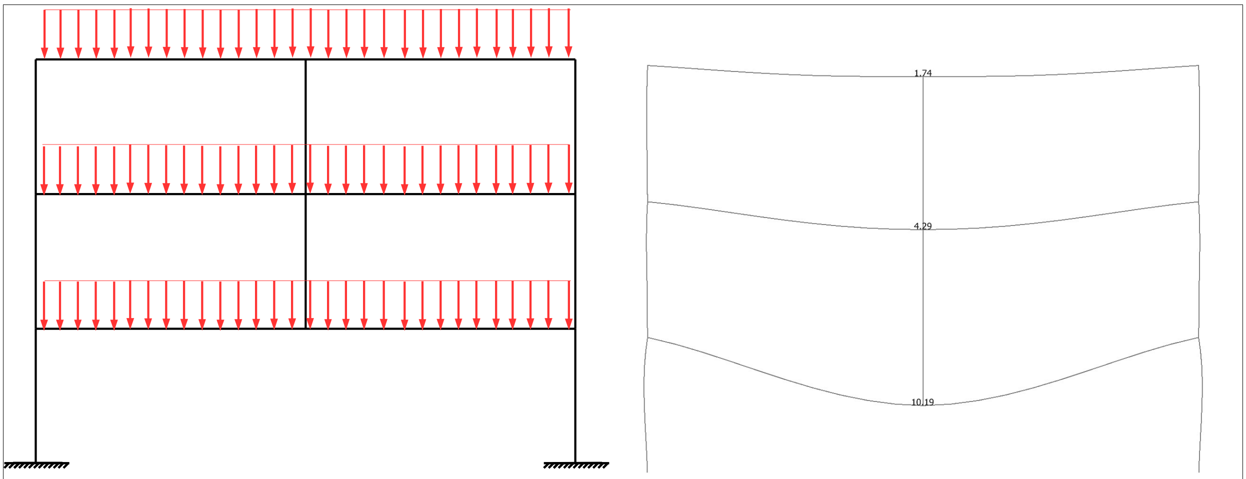
$$M_{B3}=M_{ST1B3}+M_{ST2B3}+M_{ST3B3}=0+0+(-2.18)=-2.18\,\text{kNm}$$

Fig. 7.5.1.4 shows the final accumulated displacement field at the end of the stage calculation in FEM-Design. Here you can see the results at specific points of the beams by the symmetry axis of the structure:

$$e_{B1FEM-CS}=10.18\,\mathrm{mm}$$

$$e_{B2FEM-CS}=4.29\,\mathrm{mm}$$

$$e_{B3FEM-CS}=1.74\,\mathrm{mm}$$



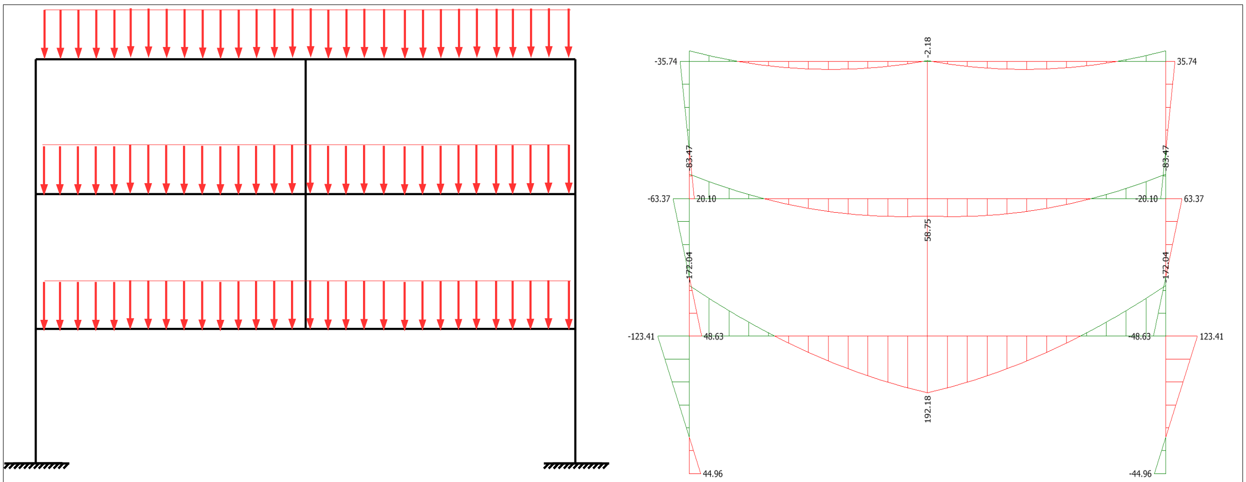
7.5.1.4 – The final accumulated translational displacements [mm] *according to the construction stage incremental tracking method*

Fig. 7.5.1.5 shows the final accumulated bending moments at the end of the stage calculation in FEM-Design. Here you can see the results at specific points of the beams by the symmetry axis of the structure:

$$M_{B1FEM-CS}=192.18\,\mathrm{kNm}$$

$$M_{B2FEM-CS}=58.75\,\mathrm{kNm}$$

$$M_{B3FEM-CS}=-2.18\,\mathrm{kNm}$$



7.5.1.5 – The final accumulated bending moment diagram [kNm] *according to the construction stage incremental tracking method*

The verification results are identical with the FEM-Design calculation.

At the end of this chapter let's compare the final construction stage results with the results of a calculation without construction stage calculation method (regular calculation on the frame with one statical system and load distribution). Fig. 7.5.1.6-7 show the displacements and the bending moments after a regular calculation.

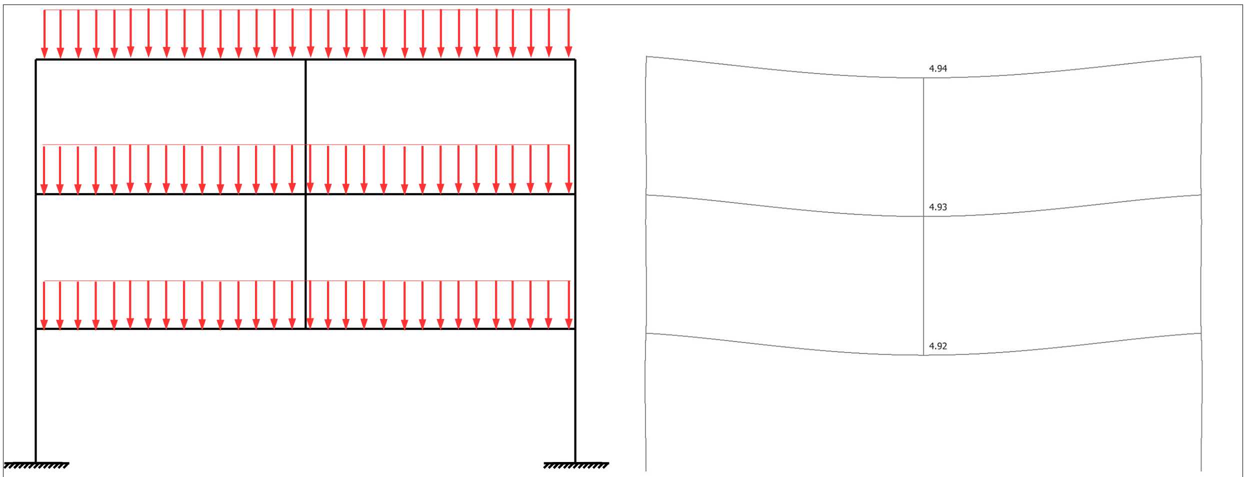
The results based on a regular calculation:

$$e_{B1FEM-withoutCS} = 4.92\,\text{mm} \quad ; \quad M_{B1FEM-withoutCS} = 81.78\,\text{kNm}$$

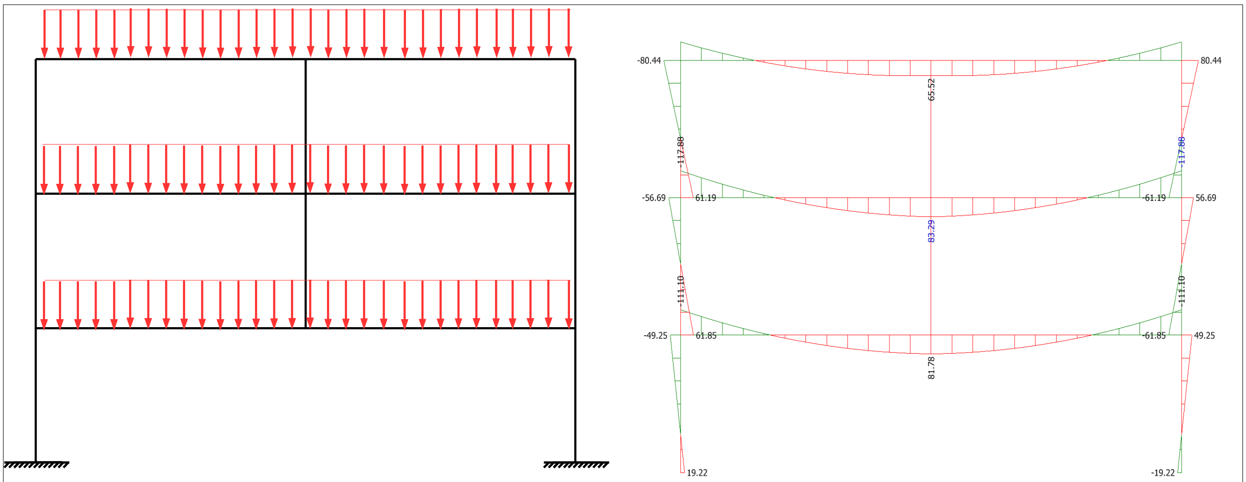
$$e_{B2FEM-withoutCS} = 4.93\,\text{mm} \quad ; \quad M_{B2FEM-withoutCS} = 83.29\,\text{kNm}$$

$$e_{B3FEM-withoutCS} = 4.94\,\text{mm} \quad ; \quad M_{B3FEM-withoutCS} = 65.52\,\text{kNm}$$

In this verification example the maximum displacement from the construction stage calculation is 2.07 times greater at the first floor (compare Fig. 7.5.1.4 with Fig. 7.5.1.6). The bending moment value is 2.35 times greater at the first floor by this verification example (compare Fig. 7.5.1.5 with Fig. 7.5.1.7).



7.5.1.6 – The translational displacements [mm] *according to the regular calculation method*



7.5.1.7 – The bending moment diagram [kNm] *according to the regular calculation method*

By a real frame structure the difference in the final results are not that much if we consider and add the live loads to the final stage results. Here in this example to show and emphasize the calculation method only a self-weight-like load was considered.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.5.1 A steel frame building with construction stages calculation.str

8 Dynamic response analysis

8.1 Footfall analysis of a concrete footbridge

Example taken from Ref. [13]. Let's take the following footbridge statical system from Fig. 8.1.1.

Inputs for the self excitation footfall analysis:

Dynamic elastic modulus of concrete	E = 38 GPa
The distributed load (load-mass conversion)	p = 18.13 kN/m
Number of considered mode shapes	N = 3
Inertia of the cross-section	$I = 0.056\ m^4$
Area of the section	$A = 0.77\ m^2$
Number of footsteps (conservative estimation)	$N_{footstep} = 100$ pcs
Mass of the walker	m = 71.36 kg
Frequency weighting curve	W_g
The excitation frequency interval	$f_{p,min}= 1$ Hz, $f_{p,max}= 2.8$ Hz
Frequency steps	steps = 100 pcs
The cut-off eigenfrequency	$f_{cut} = 15$ Hz
Damping	$\zeta = 1.5$ %
Fourier coefficients	The Concrete Centre Table 4.3

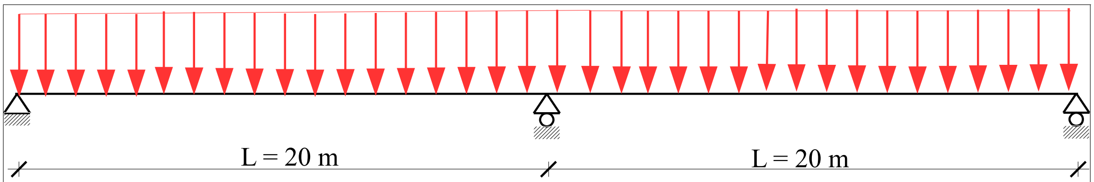


Figure 8.1.1 – The concrete footbridge with the considered load-mass conversion

The model is divided into 16 finite bar elements. The given distributed load is converted to mass with 1.0 factor (1848 kg/m) for the eigenfrequency calculation. The statical system is a beam with the given stiffness parameters and with 3 supports (see Fig. 8.1.1). All of the necessary parameters for the footfall analysis is given in the inputs. In FEM-Design the used excitation method was the self excitation method. For the self excitation method the adjusted region contained the full beam structure.

The first three mode shapes are visible in Fig. 8.1.2 based on the FEM-Design calculation. Table 8 contains the theoretical solutions about the eigenfrequencies of the first three modes according to Ref. [13] and FEM-Design results are also indicated. There are good agreements between the two results.

Mode	Theoretical (Hz)	FEM-Design (Hz)
1st	4.22	4.203
2nd	6.59	6.536
3rd	16.90	16.68

Table 8 – The first three eigenfrequencies

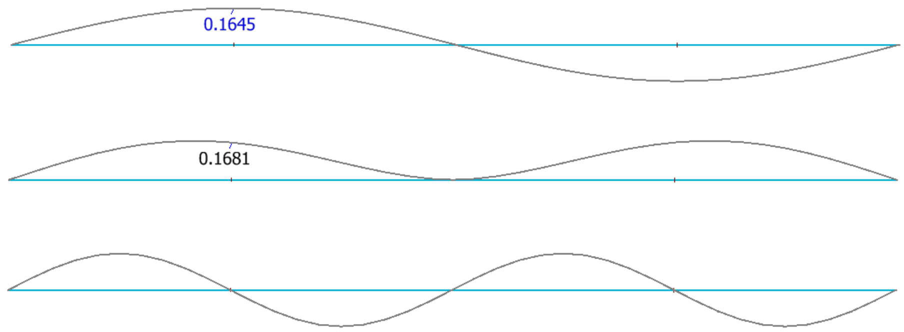


Figure 8.1.2 – The first three mode shapes [-]

This footbridge is relatively soft, therefore the steady-state acceleration will be greater than the transient. As a simple hand calculation the RMS acceleration for walking at 2.102 Hz is the following:

The amplitude of the excitation force by the second harmonic:

$$F_2=\frac{71.36}{1000}\cdot 9.81\,(0.069+0.0056\cdot 2\cdot 2.102)=0.06478\,\text{kN}$$

In this case the second harmonic of the excitation frequency causes resonance.

The dynamic magnification factor for the accelerations by the 1st mode shape and 2nd harmonic:

$$D_{1,2}=\frac{2^2\left(\frac{f_p}{f_1}\right)^2}{\sqrt{\left(1-2^2\left(\frac{f_p}{f_1}\right)^2\right)^2+4\zeta^2 2^2\left(\frac{f_p}{f_1}\right)^2}}=\frac{2^2\left(\frac{2.102}{4.203}\right)^2}{\sqrt{\left(1-2^2\left(\frac{2.102}{4.203}\right)^2\right)^2+4\cdot 0.015^2\cdot 2^2\left(\frac{2.102}{4.203}\right)^2}}=$$

$$=\frac{1}{\sqrt{0+4\cdot 0.015^2\cdot 1}}=\frac{1}{2\cdot 0.015}=33.33$$

Based on these values the RMS acceleration at mid-span (see Fig. 8.1.2 also):

$$a_{w,midspan,RMS[steady\ state]} = \frac{1}{\sqrt{2}} \mu^2_{midspan,1} \frac{F_2}{M_1} D_{1,2} W_2 = \frac{1}{\sqrt{2}} 0.1645^2 \frac{0.06478}{1} 33.33 \cdot 1.0 = 0.04131 \frac{\mathrm{m}}{\mathrm{s}^2}$$

In Ref. [13] the peak acceleration value is a_{peak} = 0.06 m/s^2, therefore the comparable RMS value is:

$a_{RMS} = \frac{a_{peak}}{\sqrt{2}} = \frac{0.06}{\sqrt{2}} = 0.04243 \frac{\mathrm{m}}{\mathrm{s}^2}$ and the response factor based on Ref. [13]: $\mathrm{R} = 8.5$

Based on the FEM-Design calculation these two values are (see Fig. 8.1.3 as well):

$a_{RMS,FEM} = 0.0443 \frac{\mathrm{m}}{\mathrm{s}^2}$ and $\mathrm{R} = 8.86$.

There are good agreements between the results. The difference comes from the fact that not only the first mode shape has effect on the accelerations however in Ref. [13] and the hand calculation here considered only the first mode.

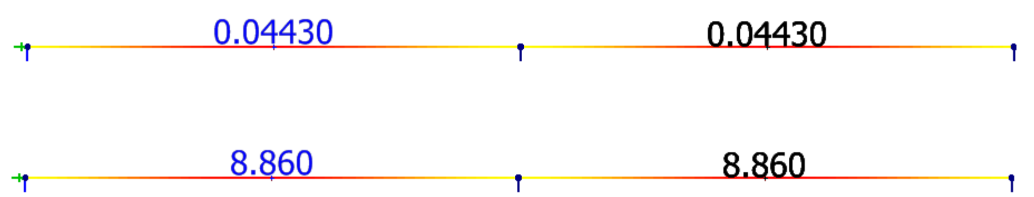


Figure 8.1.3 – The acceleration [m/s^2] *and the response factor* [-] *in FEM-Design*

Another interesting result could be the frequency curve. Fig. 8.1.4 shows the accelerations at the midspan point in function of the excitation frequencies. The red line is the steady-state response and the green one is the transient. Based on Fig. 8.1.4 we can say that in this example the transient response is really negligible compared to the steady-state response. The frequency curve clearly shows the resonance excitation frequencies where the peak RMS accelerations arise.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/8.1 Footfall analysis of a concrete footbridge.str

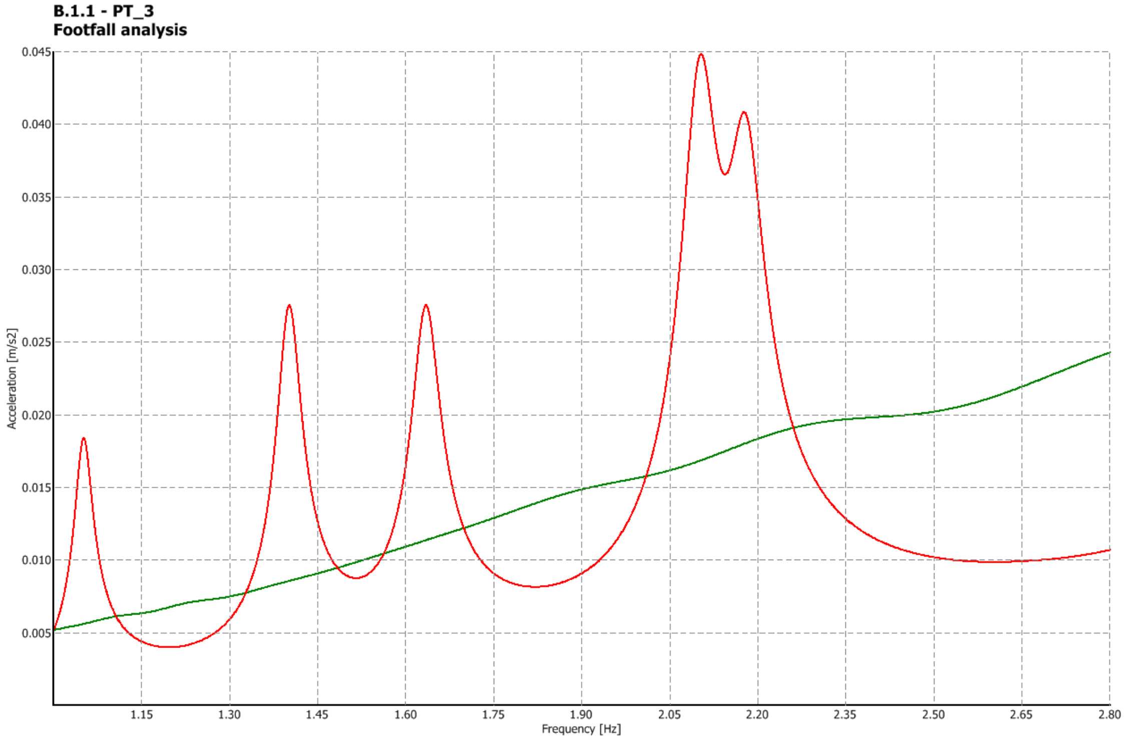


Figure 8.1.4 – Accelerations in function of excitation frequencies in FEM-Design
Red: steady-state response
Green: transient response

8.2 Footfall analysis of a composite floor

Example taken from Ref. [14]. Let's take a 130 mm deep normal weight concrete slab on top of 1.2 mm thick re-entrant deck. Slabs supported by 6.0 m span secondary beams at 2.48 m cross-centres which, in turn, are supported by 7.45 m span castellated primary beams in orthogonal direction, see Fig. 8.2.1. The input data and the geometry are available in Ref. [14].

Inputs for the self excitation footfall analysis:

Excitation region (see Fig. 8.2.1)	The whole floor slab
The distributed load (load-mass conversion)	$p = 4.48$ kN/m^2
Number of footsteps (conservative estimation)	$N_{footstep} = 100$ pcs
Mass of the walker	$m = 76$ kg
Frequency weighting curve	W_g
The excitation frequency interval	$f_{p,min} = 1.8$ Hz, $f_{p,max} = 2.2$ Hz
Frequency steps	steps = 100 pcs
The cut-off eigenfrequency	$f_{cut} = 15$ Hz
Damping	$\zeta = 4.68$ %
Fourier coefficients	SCI P354 Table 3.1

In Ref. [14] with the finite element calculation the first fundamental natural frequency was:

$f_1 = 10.80\,\mathrm{Hz}$

In Ref. [14] with the finite element calculation the response factor was:

$\mathrm{R} = 3.18$

With the given parameters above and considering the geometry and the material properties based on Ref. [14] FEM-Design calculation gives the following results (see Fig. 8.2.1 also):

$f_{FEM} = 10.82\,\mathrm{Hz}$ and $\mathrm{R} = 3.82$

We can say that there are good agreements between the results. However, it should be noted that in Ref. [14] the results of the calculation is given, but the details of the finite element model and calculation method is unclear, therefore there may be differences in the modeling methods. By this example it is very hard to say that the result in Ref. [14] is relevant because the hand calculation is quite different than the FEM calculation what was published in Ref. [14]. Based on our opinion the indicated FEM result in Ref. [14] belongs to the transient response as well as the result in FEM-Design.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/8.2 Footfall analysis of a composite floor.str

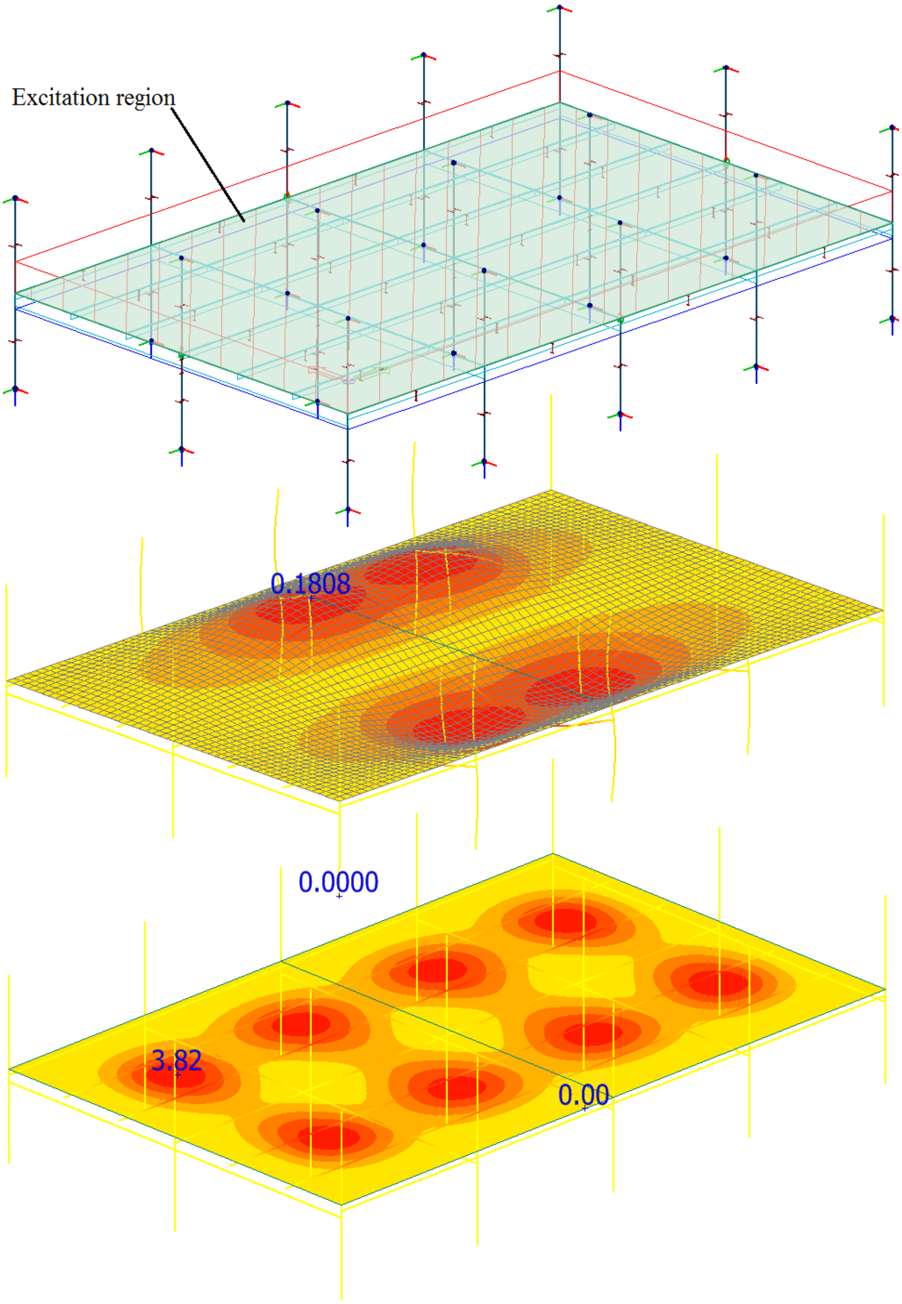


Figure 8.2.1 – The model, the first mode shape [-] *and the response factor* [-] *results in FEM-Design*

8.3 Footfall analysis of a lightweight floor

Example taken from Ref. [14]. Let's take a chipboard flooring on lightweight steel beams, see Fig. 8.3.1. The input data and the geometry are available in Ref. [14].

Inputs for the full excitation footfall analysis:

Excitation point (see Fig. 8.3.1)	In the middle of the floor
The distributed load (load-mass conversion)	$p = 0.69$ kN/m^2
Number of footsteps (conservative estimation)	$N_{footstep} = 100$ pcs
Mass of the walker	$m = 76$ kg
Frequency weighting curve	W_g
The excitation frequency interval	$f_{p,min} = 1.8$ Hz, $f_{p,max} = 2.2$ Hz
Frequency steps	steps = 100 pcs
The cut-off eigenfrequency	$f_{cut} = 15$ Hz
Damping	$\zeta = 5.0$ %
Fourier coefficients	SCI P354 Table 3.1

In Ref. [14] with the finite element calculation the first fundamental natural frequency was:

$f_1 = 16.31\,\text{Hz}$

In Ref. [14] with the finite element calculation the response factor was:

$\text{R} = 53.9$

In FEM-Design the average finite element size was 0.40 m. With the given parameters above and considering the geometry and the material properties based on Ref. [14] FEM-Design calculation gives the following results (see Fig. 8.3.1 also):

$f_{FEM} = 16.13\,\text{Hz}$ and $\text{R} = 53.87$

Fig. 8.3.2 shows the response factors in function of the given interval of the excitation force based on FEM-Design calculation.

We can say that there are good agreements between the results. However, it should be noted that in Ref. [14] the results of the calculation is given, but the details of the finite element model and calculation method is unclear, therefore there may be differences in the modeling methods.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/8.3 Footfall analysis of a lightweight floor.str

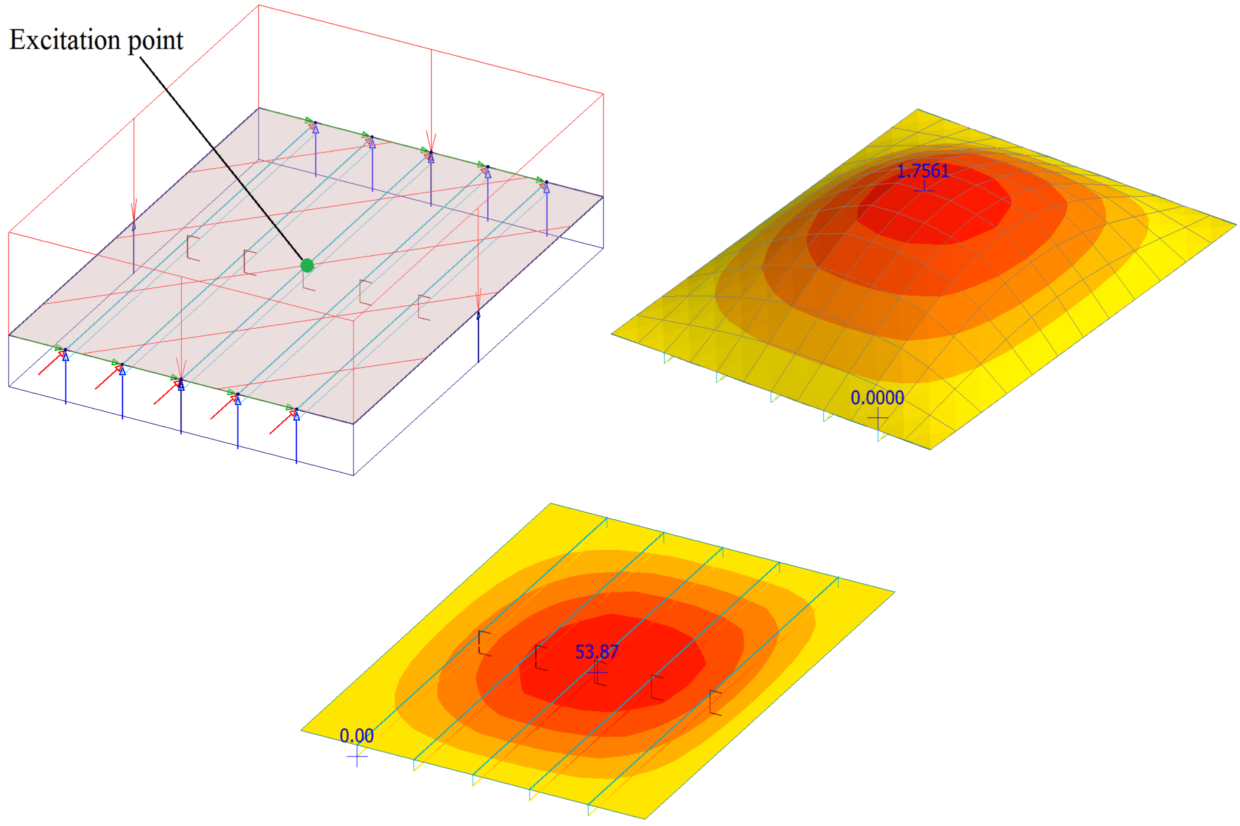


Figure 8.3.1 – The model, the first mode shape [-] *and the response factor* [-] *results in FEM-Design*

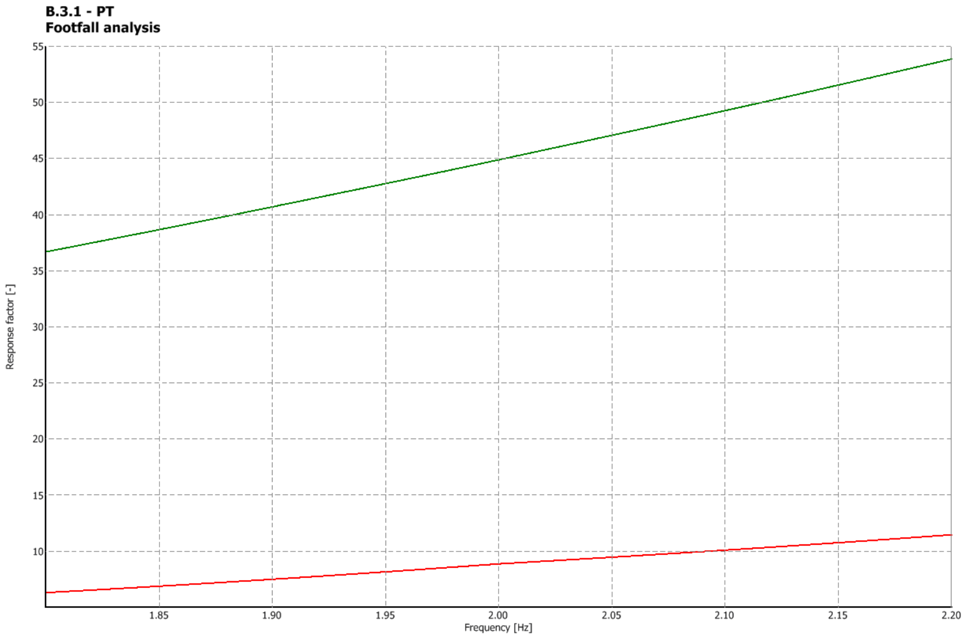


Figure 8.3.2 – Response factor in function of excitation frequencies in FEM-Design
Red: steady-state response
Green: transient response

8.4 Footfall analysis of a small stage with rhythmic crowd load

This calculation will be presented according to Danish Annex (Ref. [15]). The floor is a simply supported concrete slab. The half of the slab is a stage where the rhythmic crowd activity will be considered (see Fig. 8.4.1).

Inputs for the rhythmic crowd load footfall analysis:

Elastic modulus of concrete	$E = 31$ GPa, $\nu = 0.2$
Thickness of the concrete slab	$t = 250$ mm
Self-weight plus the considered imposed load	$p = 6.75$ kN/m^2
Mean static crowd load (on half of the slab, Fig. 8.4.1)	$F_p = 1.0$ kN/m^2
The excitation frequency	$f_p = 3$ Hz
The cut-off eigenfrequency	$f_{cut} = 30$ Hz
Damping	$\zeta = 1.9$ %
Effective number of people	$n_e = 20$
Fourier coefficients	According to Danish Annex Reduced possibility to move about

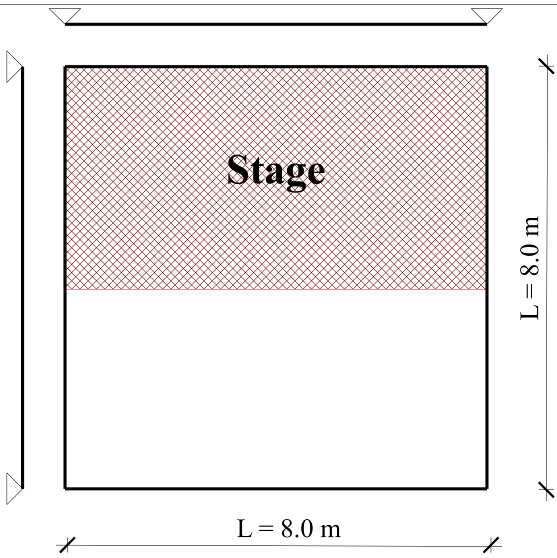


Figure 8.4.1 – The slab with the stage

The first eigenfrequency (based on finite element calculation):

$$f_1 = 12\,\mathrm{Hz}$$

In the Danish Annex the logarithmic decrement is given instead of critical damping ratio. The logarithmic decrement with the given critical damping ratio from the inputs:

$$(\delta_s + \delta_p) = 2\pi\,\zeta = 2\pi\,0.019 = 0.12$$

The frequency response factor in the Danish Annex is given with:

$$H_j = \frac{1}{\sqrt{\left(1-\left(\frac{j\cdot f_p}{f_1}\right)^2\right)^2 + \left(\frac{(\delta_s+\delta_p)\, j\cdot f_p}{\pi\, f_1}\right)^2}}$$, therefore:

$$H_1 = \frac{1}{\sqrt{\left(1-\left(\frac{1\cdot 3}{12}\right)^2\right)^2 + \left(\frac{0.12\cdot 1\cdot 3}{\pi\, 12}\right)^2}} = 1.067 \quad ;$$

$$H_2 = \frac{1}{\sqrt{\left(1-\left(\frac{2\cdot 3}{12}\right)^2\right)^2 + \left(\frac{0.12\cdot 2\cdot 3}{\pi\, 12}\right)^2}} = 1.333 \quad ;$$

$$H_3 = \frac{1}{\sqrt{\left(1-\left(\frac{3\cdot 3}{12}\right)^2\right)^2 + \left(\frac{0.12\cdot 3\cdot 3}{\pi\, 12}\right)^2}} = 2.281 \quad .$$

The considered Fourier coefficients including the size reduction factor:

$$\alpha_1 K_1 = 0.40 \quad ;$$

$$\alpha_2 K_2 = 0.25\sqrt{0.1+(1-0.1)\frac{1}{20}} = 0.0952 \quad ;$$

$$\alpha_3 K_3 = 0.05\sqrt{0.01+(1-0.01)\frac{1}{20}} = 0.0122 \quad .$$

The dynamic magnification factor for displacements (according to Danish Annex):

$$k_F = \sqrt{\sum_{j=1}^{3}\left(\alpha_j K_j H_j\right)^2} = \sqrt{(0.4\cdot 1.067)^2 + (0.0952\cdot 1.333)^2 + (0.0122\cdot 2.281)^2}$$

$$k_F = 0.4461$$

The acceleration response factor (according to Danish Annex):

$$k_a = \sqrt{\frac{1}{2}\sum_{j=1}^{3}\left(j^2 \alpha_j K_j H_j\right)^2} = \frac{1}{\sqrt{2}}\sqrt{(1^2\cdot 0.4\cdot 1.067)^2 + (2^2\cdot 0.0952\cdot 1.333)^2 + (3^2\cdot 0.0122\cdot 2.281)^2}$$

$$k_a = 0.5013$$

The maximum deflection of the slab under the mean static crowd load on the half of the slab (based on a finite element calculation, see Fig. 8.4.2):

$$u_p = 0.2132\,\mathrm{mm}$$

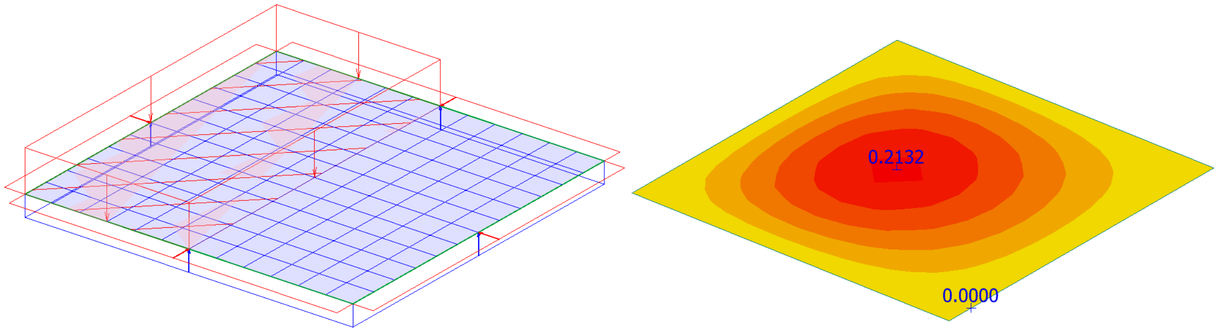


Figure 8.4.2 – The slab with the crowd load and the displacements in [mm] *under it in FEM-Design*

The RMS acceleration of the structure induced by the vertical dynamic load (according to Danish Annex):

$$a_a = k_a (2\pi f_p)^2 u_p = 0.5013 \cdot (2\pi 3)^2 0.2132/1000 = 0.03797 \frac{\mathrm{m}}{\mathrm{s}^2}$$

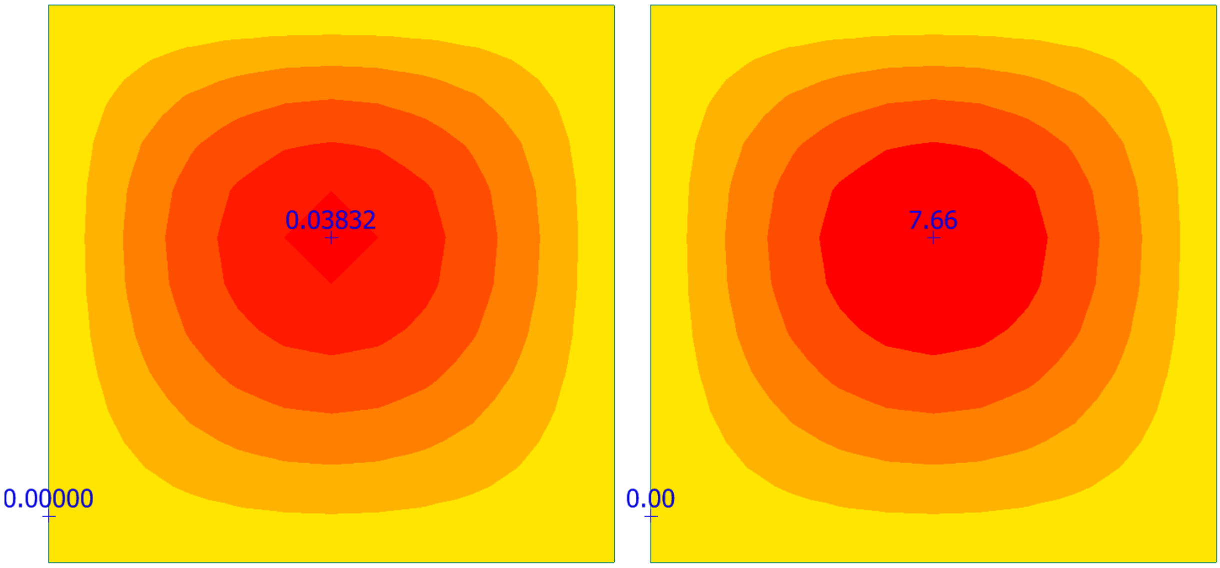
The accelerations and the dynamic magnification factors for displacements based on the FEM-Design calculation (see Fig. 8.4.3):

$$a_{FEM} = 0.03832 \frac{\mathrm{m}}{\mathrm{s}^2}$$

$$k_{FEM} = 0.446$$

The difference between the hand calculation and FEM-Design calculation is less than 1%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/8.4 Footfall analysis of a small stage with rhythmic crowd load.str



No.	Frequency [Hz]	Dynamic magnification factor [-]
1	12.0	0.446

Figure 8.4.3 – The accelerations in [m/s²] *and the response factors* [-] *in FEM-Design*

8.5 Dynamic response of a moving mass point on a simply supported beam

Example is taken from Ref. [18]. Let's take a simply supported beam with one moving mass point on it, see. Fig. 8.5.1.

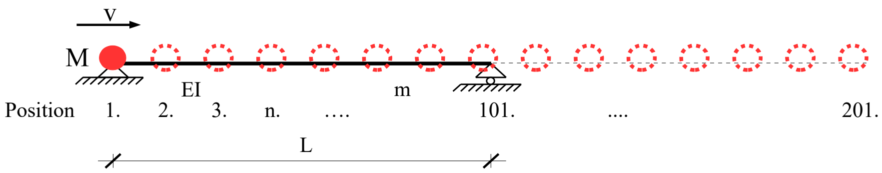


Figure 8.5.1 – The geometry and the moving mass positions on the path during the calculation

Inputs (as you can see in this case the moving object will be a "force" but its mass will be considered during the dynamic calculation):

Analyzed position	Mid-span of the beam
The distributed mass of the beam	m = 312 kg/m
Mass of the moving mass	M = 1248 kg
Elastic modulus of the beam	E= 206 GPa
Span length	L = 20 m
Analyzed constant velocities of the moving mass	v_1 = 18.12 km/h
	v_2 = 36.24 km/h
	v_3 = 72.48 km/h
Critical damping ratio	ξ = 0 %
Path length	L_p = 40 m
Cross-section	200 mm x 200 mm square

In FEM-Design by this example during the direct integration calculation the damping was neglected (thus in the Rayleigh damping matrix the α = 0 and β = 0), therefore the system is undamped. In the FEM model we split the beam into 50 parts to consider a relevant continuous mass distribution, and 201 mass point positions, see Fig. 8.5.1. The results belong to those positions which are not on the structure will give us the free vibration response of the structure, see Fig. 8.5.4.

The first three eigenfrequencies of a simply supported beam with continuous distributed mass.

$$f_1=\frac{\pi}{2\cdot L^2}\sqrt{\frac{EI}{m}}=\frac{\pi}{2\cdot 20^2}\sqrt{\frac{206000000000\cdot 0.2^4/12}{312}}=1.165\,\mathrm{Hz}$$

$$f_2=\frac{\pi}{2\cdot\left(\frac{L}{2}\right)^2}\sqrt{\frac{EI}{m}}=\frac{\pi}{2\cdot\left(\frac{20}{2}\right)^2}\sqrt{\frac{206000000000\cdot 0.2^4/12}{312}}=4.661\,\mathrm{Hz}$$

$$f_3=\frac{\pi}{2\cdot\left(\frac{L}{3}\right)^2}\sqrt{\frac{EI}{m}}=\frac{\pi}{2\cdot\left(\frac{20}{3}\right)^2}\sqrt{\frac{206000000000\cdot0.2^4/12}{312}}=10.49\,\mathrm{Hz}$$

According to FEM-Design calculation the first three eigenfrequencies of the beam are as follows, see Fig. 8.5.2:

$f_{FEM\,1}=1.165\,\mathrm{Hz}$; $f_{FEM\,2}=4.658\,\mathrm{Hz}$ $f_{FEM\,3}=10.47\,\mathrm{Hz}$

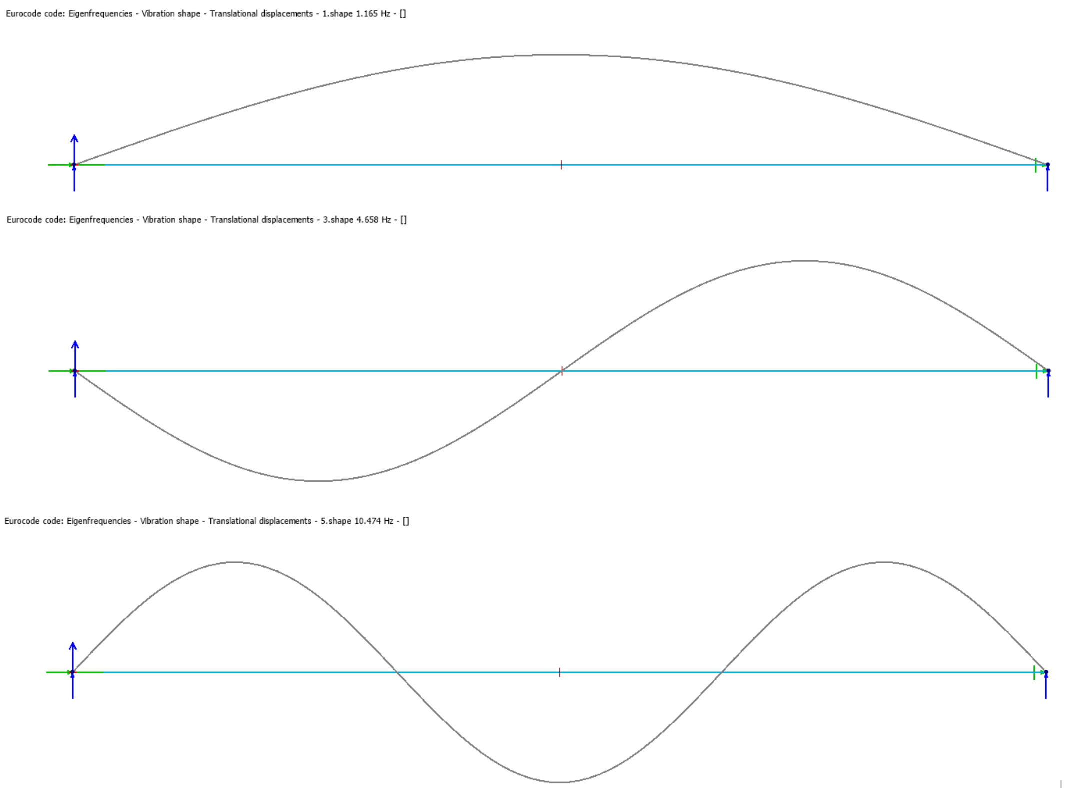


Figure 8.5.2 – The first three eigenshapes in FEM-Design

The critical velocity:

$$v_{cr}=2\,L\,f_1=2\cdot20\cdot1.165=46.6\,\frac{\mathrm{m}}{\mathrm{s}}=167.76\,\frac{\mathrm{km}}{\mathrm{h}}$$

The analytical solutions based on Ref. [18] are the normalised dynamic factors of the beam mid-span under different velocities. These values are very close to the FEM-Design numerical calculations.

The maximum static deflection of the mid-span if the mass is at the mid-span:

$$e_{max}=\frac{F\,L^3}{48\,EI}=\frac{M\,g\,L^3}{48\,EI}=\frac{1248\cdot9.81\cdot20^3}{48\cdot206000000000\cdot0.2^4/12}=0.07429\,\mathrm{m}$$

The maximum static deflection at mid-span based on FEM-Design calculation (see also Fig. 8.5.4):

$$e_{max}^{FEM} = 0.07431\ \mathrm{m}$$

The maximum normalized dynamic factor at the mid-span under v_1 velocity according to Ref. [18]:

$$\left(\frac{Z_1^{\mathrm{dynamic}}}{Z_{\mathrm{max}}^{\mathrm{static}}}\right)_{\mathrm{max}} = 1.1$$

This value based on FEM-Design, see Fig. 8.5.3-4:

$$\left(\frac{Z_1^{\mathrm{FEMdynamic}}}{Z_{\mathrm{max}}^{\mathrm{FEMstatic}}}\right)_{\mathrm{max}} = 1.096$$

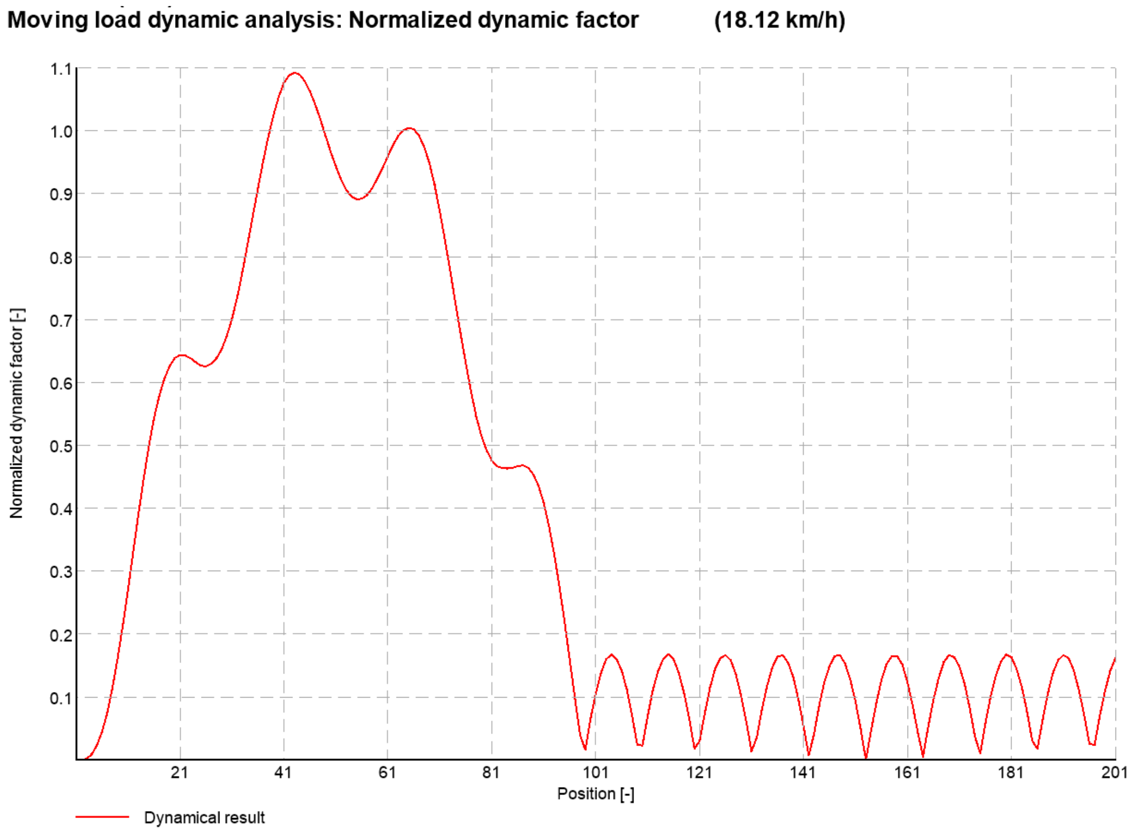


Figure 8.5.3 – The normalized dynamic factor diagram based on FEM-Design with v_1 velocity

Fig. 8.5.4 shows the static and dynamic displacements at the mid-span under the different load positions with v_1 velocity.

In Fig. 8.5.4 we can see the static and dynamic maximum deflection at the mid-span.

$$e_{dynamic\,max}^{FEM\ v_1} = 0.08139\ \mathrm{m}$$

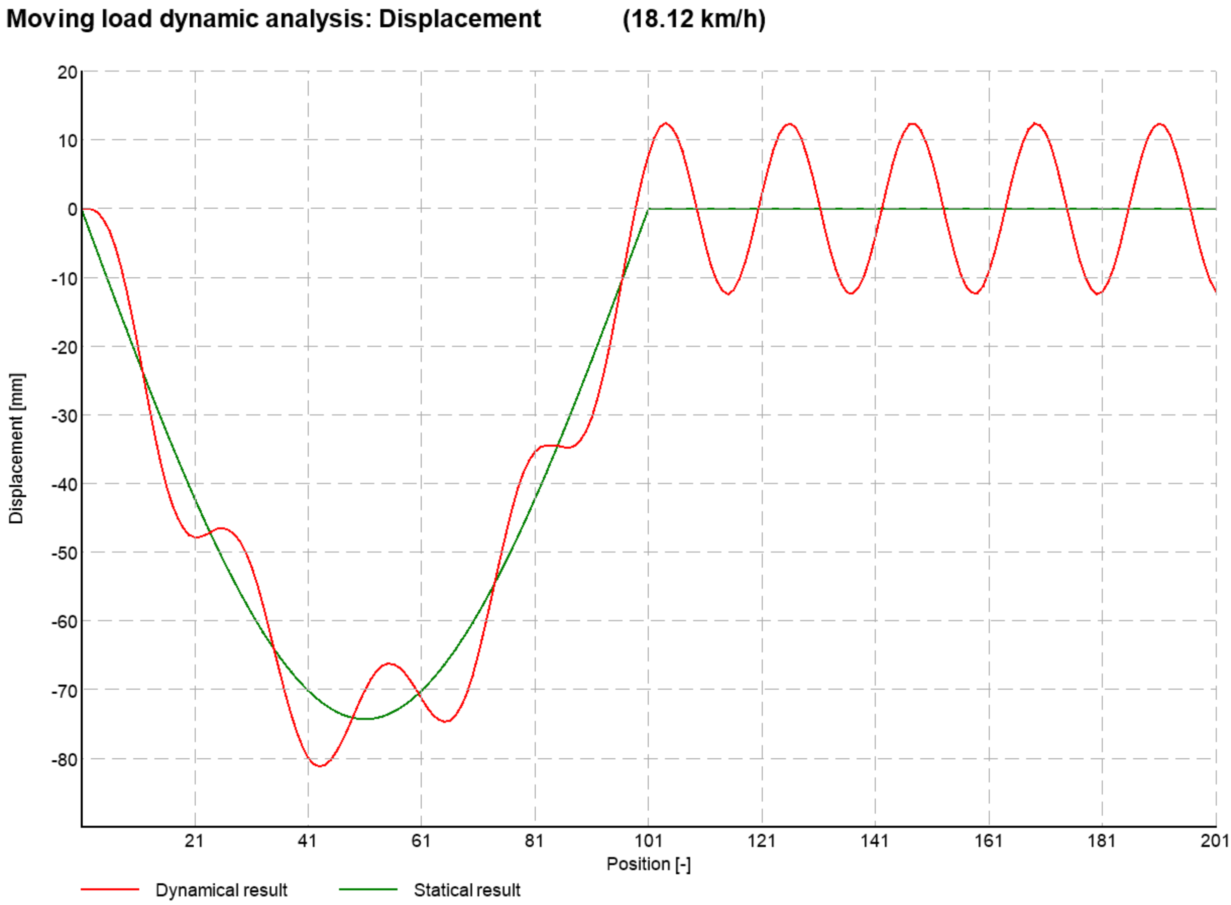


Figure 8.5.4 – The dynamic and static displacements of the mid-span in function of mass positions with v_1 velocity with FEM-Design

The maximum normalized dynamic factor at the mid-span under v_2 velocity:

$$\left(\frac{Z_2^{\text{dynamic}}}{Z_{\text{max}}^{\text{static}}}\right)_{\text{max}} = 1.21$$

This value based on FEM-Design, see Fig. 8.5.5:

$$\left(\frac{Z_2^{\text{FEMdynamic}}}{Z_{\text{max}}^{\text{FEMstatic}}}\right)_{\text{max}} = 1.21$$

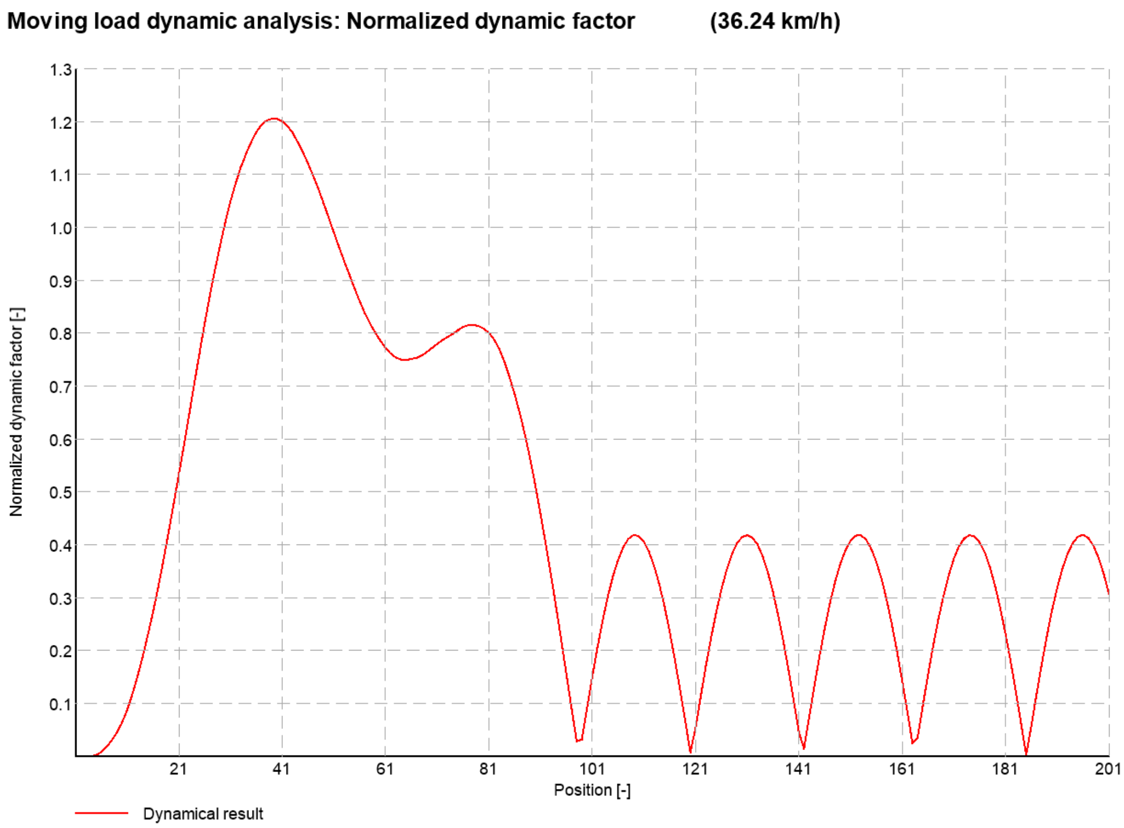


Figure 8.5.5 – The normalized dynamic factor diagram based on FEM-Design with v_2 velocity

Fig. 8.5.6 shows the static and dynamic displacements at the mid-span under the different load positions with v_2 velocity.

About Fig. 8.5.6 we can see the static and dynamic maximum deflection at the mid-span.

$$e_{dynamic\ max}^{FEM\ v_2} = 0.08991\ \text{m}$$

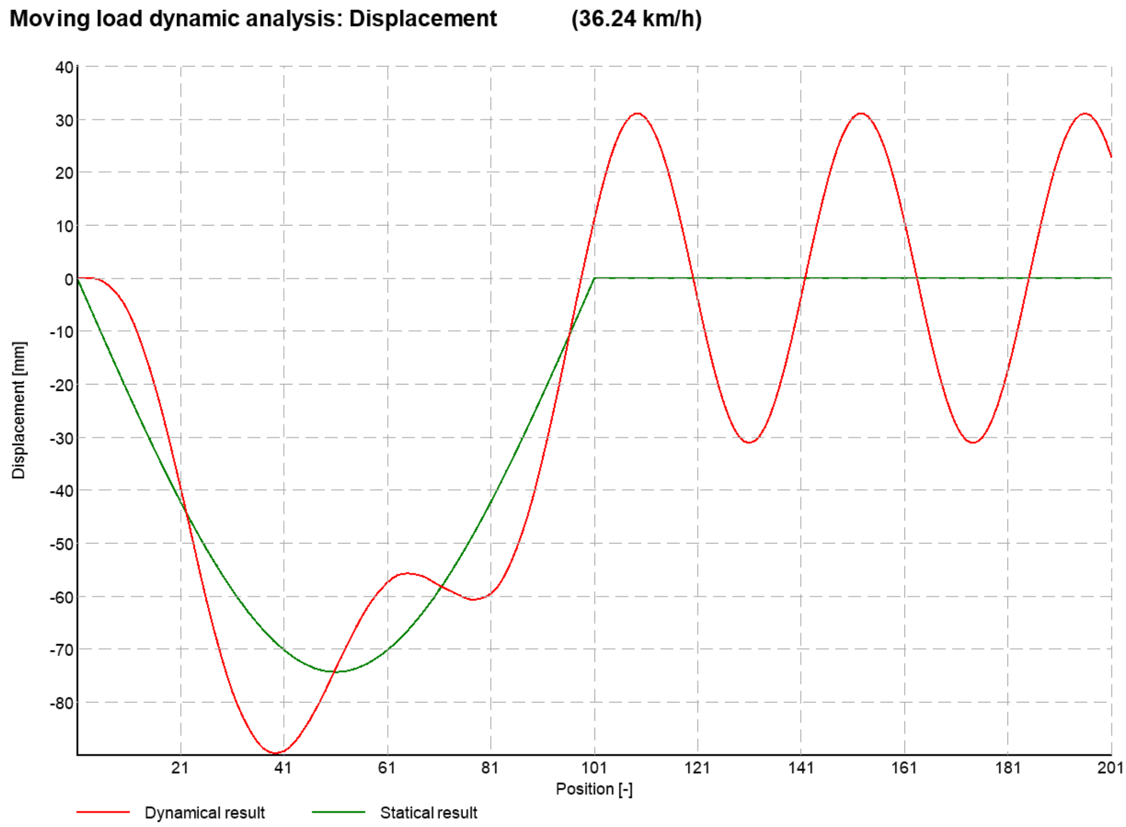


Figure 8.5.6 – The dynamic and static displacements of the mid-span in function of mass positions with v_2 velocity with FEM-Design

The maximum normalized dynamic factor at the mid-span under v_3 velocity:

$$\left(\frac{Z_3^{\text{dynamic}}}{Z_{\text{max}}^{\text{static}}}\right)_{\text{max}} = 1.85$$

This value based on FEM-Design, see Fig. 8.5.7:

$$\left(\frac{Z_3^{\text{FEMdynamic}}}{Z_{\text{max}}^{\text{FEMstatic}}}\right)_{\text{max}} = 1.76$$

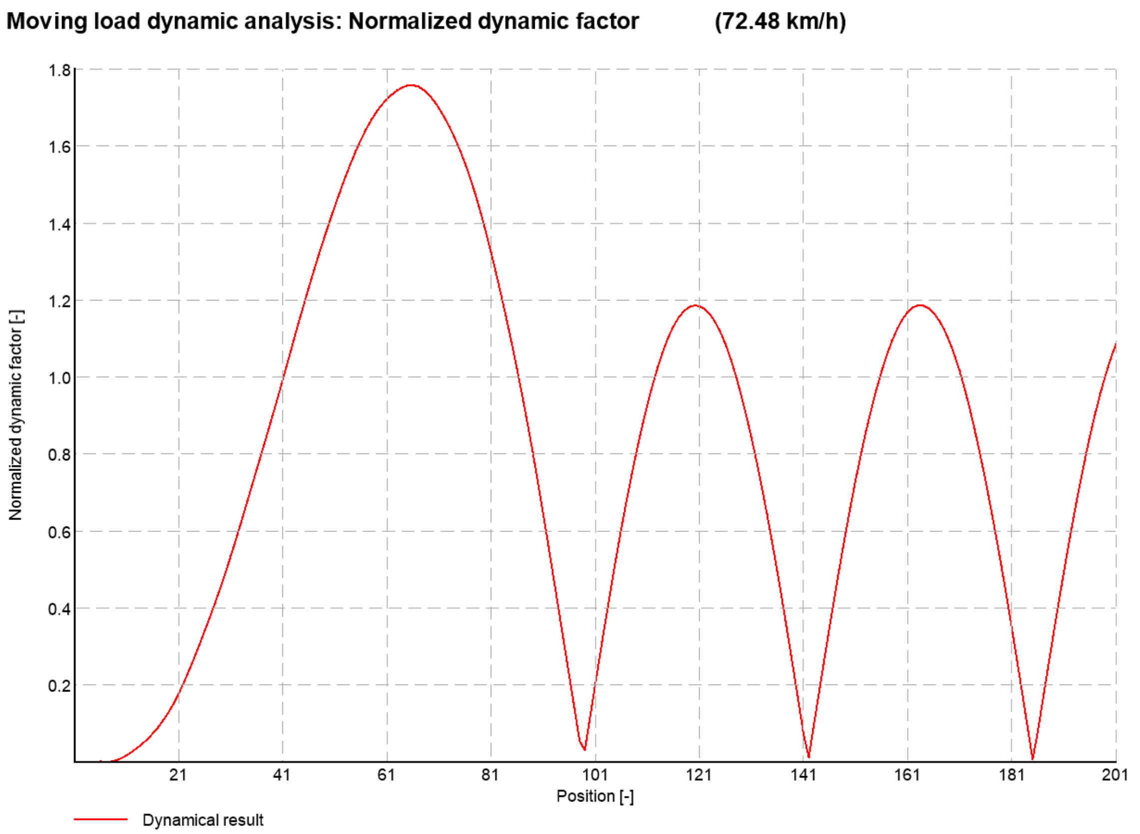


Figure 8.5.7 – The normalized dynamic factor diagram based on FEM-Design with v_3 velocity

Fig. 8.5.8 shows the static and dynamic displacements at the mid-span under the different load positions with v_3 velocity.

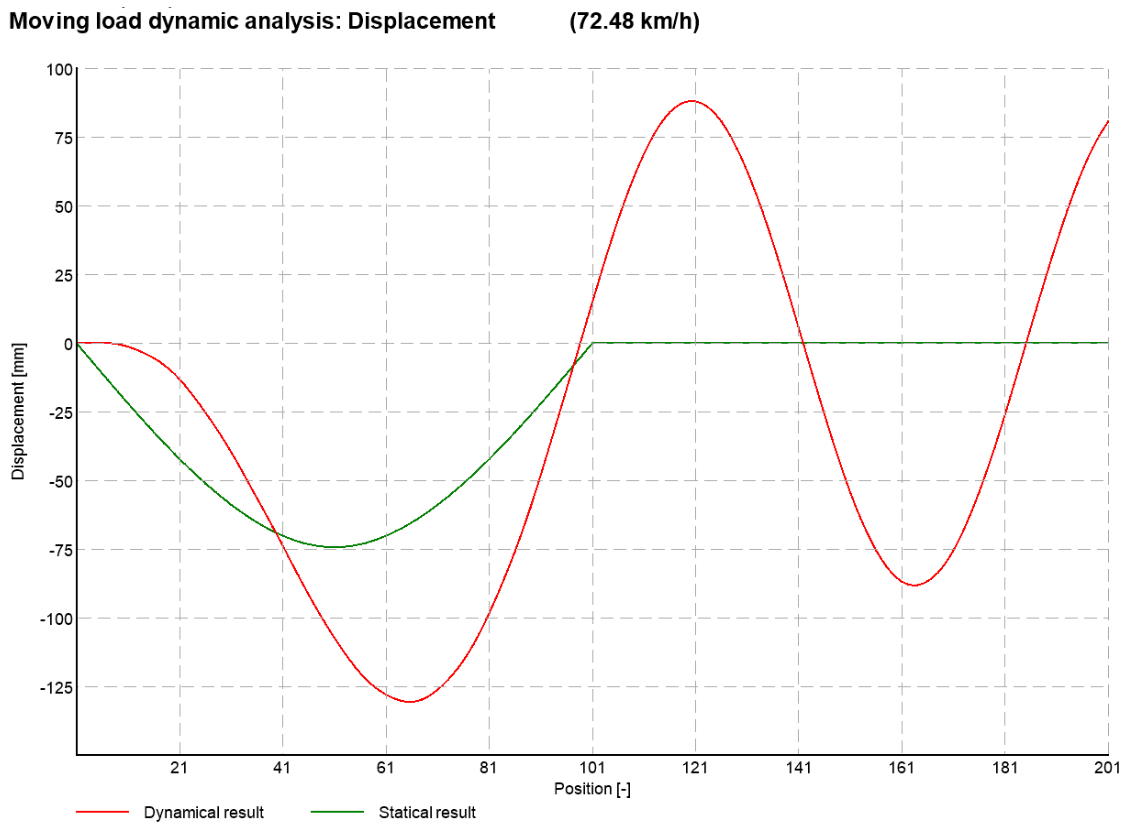


Figure 8.5.8 – The dynamic and static displacements of the mid-span in function of mass positions with v_3 velocity with FEM-Design

About Fig. 8.5.8 we can see the static and dynamic maximum deflection at the mid-span.

$$e_{dynamic\,max}^{FEM\ v_3} = 0.1310\,\mathrm{m}$$

We can say that there are good agreements between the results. On the FEM-Design dynamic deflection results we can see the free vibration behaviour of the structure as well.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst190x/models/8.5 Dynamic response of a moving mass point on a simply supported beam.str

8.6 Dynamic response of a moving single force on a simply supported beam

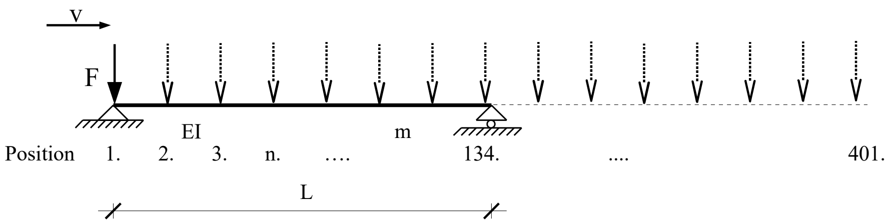


Figure 8.6.1 – The geometry and the moving load positions on the path during the calculation

Inputs:

Analyzed position	Mid-span of the beam
The distributed mass of the beam	μ = 7950 kg/m
The concentrated moving force	F = 480 kN
Bending stiffness	EI = $4.26{*}10^7$ kNm^2
Span length	L = 30 m
Analyzed constant velocities of the moving force	v_1 = 50 m/s = 180 km/h
	v_2 = 100 m/s = 360 km/h
Damping	ξ = 0 %
Path length	L_p = 90 m

In this calculation we neglect the mass effect of the moving force, because based on Ref. [19] we can get the analytical solution of the problem directly. Fig. 8.6.1 shows the problem with the considered moving force, positions and geometry.

The first eigen and angular natural frequency based on closed form with Bernoulli beam theory:

$$f_1=\frac{\pi}{2\cdot L^2}\sqrt{\frac{EI}{m}}=\frac{\pi}{2\cdot 30^2}\sqrt{\frac{4.26\cdot 10^7}{7.95}}=4.040\text{ Hz} \quad ; \quad \omega_{01}=2\pi f_1=25.38\frac{\text{rad}}{\text{s}}$$

The first eigen and angular natural frequency based on FEM-Design calculation with Timoshenko beam theory:

$$f_{FEM\ 1}=3.997\text{ Hz} \quad ; \quad \omega_{FEM\ 01}=2\pi f_1=25.11\frac{\text{rad}}{\text{s}}$$

The analytical solution of the deflection of the beam in function of time and load position according to Ref. [19] if the moving force is on the structure can be calculated with the following formula:

$$e(x,t)=\sum_{r=1}^{\infty} v_r(x)\eta_r(t)$$

where:

$$v_r(x)=\sqrt{\frac{2}{\mu L}}\sin\frac{r\pi x}{L} \quad ; \quad \eta_r(t)=\frac{F}{\omega_{0r}}\sqrt{\frac{2}{\mu L}}\frac{1}{\left(\frac{r\pi v}{L}\right)^2-\omega_{0r}^2}\left(\frac{r\pi v}{L}\sin\omega_{0r}t-\omega_{0r}\sin\frac{r\pi v}{L}t\right)$$

In addition to the above mentioned parameters, r is the number of the eigenfrequencies.

At the midspan only the odd number of eigenfrequencies have effect on the final solution. In this case the dominant eigenfrequency is the first one, thus at the analytical solution we will use only the first eigenfrequency by the summation. In the complete analytical solution all of the eigenfrequencies have effect on the final solution, but in this case the rest of them have neglectable influence.

If $0 < t < L/v$ the approximate deflection at the mid-span in function of time:

$$e\left(\frac{L}{2},t\right)=\frac{2}{\mu L}\frac{F}{\omega_{01}}\frac{1}{\left(\frac{1\pi v}{L}\right)^2-\omega_{01}^2}\left(\frac{1\pi v}{L}\sin\omega_{01}t-\omega_{01}\sin\frac{1\pi v}{L}t\right)$$

The derivative of this function (velocity):

$$\dot{e}\left(\frac{L}{2},t\right)=\frac{2}{\mu L}\frac{F}{\omega_{01}}\frac{1}{\left(\frac{1\pi v}{L}\right)^2-\omega_{01}^2}\frac{\omega_{01}\pi v}{L}\left(\cos\omega_{01}t-\cos\frac{\pi v}{L}t\right)$$

In that time value when the force is leaving the structure the initial values of the remaining free vibration part of the solution:

In case of v_1:

$$t_1=\frac{L}{v_1}=\frac{30}{50}=0.6\,\mathrm{s}$$

Displacement:

$$e_{\mathrm{init1}}=e\left(\frac{L}{2},0.6\right)=\frac{2}{7950\cdot 30}\frac{480000}{25.38}\frac{1}{\left(\frac{1\pi 50}{30}\right)^2-25.38^2}\left(\frac{1\pi 50}{30}\sin 25.38\cdot 0.6-25.38\sin\frac{1\pi 50}{30}\cdot 0.6\right)=$$
$$=-0.0006217\,\mathrm{m}$$

Velocity:

$$\dot{e}_{\mathrm{init1}}=\dot{e}\left(\frac{L}{2},0.6\right)=\frac{2}{7950\cdot 30}\frac{480000}{25.38}\frac{1}{\left(\frac{1\pi 50}{30}\right)^2-25.38^2}\frac{25.38\cdot\pi 50}{30}\left(\cos 25.38\cdot 0.6-\cos\frac{\pi 50}{30}0.6\right)$$
$$=-0.003861\,\mathrm{m/s}$$

In case of v_2:

$$t_2=\frac{L}{v_2}=\frac{30}{100}=0.3\,\mathrm{s}$$

Displacement:

$$e_{\mathrm{init2}}=e\left(\frac{L}{2},0.3\right)=-0.0030183\,\mathrm{m}$$

Velocity:

$$\dot{e}_{\mathrm{init2}}=\dot{e}\left(\frac{L}{2},0.3\right)=-0.097608\,\mathrm{m/s}$$

After that the moving force has just reached the end of the beam the dynamic behaviour of the beam will be a free vibration (if no damping is considered). With the mentioned initial values based on that time value when the force will be at the end of the beam the function of the free vibration can be given:

If $L/v < t$ the approximate deflection at the mid-span in function of time:

$$e\left(\frac{L}{2},t\right)=e_{\mathrm{init}}\cos\left[\omega_{01}\left(t-\frac{L}{v}\right)\right]+\frac{\dot{e}_{\mathrm{init}}}{\omega_{01}}\sin\left[\omega_{01}\left(t-\frac{L}{v}\right)\right]$$

Fig. 8.6.2 shows the deflection function of the given analytical solution with the two different velocities.

Fig. 8.6.3-4 shows the dynamic and static deflection of the mid-span under the moving load based on FEM-Design calculation with the two different velocities.

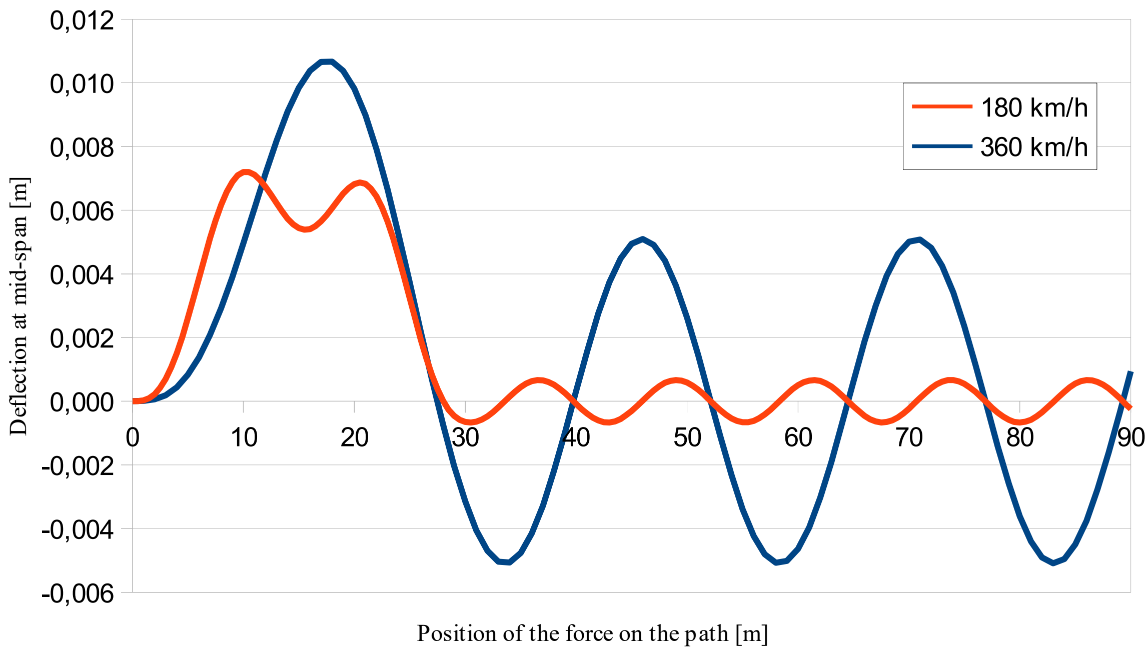


Figure 8.6.2 – The dynamic deflection of the mid-span with the two given velocities according to the analytical solution

We can see in Fig. 8.6.2-4 that after the position when the load has just left the beam the dynamic response is free vibration.

According to the given solutions we can say that the FEM-Design result are identical with the analytical solution.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst190x/models/8.6 Dynamic response of a moving single force on a simply supported beam.str

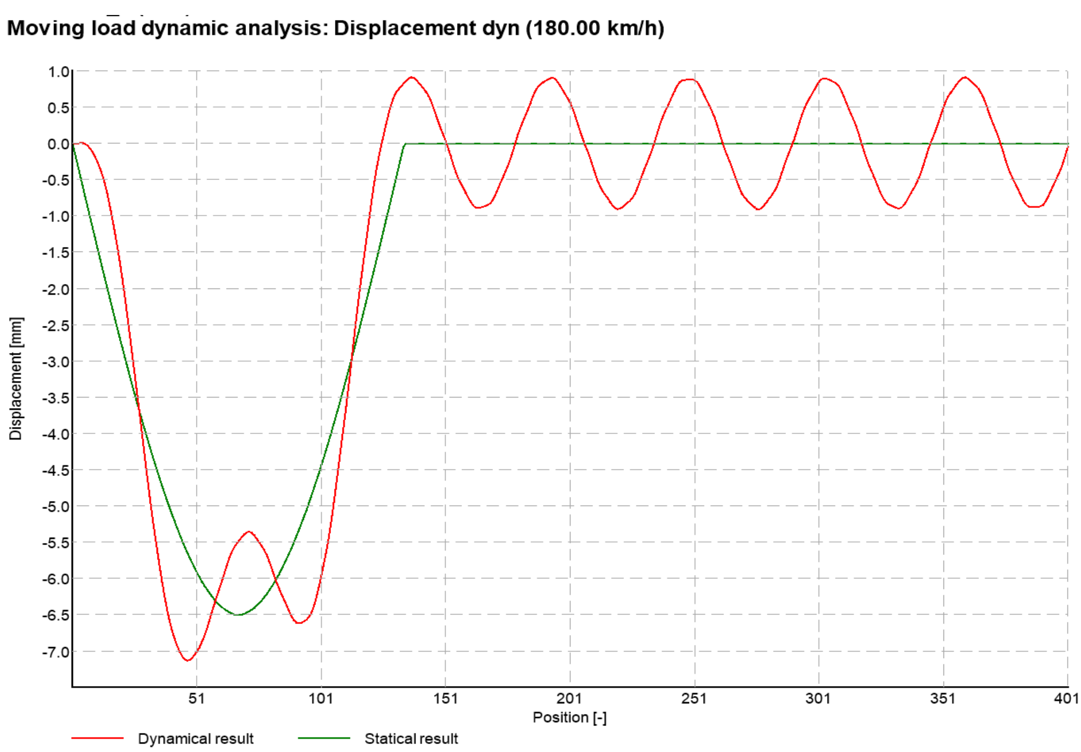


Figure 8.6.3 – The dynamic and static displacements of the mid-span in function of force positions with v_1 velocity with FEM-Design

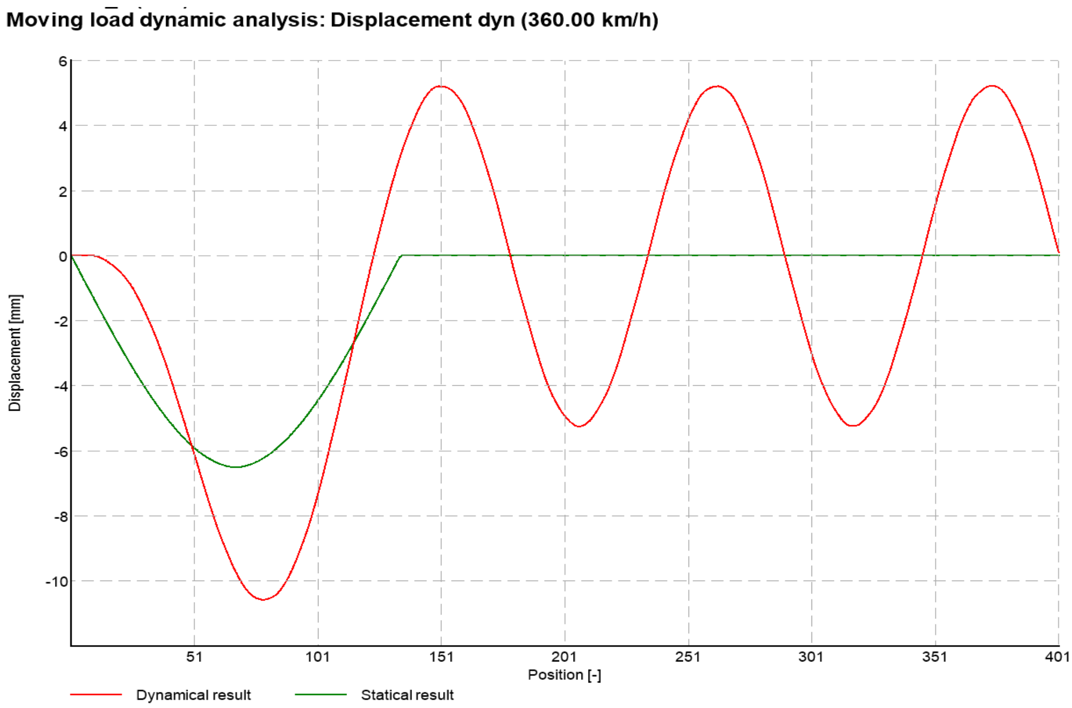


Figure 8.6.4 – The dynamic and static displacements of the mid-span in function of force positions with v_2 velocity with FEM-Design

8.7 Dynamic response of a moving force group on a simply supported beam

Inputs:

Analyzed position	Mid-span of the beam
The distributed mass of the beam	m = 7950 kg/m
Bending stiffness	$EI = 4.26*10^7$ kNm^2
Span length	L = 30 m
Analyzed velocities of the moving force groups	Case a): v_1 = 50 m/s = 180 km/h
	Case a): v_2 = 100 m/s = 360 km/h
	Case b): v_1 = 100 m/s = 360 km/h
	Case b): v_2 = 120 m/s = 432 km/h
Critical damping ratio	ξ = 1.5 % = 0.015
Path length	L_p = 500 m

In this calculation we neglect the mass effect of the moving force, because based on Ref. [19] we have benchmark results to compare our FEM-Design results. In this example the structure is the same as in the previous example, only the moving load differs from it.

In this example we considered two different load groups as the moving loads. Fig. 8.7.1 top shows case a) and bottom shows case b). In both cases we modeled 12 motor trains group as the moving load. Case a) includes 12 individual concentrated loads (12 x 544 kN) to consider the weights of the train cars. In case b) we considered that every train cars have 4 axes, thus the load group contains 12 x 4 x 136 kN. The resultants of the load groups are the same.

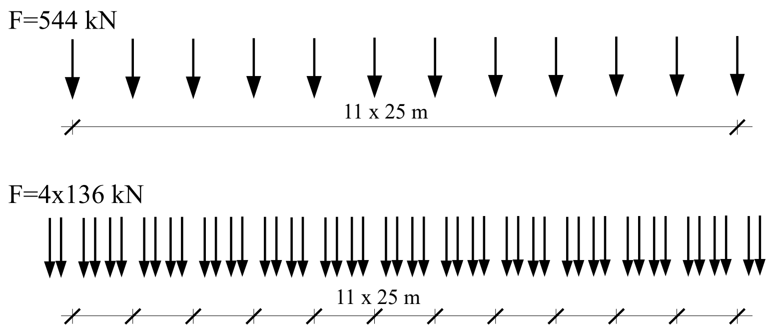


Figure 8.7.1 – The considered load groups to the dynamic analysis

Thus in these examples case a) and case b) load groups will be move on the path.

In this example we considered damping based on the equivalent Kelvin-Voigt damping.

Considering the given critical damping ratio and the first eigenfrequency of the given structure (based on the previous example) the Rayleigh damping parameters are as follows:

$$\alpha=0; \quad \beta=\frac{\xi}{\pi f_1}=\frac{0.015}{\pi 3.997}=0.001195$$

With the given input parameters Fig. 8.7.2 shows the dynamic response of the mid-span in function of the load position based on Ref. [19] in case a) load groups with $v_1 = 50$ m/s = 180 km/h and $v_2 = 100$ m/s = 360 km/h.

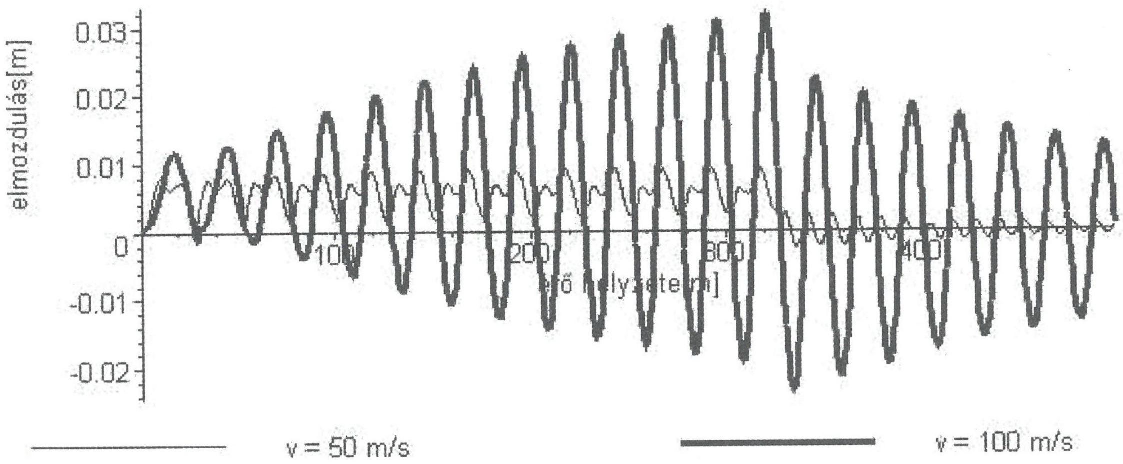


Figure 8.7.2 – The dynamic deflection of the mid-span based on Ref. [19] *in case a) with the two velocities*

Fig. 8.7.3-4 shows the dynamic response of mid-span in function of the load positions based on the FEM-Design moving load dynamic calculation in case a) load group.

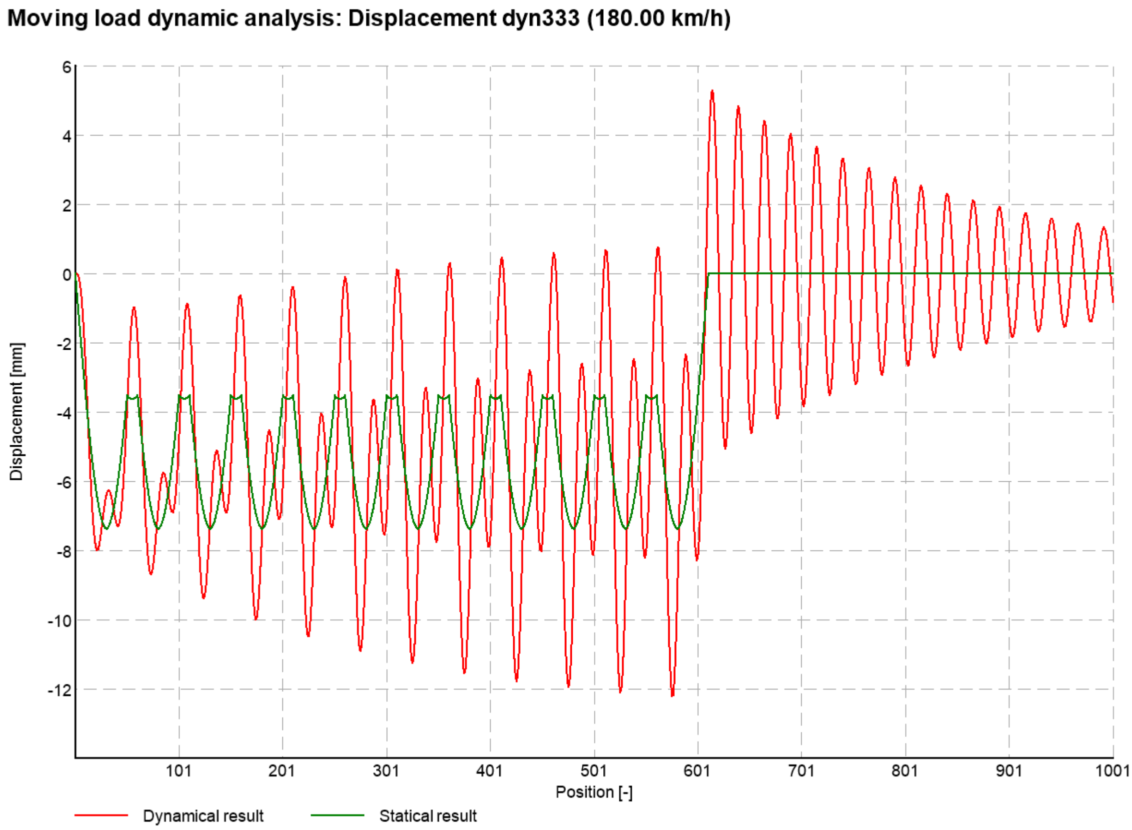


Figure 8.7.3 – The dynamic and static displacements of the mid-span in function of force positions with v_1 velocity in FEM-Design in case a)

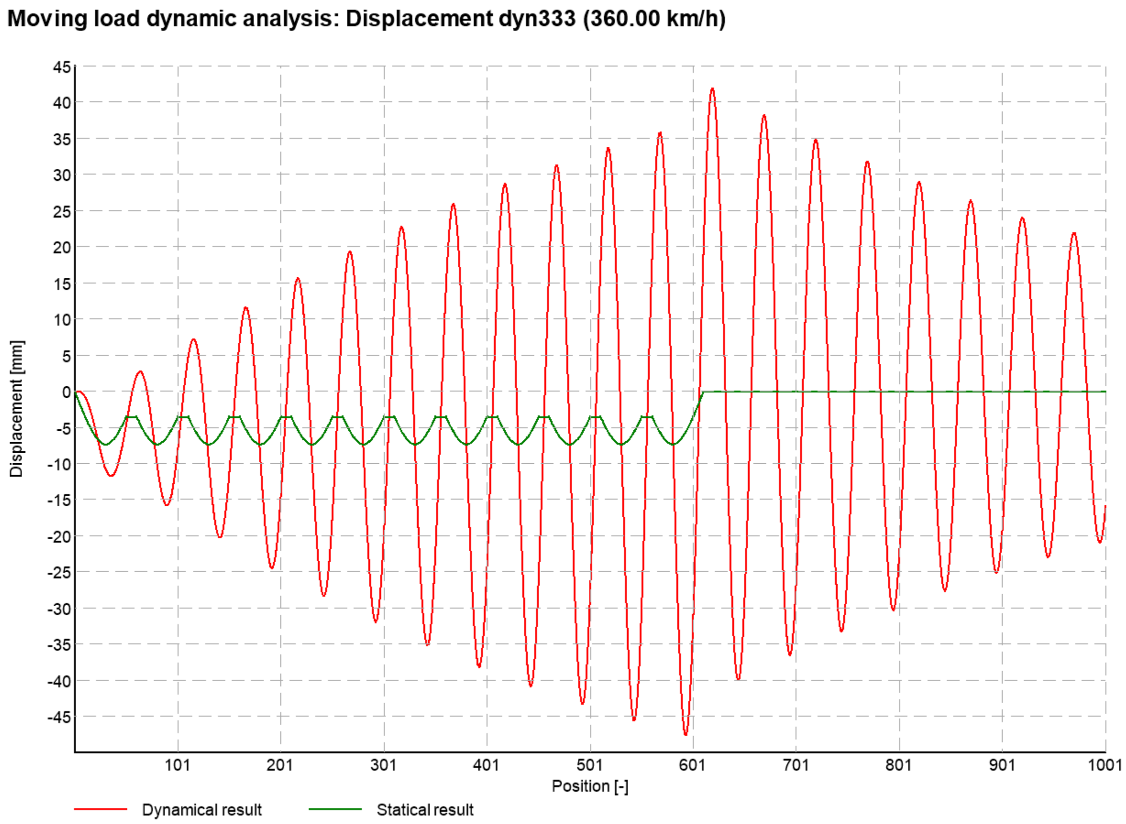


Figure 8.7.4 – The dynamic and static displacements of the mid-span in function of force positions with v_2 velocity in FEM-Design in case a)

We can say that the results are very close to each other.

Fig. 8.7.5 shows the dynamic response of the mid-span in function of the load position based on Ref. [19] in case b) load groups with $v_1 = 100$ m/s = 360 km/h and $v_2 = 120$ m/s = 432 km/h.

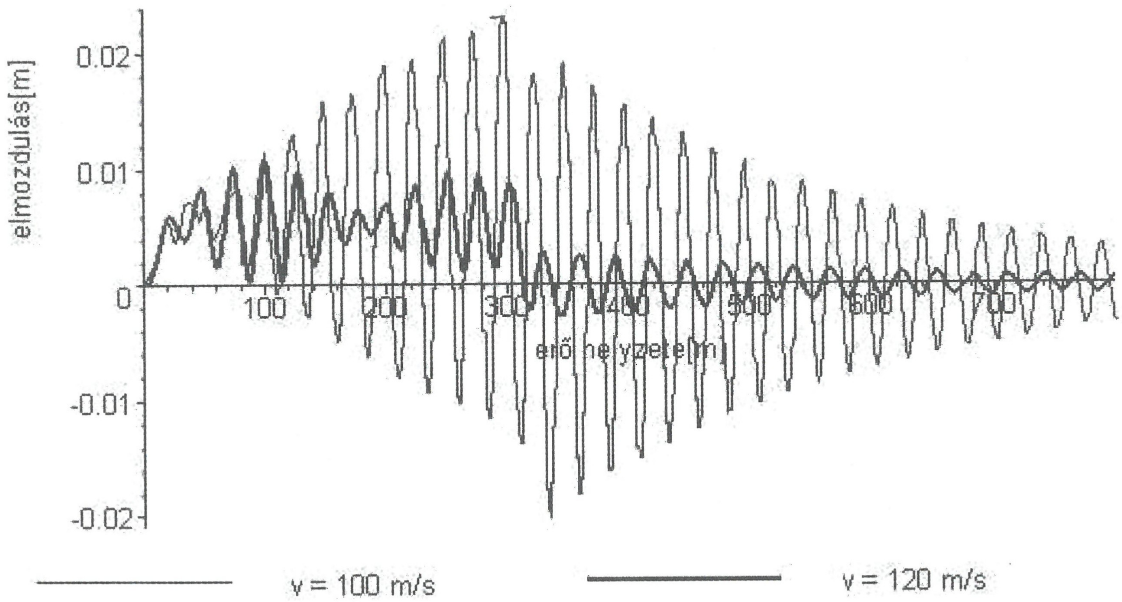


Figure 8.7.5 – The dynamic deflection of the mid-span based on Ref. [19] *in case b) with the two velocities*

Fig. 8.7.6-7 shows the dynamic response of mid-span in function of the load positions based on the FEM-Design moving load dynamic calculation in case b) load groups.

We can say that the results are very close to each other.

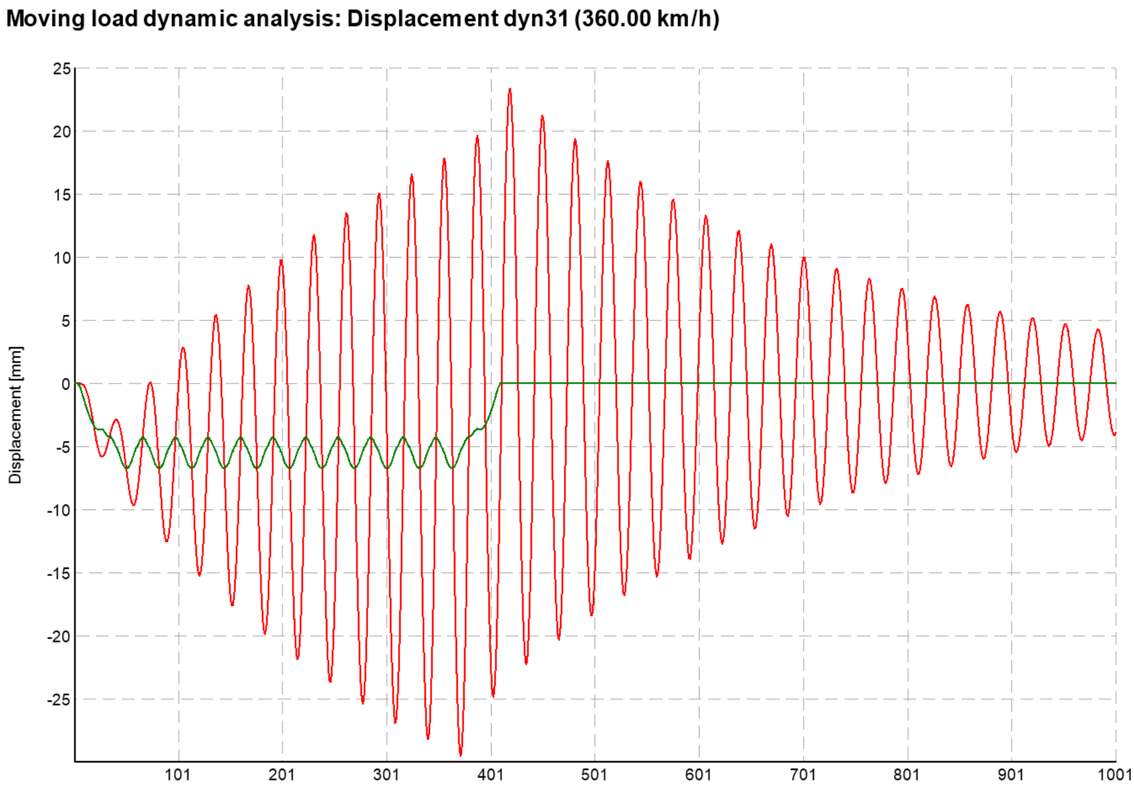


Figure 8.7.6 – The dynamic and static displacements of the mid-span in function of force positions with v_1 velocity in FEM-Design in case b)

Download link to the example file:
Case a):
http://download.strusoft.com/FEM-Design/inst190x/models/8.7 Dynamic response of a moving force group on a simply supported beam case a.str

Case b):
http://download.strusoft.com/FEM-Design/inst190x/models/8.7 Dynamic response of a moving force group on a simply supported beam case b.str

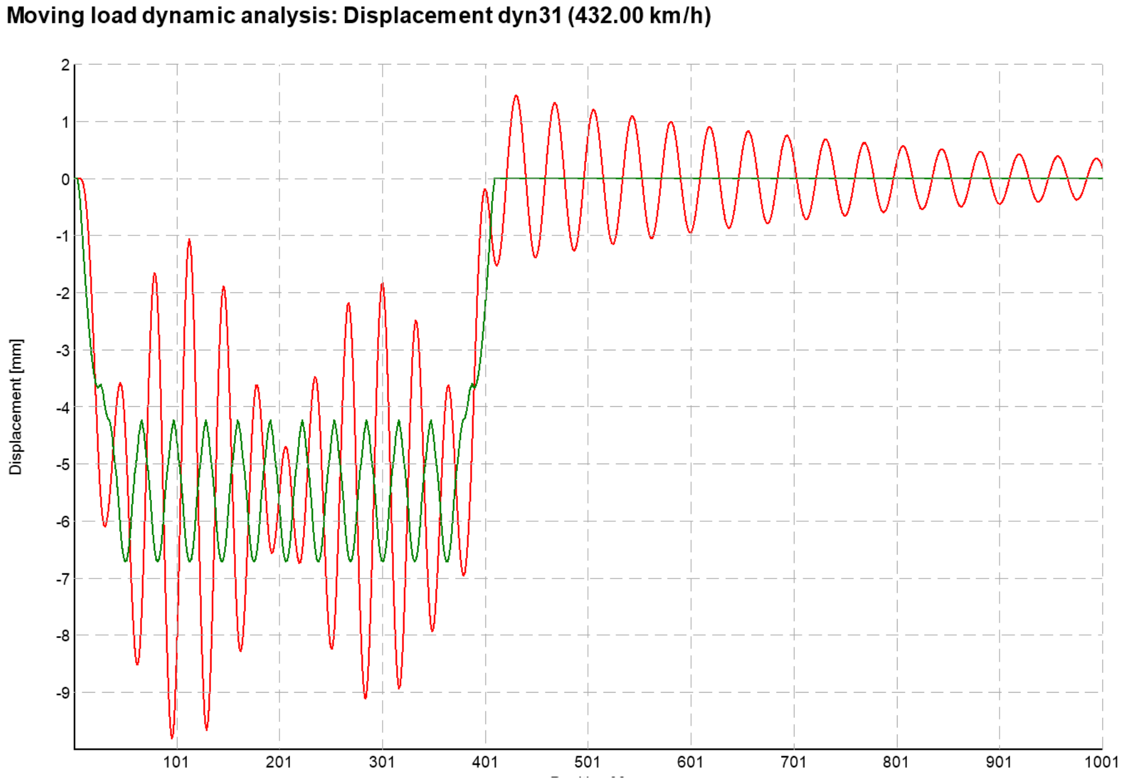


Figure 8.7.7 – The dynamic and static displacements of the mid-span in function of force positions with v_2 velocity in FEM-Design in case b)

9 Design calculations

This chapter is unfinished.

According to Eurocode standard!

9.1 Foundation design

This chapter is unfinished.

9.1.1 Design of an isolated foundation

This chapter is unfinished.

9.1.2 Design of a wall foundation

This chapter is unfinished.

9.1.3 Design of a foundation slab

This chapter is unfinished.

9.2 Reinforced concrete design

In this chapter we will show some detailed verification calculations regarding to reinforced concrete design according to EN 1992-1-1.

9.2.1 Moment capacity calculation for beams under pure bending

In this chapter we calculate the moment capacity of reinforced concrete cross sections under pure bending (uniaxial bending).

The first example will be an *under-reinforced* cross section. The second one will be a *normal-reinforced* and the third one an *over-reinforced* section. After independent "hand" calculations we compare the results with FEM-Design values.

The following input parameters are common for the different cross sections regarding to this subchapter.

Inputs:

Concrete characteristic compressive strength	$f_{ck} = 30$ N/mm^2 or 20 N/mm^2
The end of the parabolic part (material model, see EC-2)	$\varepsilon_{c2} = 0.20$ %
Ultimate limit strain of concrete (see EC-2)	$\varepsilon_{cu2} = 0.35$ %
Partial factor of concrete	$\gamma_c = 1.50$
Reinforcing steel characteristic yield strength	$f_{yk} = 420$ N/mm^2
Elastic modulus of reinforcing steel	$E_s = 200$ GPa
Ultimate limit strain of reinforcing steel	$\varepsilon_{uk} = 2.5$ %
Partial factor of reinforcing steel	$\gamma_s = 1.15$
Behaviour of plastic part (see Fig. 9.2.1.1.4)	$k = 1.05$
Concrete cover (on stirrups)	$c = 20$ mm
Stirrup diameter	$\phi_s = 8$ mm

The external dimensions are the same for every cross section (b = 300 mm, h = 500 mm).

9.2.1.1 Under-reinforced cross section

In this example we put two longitudinal rebars with 16 mm diameter at the bottom left and right corner of the strirrups (see Fig. 9.2.1.1.1, the concrete is C30/37). We neglect the effect of the hangers. Two different "hand" calculation methods provided here. First of all with aim of the most simple material models for concrete and reinforcing steel and secondly with improved material models, which FEM-Design uses also.

First we calculate the moment capacity with the following material models (see Fig. 9.2.1.1.2).

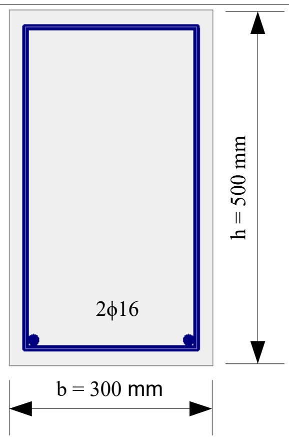


Figure 9.2.1.1.1 – The cross section of the under-reinforced case

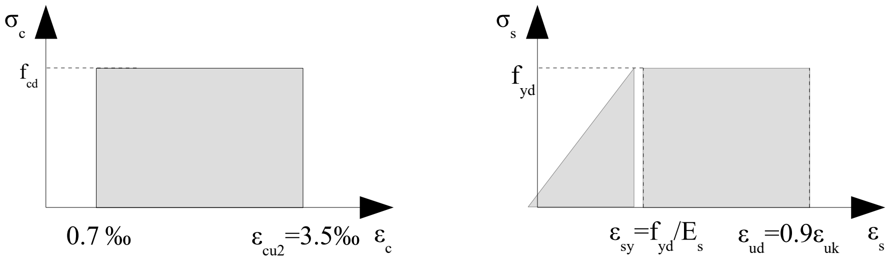


Figure 9.2.1.1.2 – The material models for the first calculation
left: concrete (only compression), right: rebars (both tension and compression)

Due to the under-reinforced section behaviour we assume that the rebars strains are at the design ultimate limit strain value.

Based on the sum of the forces we can get the height of the active compression concrete zone (see Fig. 9.2.1.1.3).

$$b\,x_c\,f_{cd} - A_s\,f_{yd} = 0$$

The compressed zone:

$$x_c = \frac{A_s\,f_{yd}}{b\,f_{cd}} = \frac{2\cdot\frac{16^2\cdot\pi}{4}\cdot\frac{420}{1.15}}{300\cdot\frac{30}{1.5}} = 24.48\,\text{mm}$$

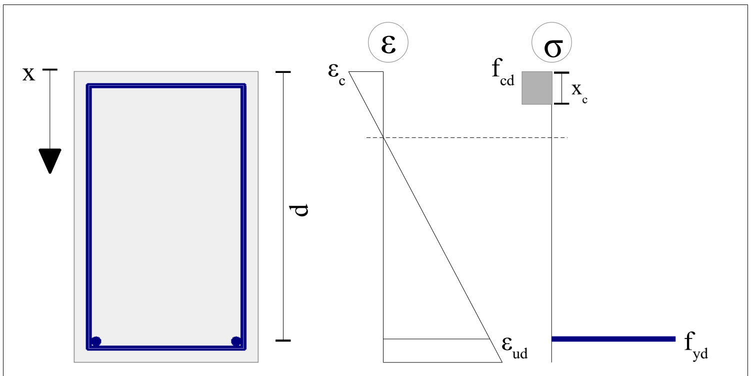


Figure 9.2.1.1.3 – The assumed strains and stresses

We need to check that the assumption of the rebar strains were proper or not.

Based on the assumption that the concrete reaches its ultimate compression strain limit then the strain in the rebars (see Fig. 9.2.1.1.3):

$$\varepsilon_s = \frac{(d - 1.25\, x_c)\,\varepsilon_{cu2}}{1.25\, x_c} = \frac{((500-20-8-8)-1.25\cdot 24.48)\cdot 0.0035}{1.25\cdot 24.48} = 4.957\,\% > \varepsilon_{ud} = 2.25\,\%$$

Thus the maximum concrete strain cannot be ε_{cu2}= 0.35%, due to this the rebars reach the ultimate strain, thus the assumption was correct, the failure mode is the rupture of the rebars (under-reinforced section). But it has no effect on the former equilibrium equation.

The moment capacity of the section with the simple material models:

$$M_{Rd,1} = b\, x_c\, f_{cd}\left(d - \frac{x_c}{2}\right) = 300\cdot 24.48\cdot \frac{30}{1.5}\cdot\left(464 - \frac{24.48}{2}\right) = 66.35\ \text{kNm}$$

Secondly we calculate with improved material models, see the following equations.

Concrete (see Fig. 9.2.1.1.4 left side):

$$\sigma_c(\varepsilon_c) = f_{cd}\left[1 - \left(1 - \frac{\varepsilon_c}{\varepsilon_{c2}}\right)^2\right] \quad \text{if } 0 \le \varepsilon_c \le \varepsilon_{c2}$$

$$\sigma_c(\varepsilon_c) = f_{cd} \quad \text{if } \varepsilon_{c2} < \varepsilon_c \le \varepsilon_{cu2}$$

Rebars (see Fig. 9.2.1.1.4 right side):

$$\sigma_s(\varepsilon_s)=\varepsilon_s E_s \qquad if \quad \varepsilon_s \le \frac{f_{yd}}{E_s}$$

$$\sigma_s(\varepsilon_s)=f_{yd}+(k-1)f_{yd}\frac{\varepsilon_s-\frac{f_{yd}}{E_s}}{\varepsilon_{ud}-\frac{f_{yd}}{E_s}} \qquad if \quad \frac{f_{yd}}{E_s}<\varepsilon_s\le\varepsilon_{ud}$$

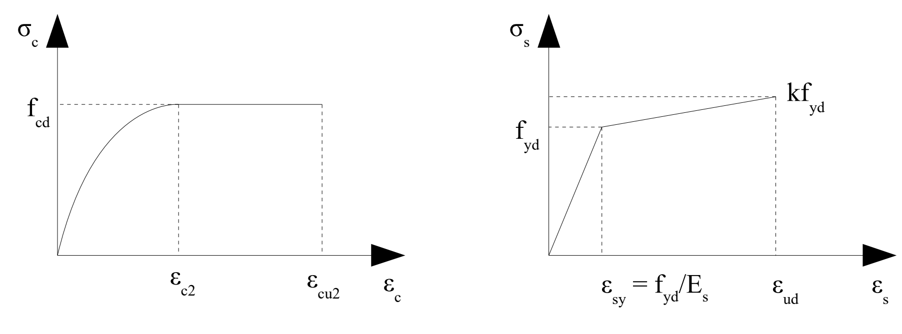


Figure 9.2.1.1.4 – The material models for the second calculation
left: *concrete (only compression),* right: *rebar (both tension and compression)*

The assumed stress and strain distributions (Fig. 9.2.1.1.5).

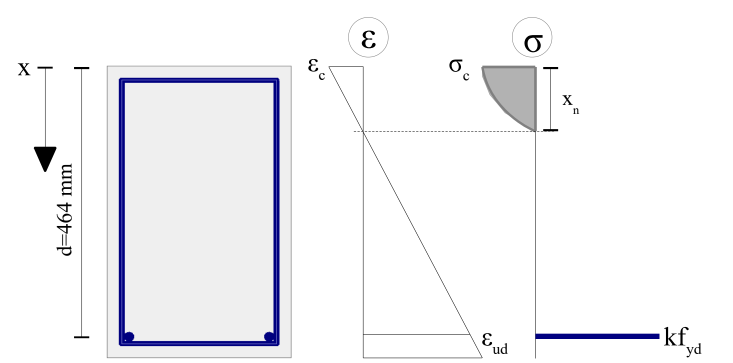


Figure 9.2.1.1.5 – The assumed strains and the stresses

The concrete strains depends on the curvature and based on the improved material models this led to a nonlinear equations.

$$\varepsilon(x)=\kappa\cdot(x-x_n)$$

The sum of the forces:

$$N_c+N_s=0$$

Resultant force in concrete:

$$N_c=b\int_0^h \sigma_c(\varepsilon_c)dx$$

Resultant force in rebars:

$$N_s = \sum_{i=1}^{2} A_{si} \sigma_{si}(\varepsilon_{si})$$

We solved the equation system with independent numerical method as a “hand” calculation.

The position of the neutral axis:

$$x_n = 38.18\,\mathrm{mm}$$

The stress and strain values which belong to the equilibrium state are shown in Fig. 9.2.1.1.6.

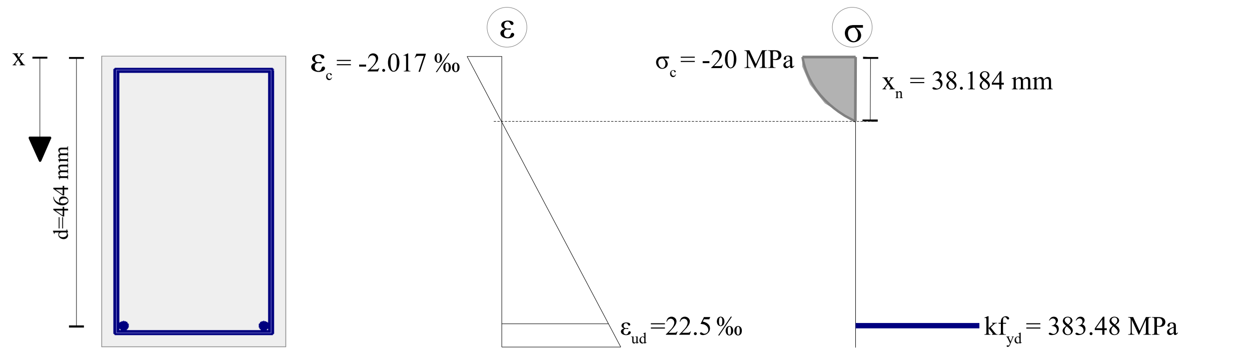


Figure 9.2.1.1.6 – The characteristic values of strains and stresses

The resultant of the concrete stress volume according to the independent “hand” calculation:

$$N_c = b \int_0^h \sigma_c(\varepsilon_c)\,dx = -153.4\,\mathrm{kN}$$

The resultant force in the rebars:

$$N_s = \sum_{i=1}^{2} A_{si} \sigma_{si}(\varepsilon_{si}) = 2\,\frac{16^2 \cdot \pi}{4}\,\frac{1.05 \cdot 420}{1.15} = 154.2\,\mathrm{kN}$$

The difference between the compression and tension forces is less than 1% therefore this is the correct position of the neutral axis and curvature according to the improved material models.

The centroid of the concrete stress volume measured from the top of the section:

$$x_c = \frac{b \int_0^h x\,\sigma_c(\varepsilon_c)\,dx}{N_c} = 14.34\,\mathrm{mm}$$

The moment around neutral axis provided by the concrete:

$$M_c = N_c(x_n - x_c) = 153.4 \cdot (38.18 - 14.34) = 3.657\,\text{kNm}$$

Rebars moment around neutral axis:

$$M_s = \sum_{i=1}^{2} A_{si}\sigma_{si}(\varepsilon_{si})(d_i - x_n) = 2 \cdot 201.1 \cdot 383.5(464 - 38.18) = 65.64\,\text{kNm}$$

The moment capacity with improved material models:

$$M_{Rd,2} = M_c + M_s = 3.657 + 65.64 = 69.30\,\text{kNm}$$

Ratio between the two hand calculations with the different material models:

$$\frac{M_{Rd,2}}{M_{Rd,1}} = \frac{69.30}{66.35} = 1.044$$

The moment capacity of the same section with FEM-Design (Fig. 9.2.1.1.7):

$$M_{Rd,FEM} = 68.98\,\text{kNm}$$

The difference between the two hand calculations is 4%.

The FEM-Design results of the stresses and strains are shown in Fig. 9.2.1.1.7. The strain values and neutral axis position value between the hand and FE calculations are under 3%, the moment capacity difference is less than 1%.

Sections	5
LC	1
N_{Ed} [kN]	0.00
$M_{Ed,1}$ [kNm]	68.98
$M_{Ed,2}$ [kNm]	0.00
Utilization [%]	100

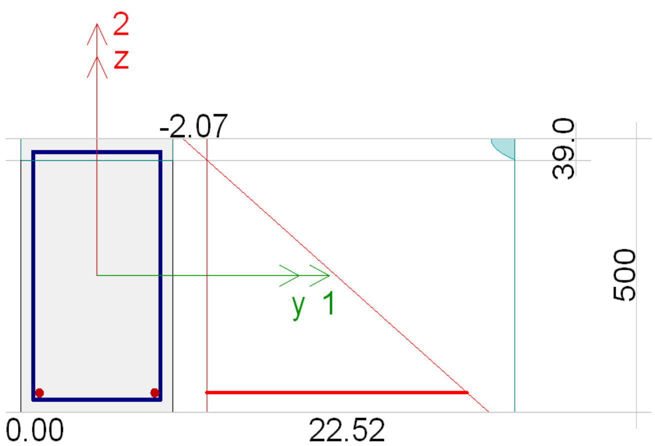


Figure 9.2.1.1.7 – The FEM-Design results (strains [‰], *neutral axis* [mm]*)*

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst180x/models/9.2.1 Moment capacity calculation for beams under pure bending.str

9.2.1.2 Normal-reinforced cross section

Here we put 6 longitudinal rebars with 16 mm diameter at the bottom of the cross section (see Fig. 9.2.1.1.8, the concrete is C30/37). We neglect the effect of the hangers. The following verification calculations will be performed with the improved material models (see Chapter 9.2.1.1).

The assumed stress and strain distributions in the section are shown in Fig. 9.2.1.1.8.

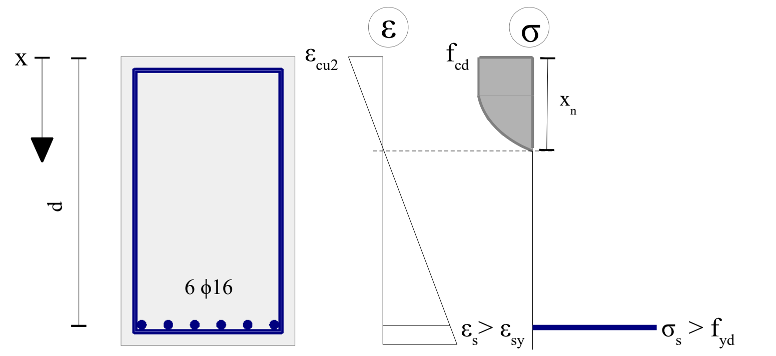


Figure 9.2.1.1.8 – Cross section and assumed strains and stresses

The concrete strain depends on the curvature and based on the improved material models this led to a nonlinear equation.

$$\varepsilon(x)=\kappa\cdot(x-x_n)$$

The sum of the forces:

$$N_c+N_s=0$$

Resultant force in concrete:

$$N_c=b\int_0^h \sigma_c(\varepsilon_c)\,dx$$

Resultant force in rebars:

$$N_s=\sum_{i=1}^{6} A_{si}\sigma_{si}(\varepsilon_{si})$$

We solved the equation system with independent numerical method as a “hand” calculation.

The neutral axis position is:

$$x_n=93.06\,\mathrm{mm}$$

The stress and strain values which belong to the equilibrium state are shown in Fig. 9.2.1.1.9.

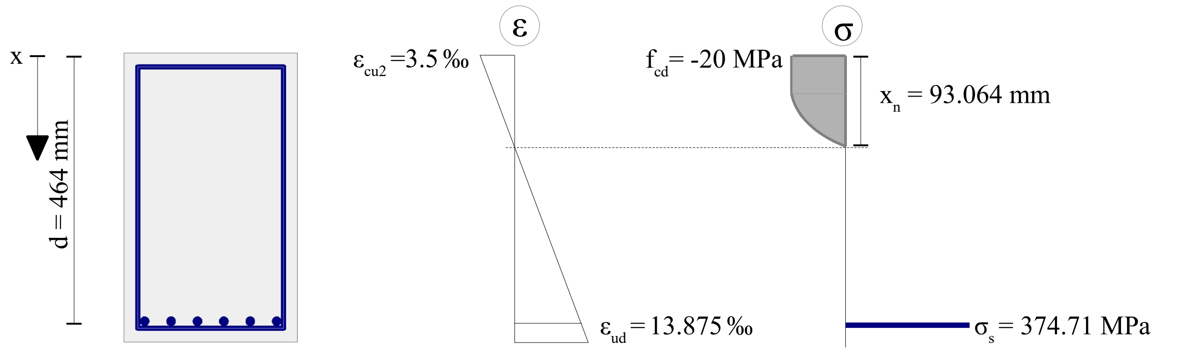


Figure 9.2.1.1.9 – The calculated strains and the stresses

The resultant of the concrete stress volume according to the independent "hand" calculation:

$$N_c = b\int_0^h \sigma_c(\varepsilon_c)\,dx = -452.03\,\text{kN}$$

The resultant force in the rebars:

$$N_s = \sum_{i=1}^{6} A_{si}\sigma_{si}(\varepsilon_{si}) = 6\frac{16^2\cdot\pi}{4}374.71 = 452.04\,\text{kN}$$

The difference between the compression and tension forces is less than 1% therefore this is the correct position of the neutral axis and curvature according to the improved material models.

The centroid of the concrete stress volume measured from the top of the section:

$$x_c = \frac{b\int_0^h x\,\sigma_c(\varepsilon_c)\,dx}{N_c} = 38.71\,\text{mm}$$

The moment around neutral axis provided by the concrete:

$$M_c = N_c(x_n - x_c) = 452.03(93.06 - 38.71) = 24.57\,\text{kNm}$$

Rebars moment around neutral axis:

$$M_s = \sum_{i=1}^{6} A_{si}\sigma_{si}(\varepsilon_{si})(d_i - x_n) = 6\cdot 201.1\cdot 374.7(464 - 93.06) = 166.8\,\text{kNm}$$

The moment capacity with improved material models:

$$M_{Rd,2} = M_c + M_s = 24.57 + 166.8 = 191.4\,\text{kNm}$$

The moment capacity with FEM-Design (Fig. 9.2.1.1.10):

$M_{Rd,FEM} = 191.0\,\text{kNm}$

The FEM-Design results of the stresses and strains are shown in Fig. 9.2.1.1.10. The strain values and neutral axis position value difference between the hand and FE calculation are less than 4%, the moment capacity difference is less than 1%.

Sections	13
LC	1
N_{Ed} [kN]	0.00
$M_{Ed,1}$ [kNm]	191.00
$M_{Ed,2}$ [kNm]	0.00
Utilization [%]	100

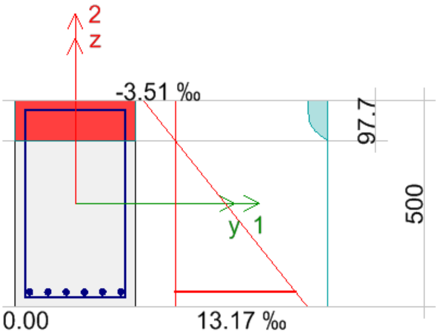


Figure 9.2.1.1.10 – The FEM-Design results (strains [‰], *neutral axis* [mm]*)*

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst180x/models/9.2.1 Moment capacity calculation for beams under pure bending.str

9.2.1.3 Over-reinforced cross section

We put 12 longitudinal rebars with 20 mm diameter at the bottom (see Fig. 9.2.1.1.11, the concrete is C20/25). We neglect the effect of the hangers. The following verification calculations will be performed with the improved material models (see Chapter 9.2.1.1).

The assumed stress and strain distributions in the section are shown in Fig. 9.2.1.1.11.

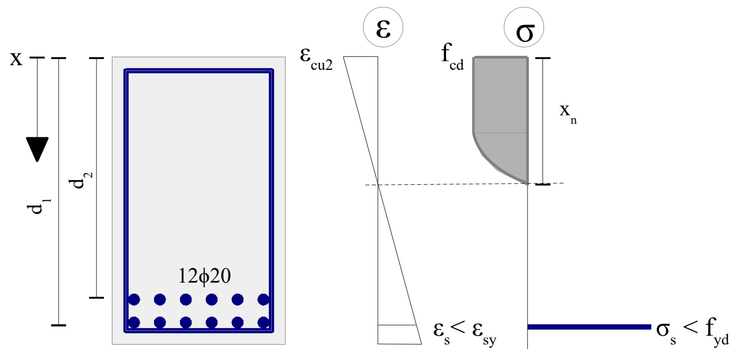


Figure 9.2.1.1.11 – Cross section and assumed strains and the stresses

The concrete strain depends on the curvature and based on the improved material models this led to a nonlinear equation.

$$\varepsilon(x) = \kappa \cdot (x - x_n)$$

The sum of the forces:

$$N_c + N_s = 0$$

Resultant force in concrete:

$$N_c = \int_0^h b\,\sigma_c(\varepsilon_c)\,dx$$

Resultant force in rebars:

$$N_s = \sum_{i=1}^{12} A_{si}\,\sigma_{si}(\varepsilon_{si})$$

We solved the equation system with independent numerical method as a “hand” calculation.

The neutral axis position is:

$$x_n = 317.95\,\mathrm{mm}$$

The stress and strain values which belong to the equilibrium state are shown in Fig. 9.2.1.1.12.

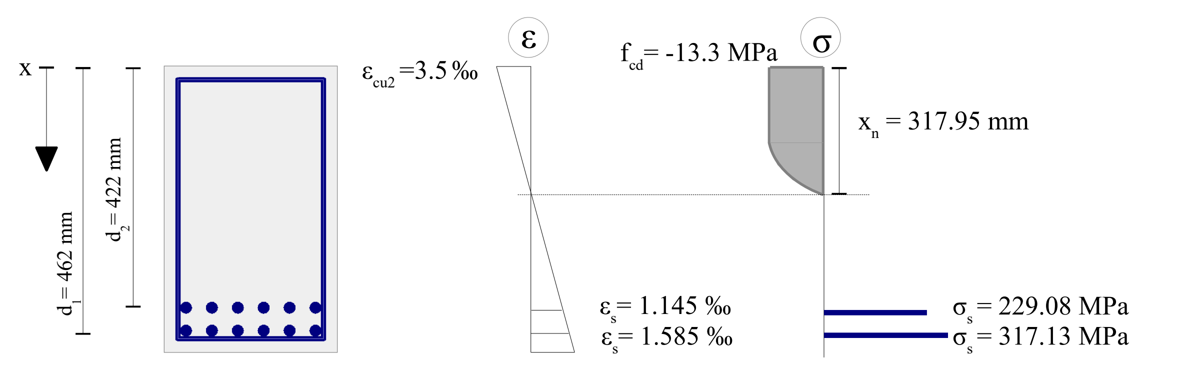


Figure 9.2.1.1.12 – The calculated strains and the stresses

The resultant of the concrete stress volume according to the independent "hand" calculation:

$$N_c = b\int_0^h \sigma_c(\varepsilon_c)\,dx = -1029.6\,\text{kN}$$

The resultant force in the rebars:

$$N_s = \sum_{i=1}^{12} A_{si}\sigma_{si}(\varepsilon_{si}) = 6\frac{20^2\cdot\pi}{4}\cdot 317.13 + 6\frac{20^2\cdot\pi}{4}\cdot 229.08 = 1029.06\,\text{kN}$$

The difference between the compression and tension forces less than 1% therefore this is the correct position of the neutral axis and curvature according to the improved material models.

Centroid of the concrete stress volume measured from the top of the section:

$$x_c = \frac{\int_0^h b\,x\,\sigma_c(\varepsilon_c)\,dx}{N_c} = 132.3\text{ mm}$$

The moment around neutral axis provided by the concrete:

$$M_c = N_c(x_n - x_c) = 1029.6(317.95 - 132.3) = 191.1\,\text{kNm}$$

Rebars moment around neutral axis:

$$M_s = \sum_{i=1}^{12} A_{si}\sigma_{si}(\varepsilon_{si})(d_i - x_n) = 6\cdot 314.1[317.1(462 - 317.95) + 229.1(422 - 317.95)] = 131.0\,\text{kNm}$$

The moment capacity with improved material models:

$$M_{Rd,2} = M_c + M_s = 191.1 + 131.1 = 322.2\,\text{kNm}$$

The moment capacity with FEM-Design (Fig. 9.2.1.1.13):

$M_{Rd,FEM} = 314.0\,\text{kNm}$

The FEM-Design results of the stresses and strains are shown in Fig. 9.2.1.1.13. The strain values and neutral axis position value difference between the hand and FE calculation are less than 2%, the moment capacity difference is less than 3%.

Sections	13
LC	1
N_{Ed} [kN]	0.00
$M_{Ed,1}$ [kNm]	314.00
$M_{Ed,2}$ [kNm]	0.00
Utilization [%]	100

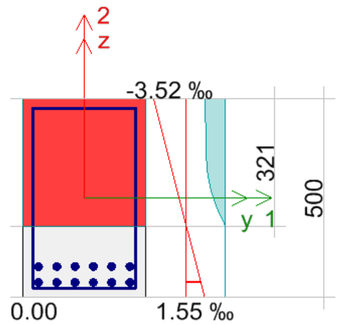


Figure 9.2.1.1.13 – The FEM-Design results (strains [‰], *neutral axis* [mm]*)*

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst180x/models/9.2.1 Moment capacity calculation for beams under pure bending.str

9.2.2 Required reinforcement calculation for a slab

In this example we calculate the required reinforcements of a slab due to *elliptic* and *hyperbolic* bending conditions. First of all the applied reinforcement is *orthogonal* and then the applied reinforcement is *non-orthogonal*. We calculate the required reinforcement with hand calculation and then compare the results with FEM-Design values.

Inputs:

The thickness	h = 200 mm
The elastic modulus of concrete	E_{cm} = 33 GPa, C30/37
The Poisson's ratio of concrete	ν = 0.2
The design value of compressive strength	f_{cd} = 20 MPa
Elastic modulus of steel bars	E_s = 200 GPa
The design value of yield stress of steel bars	f_{yd} = 434.8 MPa
Diameter of the longitudinal reinforcement	ϕ_l = 10 mm
Nominal concrete cover	c_x = 20 mm; c_y = 30 mm
Effective heights	d_x = 175 mm; d_y = 165 mm

9.2.2.1 Elliptic bending

In the first case the bending condition is an elliptic bending. In FEM-Design the model is a slab with statically determinant support system and specific moment loads at its edges for the pure internal force state (see Fig. 9.2.2.1).

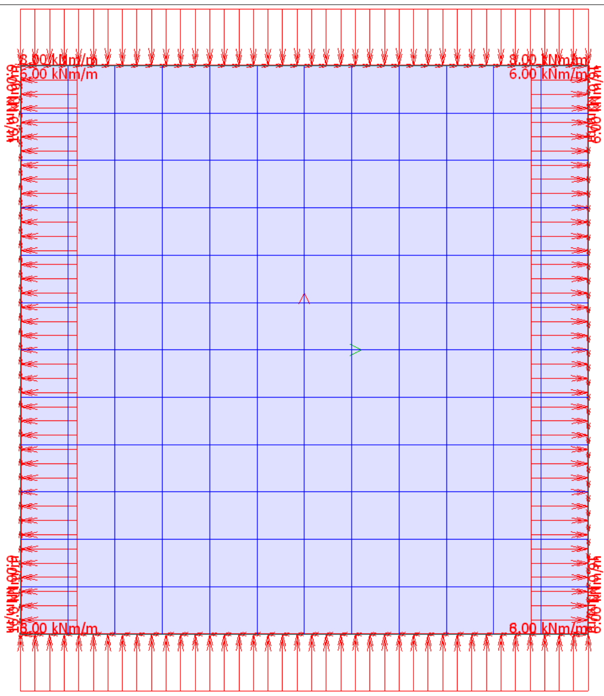


Figure 9.2.2.1 – The slab with the edge loads for pure stress state

Fig. 9.2.2.2 shows the constant internal forces in the slab due to the loads. Fig. 9.2.2.3 shows the principal moments and their directions based on the FEM-Design calculation. According to the pure stress state the principal moments and the directions are the same in each elements.

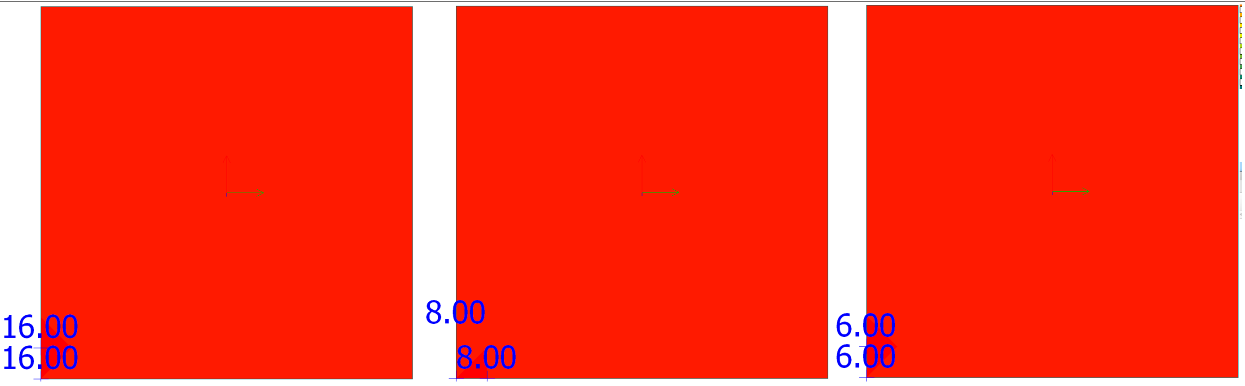


Figure 9.2.2.2 – The m_x, m_y, and m_{xy} internal forces in the slab [kNm]

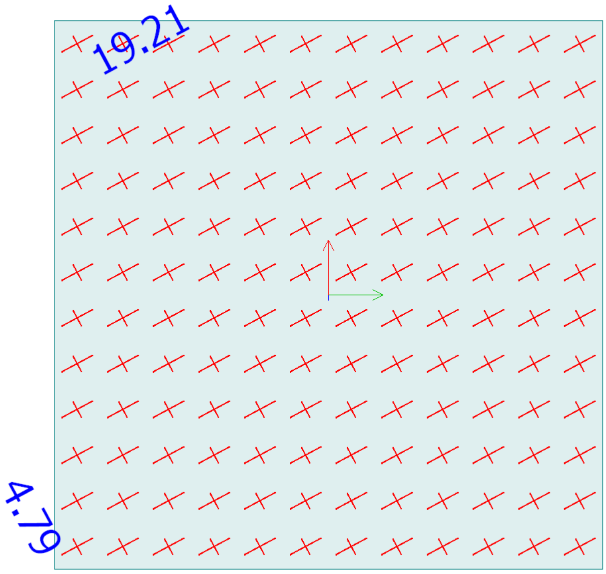


Figure 9.2.2.3 – The m_1 and m_2 principal moments and their directions in the slab [kNm]

First of all the reinforcement is orthogonal and the hand calculation and the comparison are the following:

1. Orthogonal reinforcement (φ=90°)

The reinforcement is orthogonal and their directions concide with the local system (x=ξ, y=ϑ=η).

The moments in the slab (tensor of the applied moments):

$$m_x = m_\xi = +16\,\mathrm{kNm/m}$$

$$m_y = m_\vartheta = m_\eta = +8\,\mathrm{kNm/m}$$

$$m_{xy} = m_{\xi\vartheta} = m_{\xi\eta} = +6\,\mathrm{kNm/m}$$

The first invariant of the tensor: $m_x + m_y = +24\,\mathrm{kNm/m}$

The calculation of the principal moments and their directions:

$$m_1 = \frac{m_x + m_y}{2} + \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^{\;2}} = \frac{16+8}{2} + \sqrt{\left(\frac{16-8}{2}\right)^2 + 6^2} = 19.21\,\mathrm{kNm/m}$$

$$m_2 = \frac{m_x + m_y}{2} - \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^{\;2}} = \frac{16+8}{2} - \sqrt{\left(\frac{16-8}{2}\right)^2 + 6^2} = 4.79\,\mathrm{kNm/m}$$

$$\alpha_0 = \arctan\frac{m_1 - m_x}{m_{xy}} = \arctan\frac{19.21-16}{6} = 28.15^{\circ}$$

Compare these results with Fig. 9.2.2.3. The difference is 0%.

The design moments (according to [9][10]) if the reinforcement (ξ,η) is orthogonal and their directions concide with the local co-ordinate system (x,y):

Case a)

$$m_{ud\,\xi} = m_\xi - m_\vartheta \frac{\cos\varphi}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1-2\cos\varphi}{\sin\varphi} = 16 - 8\frac{\cos 90^{\circ}}{1+\cos 90^{\circ}} + 6\frac{1-2\cos 90^{\circ}}{\sin 90^{\circ}} = +22\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = m_\vartheta \frac{1}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1}{\sin\varphi} = 8\frac{1}{1+\cos 90^{\circ}} + 6\frac{1}{\sin 90^{\circ}} = +14\,\mathrm{kNm/m}$$

This is a valid solution! Because $m_{ud\,\xi} + m_{ud\,\eta} = +36\,\mathrm{kNm/m} > m_x + m_y = +24\,\mathrm{kNm/m}$

$m_{ud\,\xi} = +22\,\mathrm{kNm/m}$ $m_{ud\,\eta} = +14\,\mathrm{kNm/m}$

Case b)

$$m_{ud\,\xi} = m_\xi + m_\vartheta \frac{\cos\varphi}{1-\cos\varphi} - m_{\xi\vartheta}\frac{1+2\cos\varphi}{\sin\varphi} = 16 + 8\frac{\cos 90^{\circ}}{1-\cos 90^{\circ}} - 6\frac{1+2\cos 90^{\circ}}{\sin 90^{\circ}} = +10\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = m_\vartheta \frac{1}{1-\cos\varphi} - m_{\xi\vartheta}\frac{1}{\sin\varphi} = 8\frac{1}{1-\cos 90^{\circ}} - 6\frac{1}{\sin 90^{\circ}} = +2\,\mathrm{kNm/m}$$

Invalid solution! Because $m_{ud\,\xi} + m_{ud\,\eta} = +12\,\mathrm{kNm/m} < m_x + m_y = +24\,\mathrm{kNm/m}$

Case ξ)

$$m_{ud\,\xi} = m_\xi - \frac{m_{\xi\vartheta}^2}{m_\vartheta} = 16 - \frac{6^2}{8} = +11.5\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = 0$$

Invalid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+11.5\,\mathrm{kNm/m}<m_x+m_y=+24\,\mathrm{kNm/m}$

Case η)

$$m_{ud\,\xi}=0$$

$$m_{ud\,\eta}=\frac{m_\xi m_\vartheta-m_{\xi\vartheta}^2}{m_\xi\sin^2\varphi+m_\vartheta\cos^2\varphi-m_{\xi\vartheta}\sin 2\varphi}=\frac{16\cdot 8-6^2}{16\cdot\sin^2 90^\circ+8\cdot\cos^2 90^\circ-6\cdot\sin(2\cdot 90^\circ)}=+5.75\,\frac{\mathrm{kNm}}{\mathrm{m}}$$

Invalid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+5.75\,\mathrm{kNm/m}<m_x+m_y=+24\,\mathrm{kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.4 and 9.2.2.5. The difference between the hand and FE calculation is 0%.

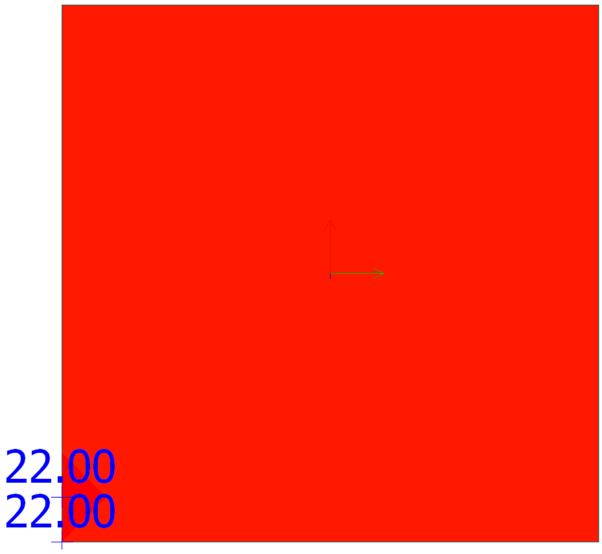


Figure 9.2.2.4 – The $m_{ud\,\xi}$ design moment for elliptic bending with orthogonal reinforcement [kNm]

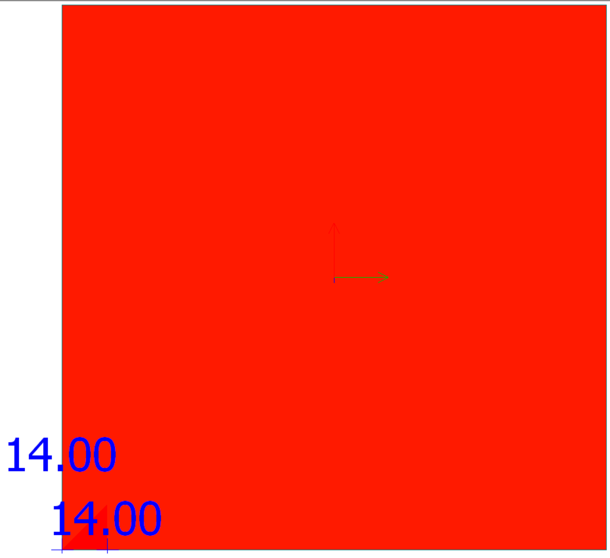


Figure 9.2.2.5 – The $m_{ud\,\eta}$ design moment for elliptic bending with orthogonal reinforcement [kNm]

Calculation of the required reinforcement based on the valid design moments:

In x (ξ) direction:

Sum of the moments:

$$m_{ud\,\xi}=f_{cd}\,x_c\left(d_x-\frac{x_c}{2}\right)\quad;\quad 22000=20\,x_c\left(175-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=6.403\,\mathrm{mm}$$

Sum of the forces:

$$x_c\,f_{cd}=a_{s\xi}\,f_{yd}\quad;\quad 6.403\cdot 20=a_{s\xi}\,434.8\quad;\quad a_{s\xi}=0.2945\,\mathrm{mm^2/mm}=294.5\,\mathrm{mm^2/m}$$

Fig. 9.2.2.6 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

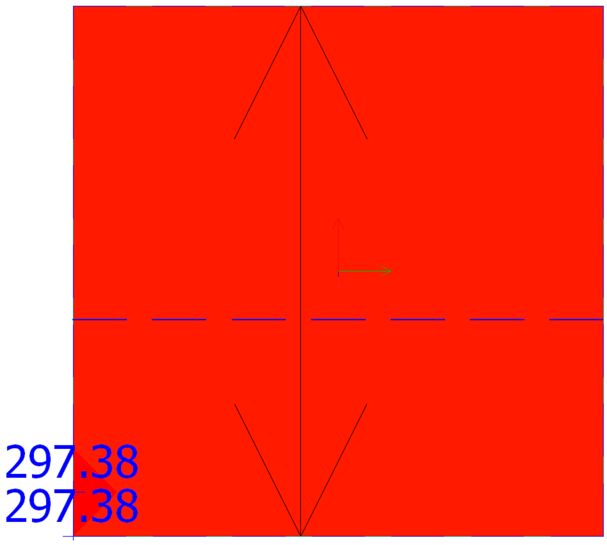


Figure 9.2.2.6 – The $a_{s\xi}$ required reinforcement at the bottom for elliptic bending with orthogonal reinforcement [mm²/m]

In y (η) direction:

Sum of the moments:

$$m_{ud\eta}=f_{cd}x_c\left(d_y-\frac{x_c}{2}\right) \quad ; \quad 14000=20x_c\left(165-\frac{x_c}{2}\right) \quad ; \quad x_c=4.298\,\mathrm{mm}$$

Sum of the forces:

$$x_c f_{cd}=a_{s\eta}f_{yd} \quad ; \quad 4.298\cdot 20=a_{s\eta}434.8 \quad ; \quad a_{s\eta}=0.1977\,\mathrm{mm^2/mm}=197.7\,\mathrm{mm^2/m}$$

Fig. 9.2.2.7 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

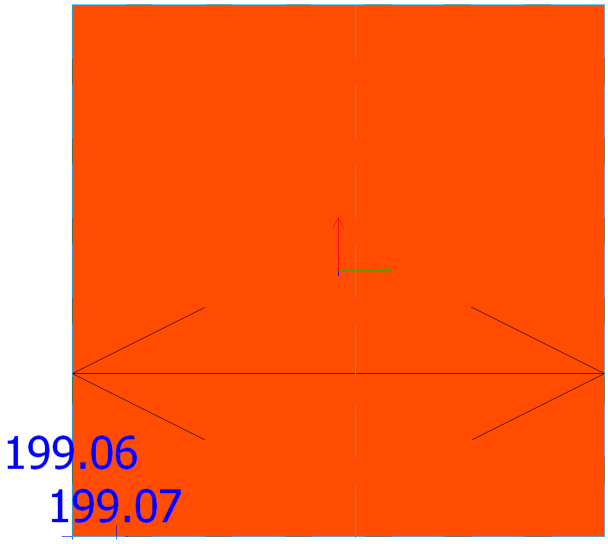


Figure 9.2.2.7 – The $a_{s\eta}$ required reinforcement at the bottom for elliptic bending with orthogonal reinforcement [mm²/m]

Secondly the reinforcement is non-orthogonal and the hand calculation and the comparison are the following:

2. Non-orthogonal reinforcement (φ=75° between ξ and η)

The reinforcement is non-orthogonal and the ξ direction concides with the local x direction. Thus y=ϑ. The angle between the ξ directional reinforcement and η directional reinforcement is φ=75°.

The moments in the slab (tensor of the applied moments):

$$m_x=m_\xi=+16\,\mathrm{kNm/m}$$

$$m_y=m_\vartheta=+8\,\mathrm{kNm/m}$$

$$m_{xy}=m_{\xi\vartheta}=+6\,\mathrm{kNm/m}$$

The first invariant of the tensor: $m_x+m_y=+24\,\mathrm{kNm/m}$

The design moments (according to [9][10]) if the reinforcement (ξ,η) is non-orthogonal:

Case a)

$$m_{ud\,\xi}=m_\xi-m_\vartheta\frac{\cos\varphi}{1+\cos\varphi}+m_{\xi\vartheta}\frac{1-2\cos\varphi}{\sin\varphi}=16-8\frac{\cos 75^o}{1+\cos 75^o}+6\frac{1-2\cos 75^o}{\sin 75^o}=+17.35\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=m_\vartheta\frac{1}{1+\cos\varphi}+m_{\xi\vartheta}\frac{1}{\sin\varphi}=8\frac{1}{1+\cos 75^o}+6\frac{1}{\sin 75^o}=+12.57\,\mathrm{kNm/m}$$

This is a valid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+29.92\,\mathrm{kNm/m}>m_x+m_y=+24\,\mathrm{kNm/m}$

$m_{ud\,\xi}=+17.35\,\mathrm{kNm/m}$ $m_{ud\,\eta}=+12.57\,\mathrm{kNm/m}$

Case b)

$$m_{ud\,\xi}=m_\xi+m_\vartheta\frac{\cos\varphi}{1-\cos\varphi}-m_{\xi\vartheta}\frac{1+2\cos\varphi}{\sin\varphi}=16+8\frac{\cos 75^o}{1-\cos 75^o}-6\frac{1+2\cos 75^o}{\sin 75^o}=+9.37\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=m_\vartheta\frac{1}{1-\cos\varphi}-m_{\xi\vartheta}\frac{1}{\sin\varphi}=8\frac{1}{1-\cos 75^o}-6\frac{1}{\sin 75^o}=+4.58\,\mathrm{kNm/m}$$

Invalid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+13.95\,\mathrm{kNm/m}<m_x+m_y=+24\,\mathrm{kNm/m}$

Case ξ)

$$m_{ud\,\xi}=m_\xi-\frac{m_{\xi\vartheta}^2}{m_\vartheta}=16-\frac{6^2}{8}=+11.5\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=0$$

Invalid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+11.5\,\mathrm{kNm/m}<m_x+m_y=+24\,\mathrm{kNm/m}$

Case η)

$$m_{ud\,\xi}=0$$

$$m_{ud\,\eta}=\frac{m_\xi m_\vartheta-m_{\xi\vartheta}^2}{m_\xi\sin^2\varphi+m_\vartheta\cos^2\varphi-m_{\xi\vartheta}\sin 2\varphi}=\frac{16\cdot 8-6^2}{16\cdot\sin^2 75^o+8\cdot\cos^2 75^o-6\cdot\sin(2\cdot 75^o)}=+7.38\frac{\mathrm{kNm}}{\mathrm{m}}$$

Invalid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+7.38\,\mathrm{kNm/m}<m_x+m_y=+24\,\mathrm{kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.8 and 9.2.2.9. The difference between the hand and FE calculation is 0%.

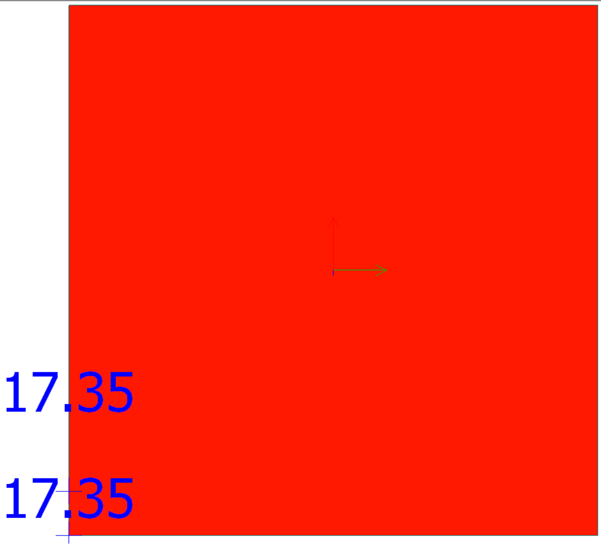


Figure 9.2.2.8 – The $m_{ud\,\xi}$ design moment for elliptic bending with non-orthogonal reinforcement [kNm]

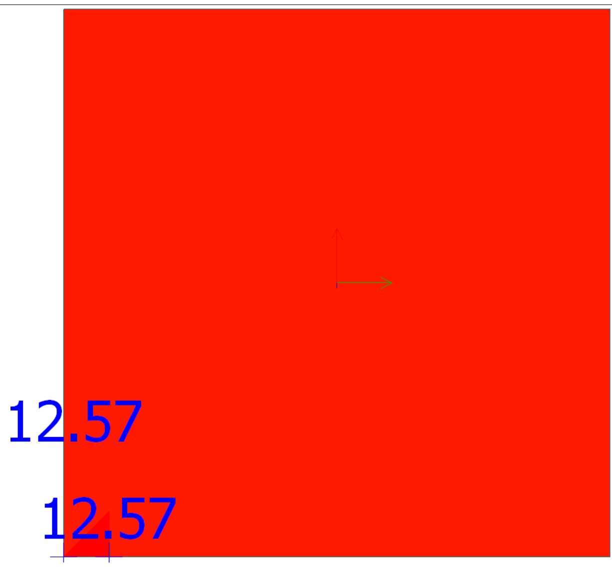


Figure 9.2.2.9 – The $m_{ud\,\eta}$ design moment for elliptic bending with non-orthogonal reinforcement [kNm]

Calculation of the required reinforcement based on the valid design moments:

In x (ξ) direction:

Sum of the moments:

$$m_{ud\,\xi} = f_{cd}\, x_c \left(d_x - \frac{x_c}{2}\right) \quad ; \quad 17350 = 20\, x_c \left(175 - \frac{x_c}{2}\right) \quad ; \quad \mathrm{x_c} = 5.029\,\mathrm{mm}$$

Sum of the forces:

$$x_c\, f_{cd} = a_{s\xi}\, f_{yd} \quad ; \quad 5.029 \cdot 20 = a_{s\xi}\, 434.8 \quad ; \quad a_{s\xi} = 0.2313\,\mathrm{mm^2/mm} = 231.3\,\mathrm{mm^2/m}$$

Fig. 9.2.2.10 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

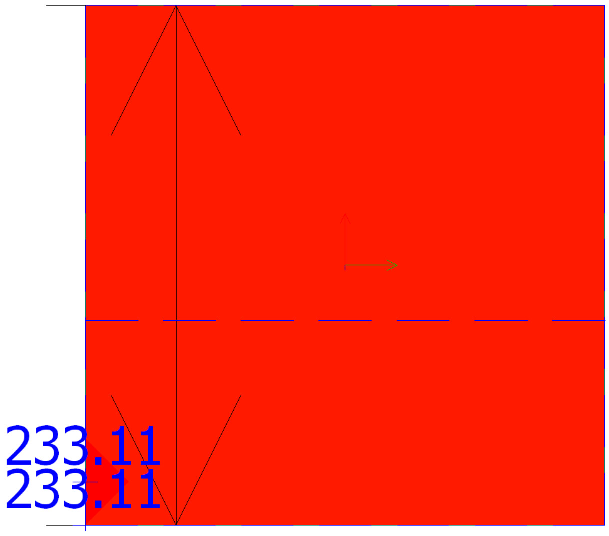


Figure 9.2.2.10 – The $a_{s\xi}$ required reinforcement at the bottom for elliptic bending with non-orthogonal reinforcement [mm²/m]

<u>In η direction:</u>

Sum of the moments:

$$m_{ud\,\eta}=f_{cd}\,x_c\left(d_y-\frac{x_c}{2}\right)\quad;\quad 12570=20\,x_c\left(165-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=3.854\,\mathrm{mm}$$

Sum of the forces:

$$x_c\,f_{cd}=a_{s\eta}\,f_{yd}\quad;\quad 3.854\cdot 20=a_{s\eta}\,434.8\quad;\quad a_{s\eta}=0.1773\,\mathrm{mm^2/mm}=177.3\,\mathrm{mm^2/m}$$

Fig. 9.2.2.11 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

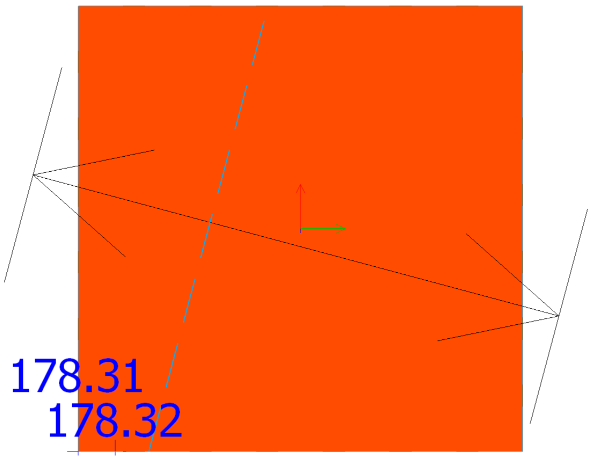


Figure 9.2.2.11 – The $a_{s\eta}$ required reinforcement at the bottom for elliptic bending with non-orthogonal reinforcement [mm²/m]

Download links to the example files:

Elliptic bending, orthogonal reinforcement:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.2.1 Required reinforcement calculation in a slab with elliptic bending and orthogonal reinforcement.str

Elliptic bending, non-orthogonal reinforcement:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.2.1 Required reinforcement calculation in a slab with elliptic bending and skew reinforcement.str

9.2.2.2 Hyperbolic bending

In the second case the bending condition is a hyperbolic bending. In FEM-Design the model is a slab with statically determinant support system and specific moment loads at its edges for the pure internal force state (see Fig. 9.2.2.12).

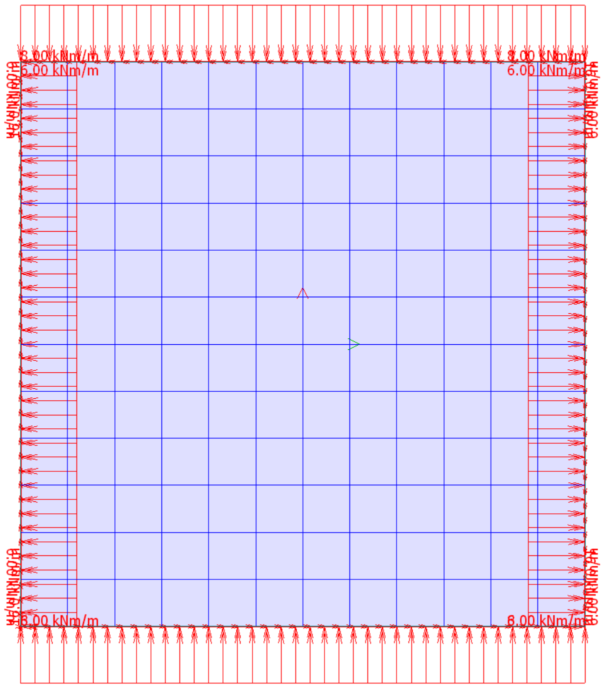


Figure 9.2.2.12 – The slab with the edge loads for pure stress state

Fig. 9.2.2.13 shows the constant internal forces in the slab due to the loads. Fig. 9.2.2.14 shows the principal moments and their directions based on the FEM-Design calculation. According to the pure stress state the principal moments and the directions are the same in each elements.

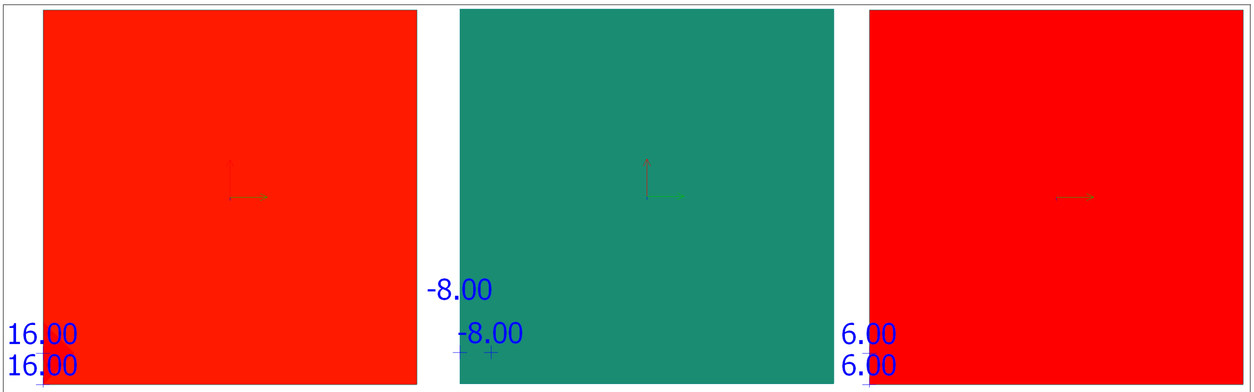


Figure 9.2.2.13 – The m_x, m_y, and m_{xy} internal forces in the slab [kNm]

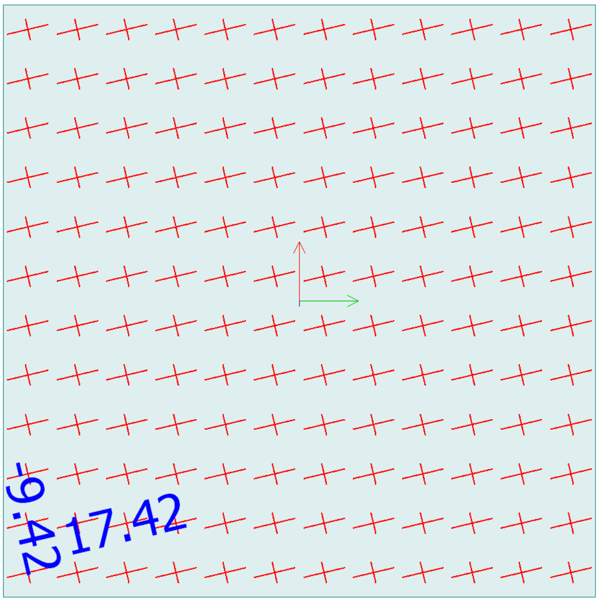


Figure 9.2.2.14 – The m_1 and m_2 principal moments and their directions in the slab [kNm]

Firstly the reinforcement is orthogonal and the hand calculation and the comparison are the following:

1. <u>Orthogonal reinforcement</u>

The reinforcement is orthogonal and their directions concide with the local system (x=ξ, y=ϑ=η).

The moments in the slab (tensor of the applied moments):

$$m_x = m_\xi = +16\,\text{kNm/m}$$

$$m_y = m_\vartheta = m_\eta = -8\,\text{kNm/m}$$

$$m_{xy} = m_{\xi\vartheta} = m_{\xi\eta} = +6\,\text{kNm/m}$$

The first invariant of the tensor: $m_x + m_y = +8\,\text{kNm/m}$

The calculation of the principal moments and their directions:

$$m_1 = \frac{m_x+m_y}{2} + \sqrt{\left(\frac{m_x-m_y}{2}\right)^2 + m_{xy}^{\ 2}} = \frac{16+(-8)}{2} + \sqrt{\left(\frac{16-(-8)}{2}\right)^2 + 6^2} = 17.42\,\text{kNm/m}$$

$$m_2 = \frac{m_x+m_y}{2} - \sqrt{\left(\frac{m_x-m_y}{2}\right)^2 + m_{xy}^{\ 2}} = \frac{16+(-8)}{2} - \sqrt{\left(\frac{16-(-8)}{2}\right)^2 + 6^2} = -9.42\,\text{kNm/m}$$

$$\alpha_0 = \arctan\frac{m_1-m_x}{m_{xy}} = \arctan\frac{17.42-16}{6} = 13.32°$$

Compare these results with Fig. 9.2.2.14. The difference is 0%.

The design moments (according to [9][10]) if the reinforcement (ξ,η) is orthogonal:

Case a)

$$m_{ud\,\xi}=m_\xi-m_\vartheta\frac{\cos\varphi}{1+\cos\varphi}+m_{\xi\vartheta}\frac{1-2\cos\varphi}{\sin\varphi}=16+8\frac{\cos 90^\circ}{1+\cos 90^\circ}+6\frac{1-2\cos 90^\circ}{\sin 90^\circ}=+22\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=m_\vartheta\frac{1}{1+\cos\varphi}+m_{\xi\vartheta}\frac{1}{\sin\varphi}=-8\frac{1}{1+\cos 90^\circ}+6\frac{1}{\sin 90^\circ}=-2\,\mathrm{kNm/m}$$

Invalid solution! Because their have different signs.

Case b)

$$m_{ud\,\xi}=m_\xi+m_\vartheta\frac{\cos\varphi}{1-\cos\varphi}-m_{\xi\vartheta}\frac{1+2\cos\varphi}{\sin\varphi}=16-8\frac{\cos 90^\circ}{1-\cos 90^\circ}-6\frac{1+2\cos 90^\circ}{\sin 90^\circ}=+10\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=m_\vartheta\frac{1}{1-\cos\varphi}-m_{\xi\vartheta}\frac{1}{\sin\varphi}=-8\frac{1}{1-\cos 90^\circ}-6\frac{1}{\sin 90^\circ}=-14\,\mathrm{kNm/m}$$

Invalid solution! Because their have different signs.

Case ξ)

$$m_{ud\,\xi}=m_\xi-\frac{m_{\xi\vartheta}^2}{m_\vartheta}=16-\frac{6^2}{-8}=+20.5\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=0$$

This is a valid solution at the bottom!

$m_{ud\,\xi}=+20.5\,\mathrm{kNm/m}$ $m_{ud\,\eta}=0\,\mathrm{kNm/m}$

Case η)

$$m_{ud\,\xi}=0$$

$$m_{ud\,\eta}=\frac{m_\xi m_\vartheta-m_{\xi\vartheta}^2}{m_\xi\sin^2\varphi+m_\vartheta\cos^2\varphi-m_{\xi\vartheta}\sin 2\varphi}=\frac{16\cdot(-8)-6^2}{16\cdot\sin^2 90^\circ+(-8)\cdot\cos^2 90^\circ-6\cdot\sin(2\cdot 90^\circ)}=-10.25\frac{\mathrm{kNm}}{\mathrm{m}}$$

This is a valid solution at the top!

$m_{ud\,\xi}=0\,\mathrm{kNm/m}$ $m_{ud\,\eta}=-10.25\,\mathrm{kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.15 and 9.2.2.16. The difference between the hand and FE calculation is 0%.

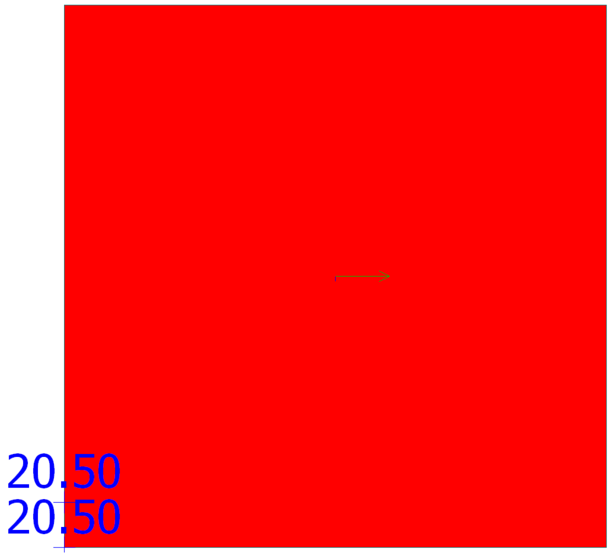


Figure 9.2.2.15 – The $m_{ud\,\xi}$ design moment for hyperbolic bending with orthogonal reinforcement [kNm]

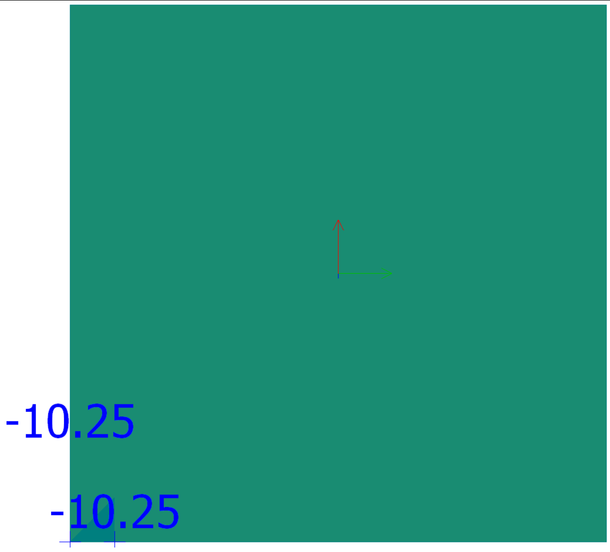


Figure 9.2.2.16 – The $m_{ud\,\eta}$ design moment for hyperbolic bending with orthogonal reinforcement [kNm]

<u>Calculation of the required reinforcement based on the valid design moments:</u>

<u>In x (ξ) direction at the bottom:</u>

Sum of the moments:

$$m_{ud\,\xi}=f_{cd}\,x_c\left(d_x-\frac{x_c}{2}\right)\quad;\quad 20500=20\,x_c\left(175-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=5.959\,\mathrm{mm}$$

Sum of the forces:

$$x_c f_{cd} = a_{s\xi} f_{yd} \quad ; \quad 5.959 \cdot 20 = a_{s\xi} 434.8 \quad ; \quad a_{s\xi} = 0.2741\,\mathrm{mm^2/mm} = 274.1\,\mathrm{mm^2/m}$$

Fig. 9.2.2.17 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

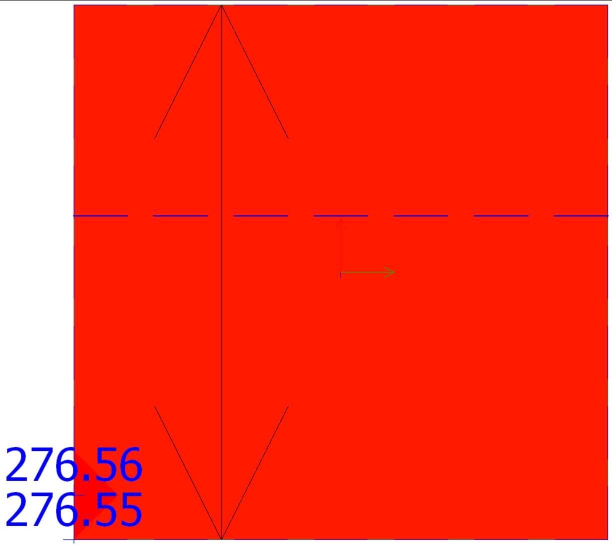


Figure 9.2.2.17 – The $a_{s\xi}$ required reinforcement at the bottom for hyperbolic bending with orthogonal reinforcement [mm²/m]

In y (η) direction at the top:

Sum of the moments:

$$m_{ud\eta} = f_{cd} x_c \left(d_y - \frac{x_c}{2} \right) \quad ; \quad 10250 = 20 x_c \left(165 - \frac{x_c}{2} \right) \quad ; \quad x_c = 3.136\,\mathrm{mm}$$

Sum of the forces:

$$x_c f_{cd} = a_{s\eta} f_{yd} \quad ; \quad 3.136 \cdot 20 = a_{s\eta} 434.8 \quad ; \quad a_{s\eta} = 0.1443\,\mathrm{mm^2/mm} = 144.3\,\mathrm{mm^2/m}$$

Fig. 9.2.2.18 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

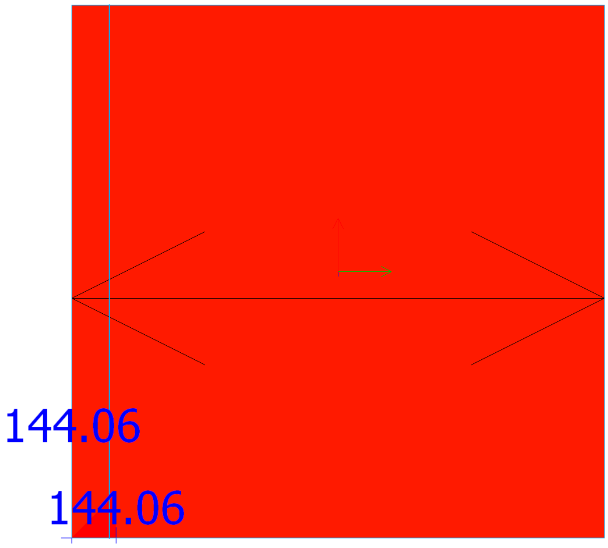


Figure 9.2.2.18 – The $a_{s\eta}$ required reinforcement at the top for hyperbolic bending with orthogonal reinforcement [mm^2/m]

Secondly the reinforcement is non-orthogonal and the hand calculation and the comparison are the following:

2. Non-orthogonal reinforcement (φ=75° between ξ and η)

The reinforcement is non-orthogonal and the ξ direction concides with the local x direction. Thus y=ϑ.

The moments in the slab (tensor of the applied moments):

$$m_x = m_\xi = +16\,\mathrm{kNm/m}$$

$$m_y = m_\vartheta = -8\,\mathrm{kNm/m}$$

$$m_{xy} = m_{\xi\vartheta} = +6\,\mathrm{kNm/m}$$

The first invariant of the tensor: $m_x + m_y = +8\,\mathrm{kNm/m}$

The design moments (according to [9][10]) if the reinforcement (ξ,η) is non-orthogonal:

Case a)

$$m_{ud\,\xi} = m_\xi - m_\vartheta \frac{\cos\varphi}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1-2\cos\varphi}{\sin\varphi} = 16 + 8\frac{\cos 75^\circ}{1+\cos 75^\circ} + 6\frac{1-2\cos 75^\circ}{\sin 75^\circ} = +20.64\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = m_\vartheta \frac{1}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1}{\sin\varphi} = -8\frac{1}{1+\cos 75^\circ} + 6\frac{1}{\sin 75^\circ} = -0.144\,\mathrm{kNm/m}$$

Invalid solution! Because their have different signs.

Case b)

$$m_{ud\,\xi}=m_{\xi}+m_{\vartheta}\frac{\cos\varphi}{1-\cos\varphi}-m_{\xi\vartheta}\frac{1+2\cos\varphi}{\sin\varphi}=16-8\frac{\cos 75^{\circ}}{1-\cos 75^{\circ}}-6\frac{1+2\cos 75^{\circ}}{\sin 75^{\circ}}=+3.78\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=m_{\vartheta}\frac{1}{1-\cos\varphi}-m_{\xi\vartheta}\frac{1}{\sin\varphi}=-8\frac{1}{1-\cos 75^{\circ}}-6\frac{1}{\sin 75^{\circ}}=-17.01\,\mathrm{kNm/m}$$

Invalid solution! Because their have different signs.

Case ξ)

$$m_{ud\,\xi}=m_{\xi}-\frac{m_{\xi\vartheta}^{2}}{m_{\vartheta}}=16-\frac{6^{2}}{-8}=+20.5\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=0$$

This is a valid solution at the bottom!

$m_{ud\,\xi}=+20.5\,\mathrm{kNm/m}$ $m_{ud\,\eta}=0\,\mathrm{kNm/m}$

Case η)

$$m_{ud\,\xi}=0$$

$$m_{ud\,\eta}=\frac{m_{\xi}m_{\vartheta}-m_{\xi\vartheta}^{2}}{m_{\xi}\sin^{2}\varphi+m_{\vartheta}\cos^{2}\varphi-m_{\xi\vartheta}\sin 2\varphi}=\frac{16\cdot(-8)-6^{2}}{16\cdot\sin^{2}75^{\circ}+(-8)\cdot\cos^{2}75^{\circ}-6\cdot\sin(2\cdot 75^{\circ})}=-14.40\frac{\mathrm{kNm}}{\mathrm{m}}$$

This is a valid solution at the top!

$m_{ud\,\xi}=0\,\mathrm{kNm/m}$ $m_{ud\,\eta}=-14.40\,\mathrm{kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.19 and 9.2.2.20. The difference between the hand and FE calculation is 0%.

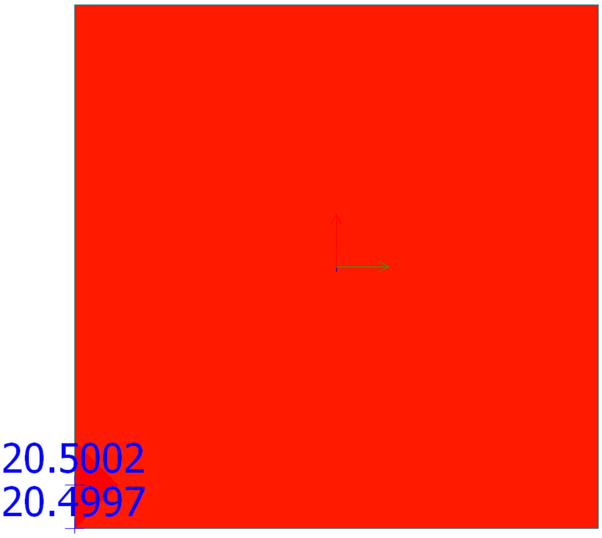


Figure 9.2.2.19 – The $m_{ud\,\xi}$ design moment for hyperbolic bending with non-orthogonal reinforcement [kNm]

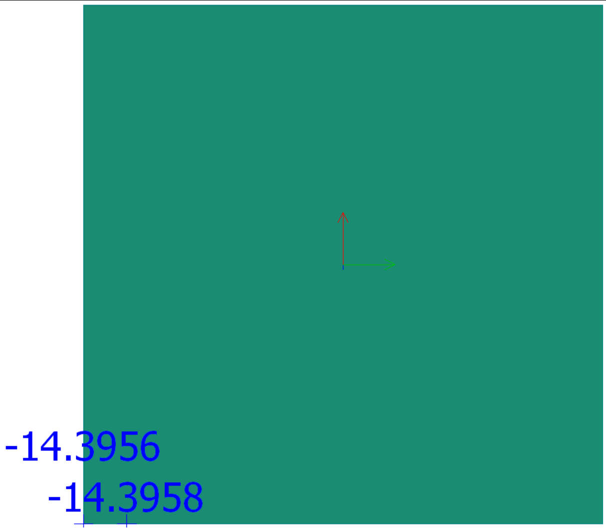


Figure 9.2.2.20 – The $m_{ud\,\eta}$ design moment for hyperbolic bending with non-orthogonal reinforcement [kNm]

<u>Calculation of the required reinforcement based on the valid design moments:</u>

<u>In x (ξ) direction at the bottom:</u>

Sum of the moments:

$$m_{ud\,\xi} = f_{cd}\, x_c \left(d_x - \frac{x_c}{2} \right) \quad ; \quad 20500 = 20\, x_c \left(175 - \frac{x_c}{2} \right) \quad ; \quad \mathrm{x_c} = 5.959\,\mathrm{mm}$$

Sum of the forces:

$$x_c\, f_{cd} = a_{s\xi}\, f_{yd} \quad ; \quad 5.959 \cdot 20 = a_{s\xi}\, 434.8 \quad ; \quad a_{s\xi} = 0.2741\,\mathrm{mm^2/mm} = 274.1\,\mathrm{mm^2/m}$$

Fig. 9.2.2.21 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

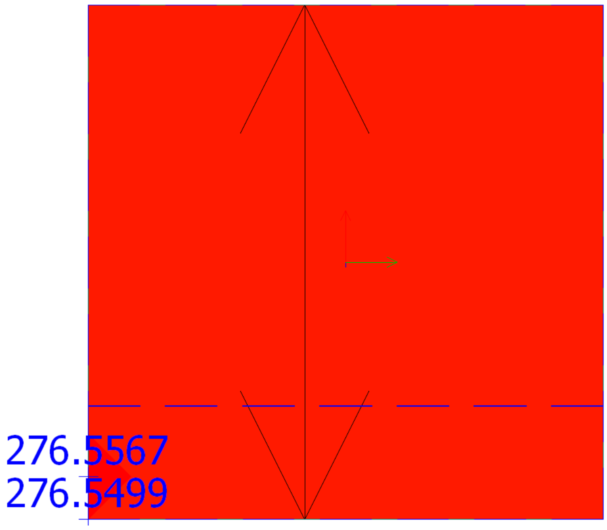


Figure 9.2.2.21 – The $a_{s\xi}$ required reinforcement at the bottom for hyperbolic bending with non-orthogonal reinforcement [mm²/m]

<u>In η direction at the top:</u>

Sum of the moments:

$$m_{ud\eta} = f_{cd} x_c \left(d_y - \frac{x_c}{2} \right) \quad ; \quad 14400 = 20\, x_c \left(165 - \frac{x_c}{2} \right) \quad ; \quad \mathrm{x_c} = 4.423\ \mathrm{mm}$$

Sum of the forces:

$$x_c f_{cd} = a_{s\eta} f_{yd} \quad ; \quad 4.423 \cdot 20 = a_{s\eta} 434.8 \quad ; \quad a_{s\eta} = 0.2034\, \mathrm{mm^2/mm} = 203.4\, \mathrm{mm^2/m}$$

Fig. 9.2.2.22 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

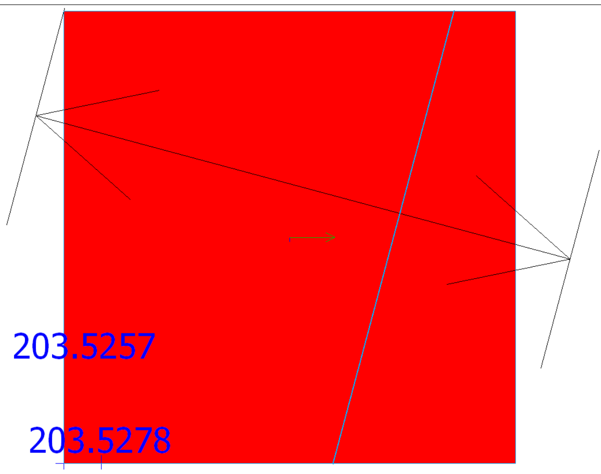


Figure 9.2.2.22 – The $a_{s\eta}$ required reinforcement at the top for hyperbolic bending with non-orthogonal reinforcement [mm²/m]

Download links to the example files

Hyperbolic bending, orthogonal reinforcement:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.2.2 Required reinforcement calculation in a slab with hyperbolic bending and orthogonal reinforcement.str

Hyperbolic bending, non-orthogonal reinforcement:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.2.2 Required reinforcement calculation in a slab with hyperbolic bending and skew reinforcement.str

9.2.3 Shear capacity calculation

In this chapter we will show detailed calculations of beams and slabs regarding to shear force.

9.2.3.1 Shear capacity of a beam

In this example we check the shear capacity of a cantilever beam (see Fig. 9.2.3.1.1). The input parameters and details are in the following table.

Inputs:

Concrete characteristic compressive strength	$f_{ck} = 25\ N/mm^2$
Beam height	h = 350 mm
Beam width	$b_w = 250mm$
Partial factor of concrete	$\gamma_c = 1.50$
Reinforcing steel characteristic yield strength	$f_{yk} = 500\ N/mm^2$
Partial factor of reinforcing steel	$\gamma_s = 1.15$
Stirrup distance	s = 200 mm
Longitudinal rebar diameter	$\phi = 20$ mm
Stirrup diameter	$\phi_s = 10$ mm
Concrete cover	c = 20 mm

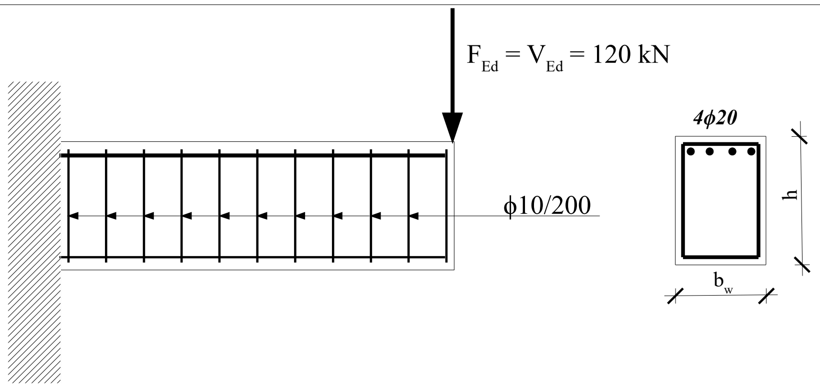


Figure 9.2.3.1.1 – The cantilever, the cross section and the design value of the shear force

First of all we need to check that we need any designed shear reinforcement or not:

$$V_{Rd,c} = \max\left\{\begin{matrix}\frac{0.18}{\gamma_c}k\left(100\rho_l f_{ck}\right)^{\frac{1}{3}} \\ v_{min}\end{matrix}\right\} b_w d = \max\left\{\begin{matrix}\frac{0.18}{1.5}\cdot 1.803\cdot\left(100\cdot 0.016\cdot 25\right)^{\frac{1}{3}} \\ 0.424\end{matrix}\right\} 250\cdot 310 = 57.3\,\mathrm{kN}$$

where:

$$d = h - c - \phi_s - \frac{\phi}{2} = 350 - 20 - 10 - \frac{20}{2} = 310\,\text{mm}$$

$$k = \min\left\{\begin{array}{c} 1+\sqrt{\frac{200}{d}} \\ 2.0 \end{array}\right\} = \min\left\{\begin{array}{c} 1+\sqrt{\frac{200}{310}} \\ 2.0 \end{array}\right\} = 1.803 \quad ;$$

$$\rho_l = \min\left\{\begin{array}{c} \frac{A_{sl}}{b_w d} \\ 0.02 \end{array}\right\} = \min\left\{\begin{array}{c} \frac{\frac{4\cdot 20^2\cdot\pi}{4}}{250\cdot 310} \\ 0.02 \end{array}\right\} = 0.016 \quad ;$$

$$v_{min} = 0.035\, k^{\frac{3}{2}} f_{ck}^{\frac{1}{2}} = 0.035\cdot 1.803^{\frac{3}{2}}\cdot 25^{\frac{1}{2}} = 0.424 \quad .$$

It is necessary to use designed shear reinforcement because:

$$V_{Rd,c} = 57.3\,\text{kN} < V_{Ed} = 120\,\text{kN}$$

Before the calculation of the designed shear reinforcement we need to check that the dimensions of the cross section is enough to bear the designed value of the shear force or not.

Thus the upper limit of the shear force bearing capacity:

$$V_{Rd,max} = \alpha_{cw} b_w z \nu f_{cd} \frac{\cot(\theta)+\cot(\alpha)}{1+\cot^2(\theta)} = 1\cdot 250\cdot 279\cdot 0.54\cdot 16.67 \frac{1.3+0}{1+1.3^2} = 303.4\,\text{kN}$$

where:

$\alpha_{cw} = 1.0$ the normal force is zero in the cantilever;

$z = 0.9\,d = 0.9\cdot 310 = 279\,\text{mm}$

$$\nu = 0.6\left(1-\frac{f_{ck}}{250}\right) = 0.6\left(1-\frac{25}{250}\right) = 0.540$$

$\alpha = 90°$

$\cot(\theta) = 1.3$ this value is adjustable in the program.

The compressed concrete strut can bear the acting shear force thus we can calculate the designed shear reinforcement.

$$V_{Ed} = 120\,\text{kN} < V_{Rd,max} = 303.4\,\text{kN}$$

The next step is to calculate the shear capacity of the beam according to the defined shear reinforcement (see Fig. 9.2.3.1.1).

$$V_{Rd,s}=\frac{z}{s}A_{sw}f_{yd}(\cot(\alpha)+\cot(\theta))\sin\alpha \;=\; \frac{279}{200}\frac{2\cdot 10^{2}\cdot\pi}{4}435(0+1.3)1 \;=\; 123.85\,\mathrm{kN}$$

$$\frac{V_{Ed}}{V_{Rd,s}}\;=\;\frac{120\,\mathrm{kN}}{123.8\,\mathrm{kN}}\;=\;97\,\%$$

The numerical results based on FEM-Design are shown in Fig. 9.2.3.1.2. The difference between the two calculations is 0%.

d_z [mm]	310
k_z [-]	1.80
$b_{w,z}$ [mm]	250
$\rho_{1,z}$ [-]	0.01621
$v_{min,z}$ [N/mm^2]	0.42
$V_{Rd,c,z}$ [kN]	57.61
$(A_{sw,z}/s)\ f_{ywd}$ [N/mm]	341.48
z_z [mm]	279
$V_{Rd,s,z}$ [kN]	123.85
$V_{Ed,z}/V_{Rd,s,z}$ [-]	0.97
A_k [mm^2]	49067
t_{ef} [mm]	73
$T_{Rd,c}$ [kNm]	8.59
$(A_{sw,min}/s)\ f_{ywd}$ [N/mm]	170.74
$T_{Rd,s}$ [kNm]	16.76
$T_{Ed}/T_{Rd,s}$ [-]	0.00
Utilization [%]	97

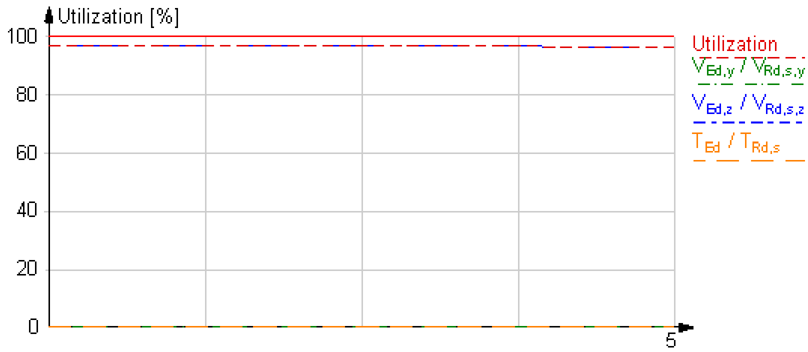


Figure 9.2.3.1.2 – The FEM-Design results

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.3.1 Shear capacity of a beam.str

9.2.3.2 Shear capacity of a slab

In this chapter we will calculate the shear capacity of a slab (see Fig. 9.2.3.2.1). In FEM-Design based on the internal forces we calculate the required shear capacity and based on the parameters of the slab and the applied longitudinal reinforcement in it the actual shear capacity is also computable.

Inputs:

Concrete characteristic compressive strength	$f_{ck} = 20$ N/mm^2
Partial factor of concrete	$\gamma_c = 1.50$
Reinforcing steel characteristic yield strength	$f_{yk} = 500$ N/mm^2
Partial factor of reinforcing steel	$\gamma_s = 1.15$
Longitudinal bar diameter in x direction	$\phi_{lx} = 14$ mm
Distance in x direction between the bars	$s_x = 150$ mm
Longitudinal bar diameter in y direction	$\phi_{ly} = 10$ mm
Distance in y direction between the bars	$s_y = 150$ mm
Concrete cover (on longitudinal bars in x direction)	$c = 20$ mm
Slab thickness	$t = 200$ mm
The effective depth of the x reinforcements	$d_x = 173$ mm
The effective depth of the y reinforcements	$d_y = 161$ mm
Normal force in x direction	$n_x = 20$ kN/m
Normal force in y direction	$n_y = 20$ kN/m
Vertical surface total load on the slab	$q_{z,Ed} = 10$ kN/m^2

Fig. 9.2.3.2.1 shows the slab. Two opposite edges of the slab are simply supported and the other two are free. The longitudinal reinforcements are also indicated in the figure.

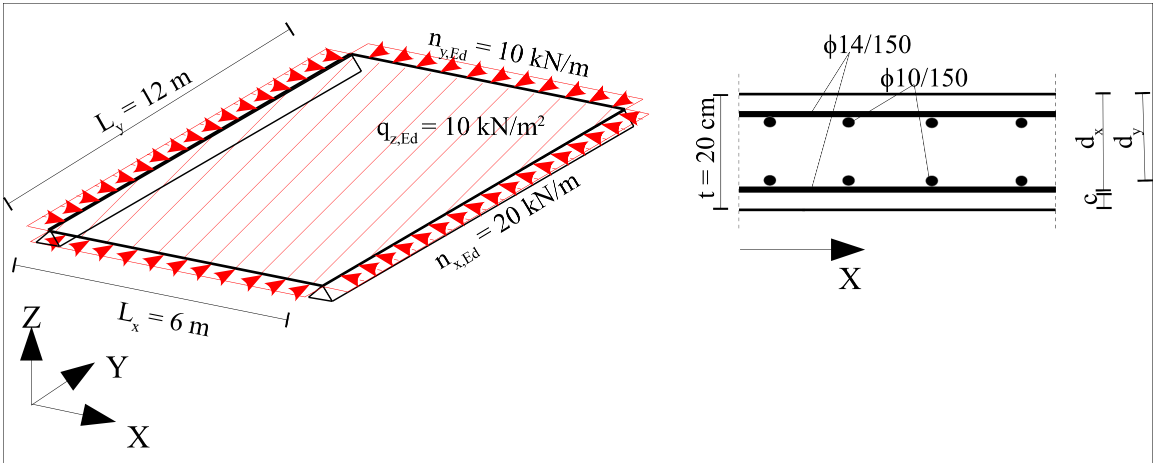


Figure 9.2.3.2.1 – The slab, the reinforcements and the design values of external forces

First of all we need to calculate the required shear capacity of the slab which depends on the internal shear forces in the two perpendicular directions of the slab local coordinate system which were also indicated in Fig. 9.2.3.2.1.

The internal forces of the slab at a corner point (based on finite element analysis with 0.5 m average element size (see Fig. 9.2.3.2.2-3):

Shear forces:

$$v_{xz}=69.25\frac{\text{kN}}{\text{m}};\, v_{yz}=17.31\frac{\text{kN}}{\text{m}}$$

Normal forces:

$$n_x=-20\frac{\text{kN}}{\text{m}};\, n_y=-10\frac{\text{kN}}{\text{m}};\, n_{xy}=0\frac{\text{kN}}{\text{m}}$$

The maximum shear force in the slab according to the two shear forces in the two directions:

$$v_{max}=\sqrt{v_{xz}^2+v_{yz}^2}=\sqrt{(69.25)^2+(17.31)^2}=71.38\frac{\text{kN}}{\text{m}}$$

The direction of the maximum shear force (right-handed coordinate system):

$$\alpha=\arctan\left(\frac{v_{yz}}{v_{xz}}\right)=\arctan\left(\frac{17.31}{69.25}\right)=+14.03°$$

The maximum shear force value could be different in every nodes and therefore the angle of it also. It means that the shear force capacity must be computed in every nodes in different directions (see the relevant formulas and equations below). Here is the calculation method for the mentioned corner point:

The shear capacity with the applied parameters in the calculated main direction:

$$v_{Rd,c} = \max\left\{\begin{matrix}\frac{0.18}{\gamma_c}k\left(100\,\rho_\alpha f_{ck}\right)^{\frac{1}{3}}+k_1\sigma_{cp_\alpha}\\ v_{min}+k_1\sigma_{cp_\alpha}\end{matrix}\right\}d_{eff} =$$

$$=\max\left\{\begin{matrix}\frac{0.18}{1.5}\cdot 2\cdot(100\cdot 0.005967\cdot 20)^{\frac{1}{3}}+0.15\cdot 0.09705\\ 0.4427+0.15\cdot 0.09705\end{matrix}\right\}167 = 94.02\frac{\text{kN}}{\text{m}}$$, where:

The longitudinal reinforcement in the main directions:

$$a_{sx}=\frac{\phi_x^2\pi}{4}\frac{1000}{s_x}=\frac{14^2\pi}{4}\frac{1000}{150}=1026\frac{\text{mm}^2}{\text{m}}$$

$$a_{sy}=\frac{\phi_y^2\pi}{4}\frac{1000}{s_y}=\frac{10^2\pi}{4}\frac{1000}{150}=523.6\frac{\text{mm}^2}{\text{m}}$$

The effective reinforcement area in the former calculated main direction:

$$a_\alpha = a_{sx}\cos^2(\alpha - 0^\circ) + a_{sy}\cos^2(\alpha - 90^\circ) = 1026\cos^2(14.03\,^\circ) + 523.6\cos^2(14.03\,^\circ - 90\,^\circ) =$$

$$= 996.5\,\frac{\mathrm{mm}^2}{\mathrm{m}}$$

Effective height:

$$d_{eff} = \frac{d_x + d_y}{2} = \frac{173 + 161}{2} = 167\,\mathrm{mm}$$

Reinforcement ratio in the main direction:

$$\rho_\alpha = \min\left\{\begin{array}{c} \dfrac{a_\alpha}{d_{eff}} = \dfrac{996.5}{167 \cdot 1000} \\ 0.02 \end{array}\right\} = 0.005967$$

The effective normal force in the direction of the maximum shear force:

$$n_\alpha = n_x\cos^2\alpha + n_y\sin^2\alpha + 2\,n_{xy}\cos\alpha\sin\alpha =$$

$$= (-20)\cos^2 14.03^\circ + (-10)\sin^2 14.03^\circ + 2 \cdot 0\cos 14.03^\circ \sin 14.03^\circ = -19.41\,\frac{\mathrm{kN}}{\mathrm{m}} \quad \text{(compression)}$$

Normal stress in the maximum shear force direction (the transformed normal force is compression):

$$\sigma_\alpha = \frac{n_\alpha}{t} = \frac{19.41}{200} = 0.09705\,\frac{\mathrm{N}}{\mathrm{mm}^2}$$

$$\sigma_{cp_\alpha} = \min\left\{\begin{array}{c} \sigma_\alpha \\ 0.2\,f_{cd} \end{array}\right\} = \min\left\{\begin{array}{c} 0.09705 \\ 2.66 \end{array}\right\} = 0.09705\,\frac{\mathrm{N}}{\mathrm{mm}^2}$$

Modifying factors:

$$k = \min\left\{\begin{array}{c} 1 + \sqrt{\dfrac{200}{d_{eff}}} \\ 2.0 \end{array}\right\} = \min\left\{\begin{array}{c} 1 + \sqrt{\dfrac{200}{167}} \\ 2.0 \end{array}\right\} = 2$$

$$v_{min} = 0.035\,k^{\frac{3}{2}} f_{ck}^{\frac{1}{2}} = 0.035 \cdot 0.15^{\frac{3}{2}} \cdot 20^{\frac{1}{2}} = 0.4427$$

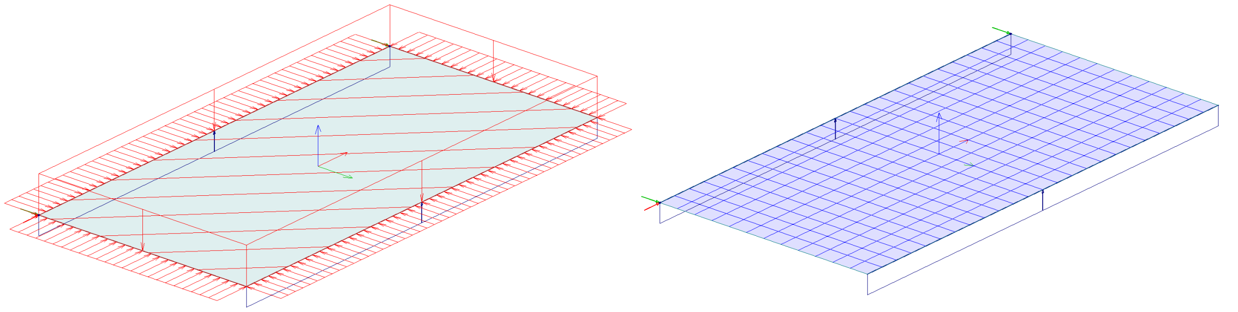


Figure 9.2.3.2.2 – The slab, loads and the finite element mesh for the problem

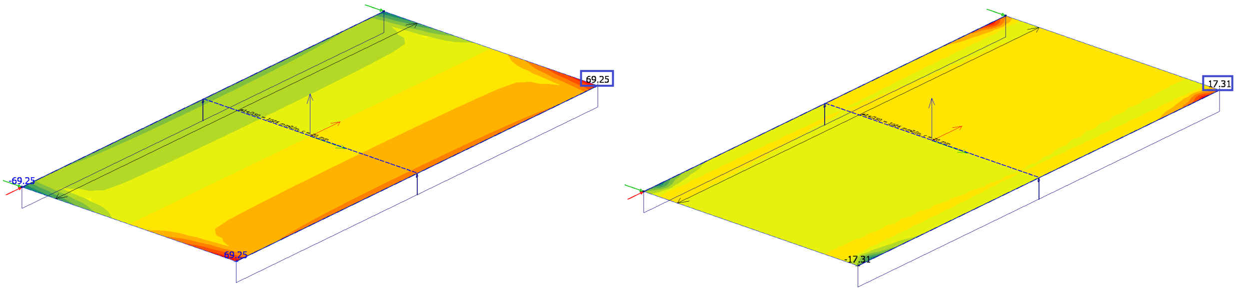


Figure 9.2.3.2.3 – The shear forces at the corner point
v_{xz}=69.25 [kN/m] (left side); v_{yz}=17.31 [kN/m] (right side);

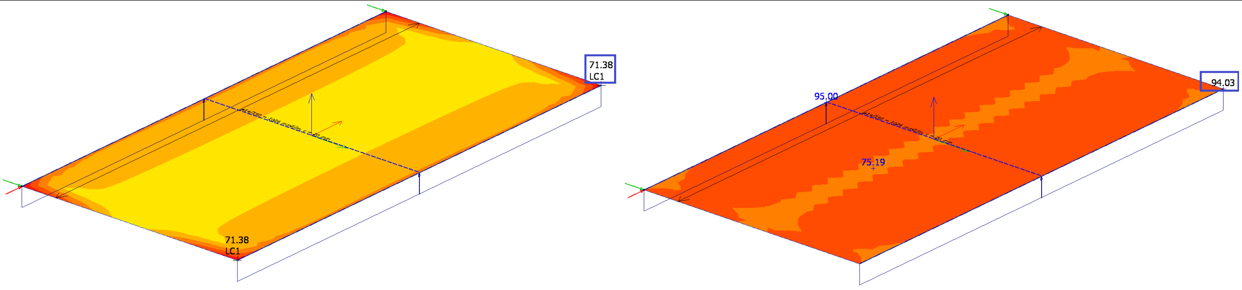


Figure 9.2.3.2.4 – The required shear force at the corner point: 71.38 [kN/m] (left side)
The applied shear force at the corner point: 94.03 [kN/m] (right side)

The results of the hand calculation are equal to the FEM-Design results (see Fig. 9.2.3.2.2-4).

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.3.2 Shear capacity of a slab.str

9.2.4 Crack width calculation of a beam

In this example we calculate the crack width of a simple supported normal-reinforced concrete beam under the given external load.

Inputs:

Concrete characteristic compressive strength	$f_{ck} = 20$ N/mm^2
Concrete effective tension strength	$f_{ct,eff} = 2.2$ N/mm^2
Concrete's Young modulus	$E_{cm} = 30$ GPa
Beam height	h = 400 mm
Beam width	b = 200 mm
Partial factor of concrete	$\gamma_c = 1.50$
Reinforcing steel characteristic yield strength	$f_{yk} = 500$ N/mm^2
Young's modulus of rebars	$E_s = 200$ GPa
Partial factor of reinforcing steel	$\gamma_s = 1.15$
Longitudinal rebar diameter	$\phi = 20$ mm
Stirrup diameter	$\phi_s = 10$ mm
Longitudinal bars effective height	d = 360 mm
Cover (on stirrups)	c = 20 mm
Partial factor of rebars	$\gamma_s = 1.15$

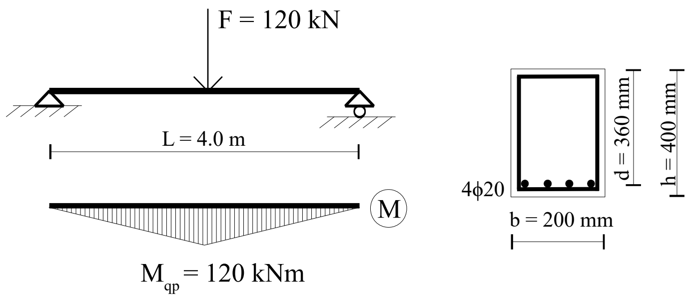


Figure 9.2.4.1 – The statical system, cross section and external loads

First we calculate the crack width with *linear-elastic* non-tension concrete material model.

The crack width is calculated with the aim of the distance of the cracks and the difference between concrete and rebar strains.

$$w_k = s_{r,max} \Delta\varepsilon = 149.48 \cdot 0.001428 = 0.2135 \text{ mm}$$

In order to calculate strains we need to calculate the cracked reinforced cross-sectional data (Stadium II.: concrete and steel are linear-elastic and the cross-section is cracked).

The position of the neutral axis according to the cracked section (Stadium II.):

$$\alpha_e = \frac{E_s}{E_{cm}} = \frac{200}{30} = 6.667 \quad ; \quad b x_{II} \frac{x_{II}}{2} = \alpha_e A_s (d - x_{II}) \quad \text{thus:} \quad x_{II} = 136.75\,\mathrm{mm}$$

The moment of inertia:

$$I_{II} = \frac{b x_{II}^3}{3} + A_s \alpha_e (d - x_{II})^2 = \frac{200 \cdot 136.75^3}{3} + \frac{4 \cdot 20^2 \cdot \pi}{4} \cdot 6.667 \cdot (360 - 136.75)^2$$

$$I_{II} = 5.878 \cdot 10^8\,\mathrm{mm}^4$$

The concrete and rebar strain difference:

$$\Delta\varepsilon = \varepsilon_{\mathrm{sm}} - \varepsilon_{\mathrm{cm}} = \max\left\{ \begin{array}{c} \dfrac{\sigma_s - k_t \dfrac{f_{ct,eff}}{\rho_{p,eff}} (1 + \alpha_e \rho_{p,eff})}{E_s} \\ 0.6 \dfrac{\sigma_s}{E_s} \end{array} \right\} = \max\left\{ \begin{array}{c} \dfrac{303.8 - 0.4 \cdot \dfrac{2.2}{0.0716} \cdot (1 + 6.667 \cdot 0.0716)}{200000} \\ 0.6 \cdot \dfrac{303.8}{200000} \end{array} \right\}$$

$$\Delta\varepsilon = 0.001428$$

Rebar stress:

$$\sigma_s = \frac{M_{qp}(d - x_{II})}{I_{II}} \alpha_e = \frac{120 \cdot 10^6 \cdot (360 - 136.75)}{5.878 \cdot 10^8} 6.667 \quad ; \quad \sigma_s = 303.8 \frac{\mathrm{N}}{\mathrm{mm}^2}$$

k_t depends the durability of the load, the given load is long-term loading:

$$k_t = 0.4$$

Effective tensile rebar ratio:

$$\rho_{p,eff} = \frac{A_s}{A_{c,eff}} = \frac{\dfrac{4 \phi^2 \pi}{4}}{b h_{c,eff}} = \frac{\dfrac{4 \phi^2 \pi}{4}}{b \cdot \min\left\{ \begin{array}{c} 2.5(h-d) \\ \dfrac{h - x_{II}}{3} \\ \dfrac{h}{2} \end{array} \right\}} = \frac{\dfrac{4 \cdot 20^2 \pi}{4}}{200 \cdot \min\left\{ \begin{array}{c} 2.5 \cdot (400 - 360) \\ \dfrac{400 - 136.75}{3} \\ \dfrac{400}{2} \end{array} \right\}} = 0.0716$$

First we need to check if the longitudinal bars are close to each other or not:

$$t_{limit}=5\left((c+\phi_s)+\frac{\phi}{2}\right)=5\cdot\left((20+10)+\frac{20}{2}\right)=200\,\text{mm} \quad ; \quad t_{actual}=40\,\text{mm}$$

$t_{limit} > t_{actual}$ therefore the rebars are close to each other, so the distance between cracks calculated as follows:

$$s_{r,max}=3.4(c+\phi_s)+0.425\,k_1 k_2\frac{\phi}{\rho_{\text{eff}}}=3.4\cdot(20+10)+0.425\cdot0.8\cdot0.5\cdot\frac{20}{0.0716}=149.48\,\text{mm}$$

k_1 depends the cohesion between the rebars and concrete, the reinforcements are ribbed:

$$k_1=0.8$$

k_2 depends the strains in the cross-section, in this case we have pure bending:

$$k_2=0.5$$

Numerical results:

$$w_{k,FEM}=0.22\,\text{mm}$$

The difference between the numerical and hand calculation is less than 3%.

Sections	6
k_1 [-]	0.80
ε_1 [-]	0.00199
ε_2 [-]	0.00000
k_2 [-]	0.50
$h_{c,ef}$ [mm]	73
$A_{c,eff}$ [mm^2]	14602
$\rho_{p,eff}$ [-]	0.09
x [mm]	181
$s_{r,max}$ [mm]	142
$(\varepsilon_{sm} - \varepsilon_{cm})$ [-]	0.001547
w_k [mm]	0.22
Utilization [%]	22

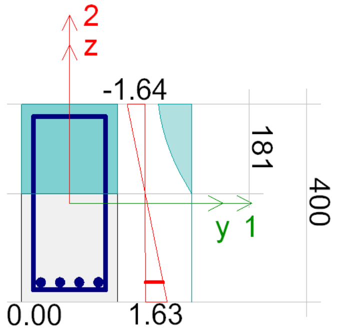


Figure 9.2.4.2 – The FEM-Design detailed results for crack width (strains [‰], neutral axis [mm])

Now we calculate the crack width with *non-linear* non-tension concrete material model:

The following figure shows (Fig. 9.2.4.3) the concrete material model (dashed curve) but FEM-Design modifies it a bit due to numerical stability (continuous curve).

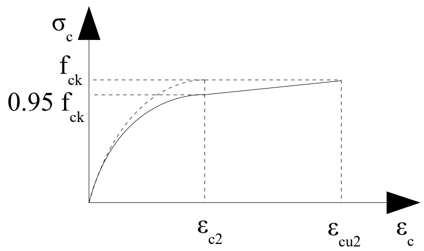


Figure 9.2.4.3 – The nonlinear concrete material model

First we need to calculate the neutral axis position and the beam curvature where the maximum moment occurs (in this case the middle cross-section). The x is measured from the top of the cross-section and its positive direction meant to the bottom of the cross-section.

The equilibrium equation respect to the forces:

$$N_c - N_s = 0$$

The force in the rebars according to that the rebars are in the linear elastic region:

$$N_s = A_s E_s \varepsilon_s = A_s E_s \kappa (d - x_n)$$

The force in the concrete according to the nonlinear compression material model:

$$N_c = b \int_0^h \sigma_c(\varepsilon) dx = b \int_0^h f_{ck} \left(1 - \left(1 - \frac{\varepsilon}{\varepsilon_{c2}}\right)^2\right) dx = b \int_0^h f_{ck} \left(1 - \left(1 - \frac{\kappa (x_n - x)}{\varepsilon_{c2}}\right)^2\right) dx$$

With numerical calculation the neutral axis and the curvature are:

$$x_n = 180.62\,\mathrm{mm} \quad ; \quad \kappa = 9.05797 \cdot 10^{-6} \frac{1}{\mathrm{mm}}$$

The compression and the tension forces:

$$N_c = b \int_0^h f_{cd} \left(1 - \left(1 - \frac{\kappa (x_n - x)}{\varepsilon_{c2}}\right)^2\right) dx = 200 \int_0^{400} 20 \left(1 - \left(1 - \frac{9.05797 \cdot 10^{-6} (180.62 - x)}{0.002}\right)^2\right) dx$$

$$N_c = 408.4\,\mathrm{kN}$$

$$N_s = 1257 \cdot 200000 \cdot 9.05797 \cdot 10^{-6} \cdot (360 - 180.62) = 408.4\,\mathrm{kN}$$

The crack width with non-linear concrete material model:

$$w_k = s_{r,max} \Delta\varepsilon = 141.57 \cdot 0.001544 = 0.2186\,\text{mm}$$

The concrete and rebar strain difference:

$$\Delta\varepsilon = \varepsilon_{\text{sm}} - \varepsilon_{\text{cm}} = \max\left\{\begin{matrix} \dfrac{\sigma_s - k_t \dfrac{f_{ct,eff}}{\rho_{p,eff}}\left(1+\alpha_e \rho_{p,eff}\right)}{E_s} \\ 0.6\dfrac{\sigma_s}{E_s} \end{matrix}\right\} = \max\left\{\begin{matrix} \dfrac{324.96 - 0.4 \cdot \dfrac{2.2}{0.0859} \cdot \left(1+6.667 \cdot 0.0859\right)}{200000} \\ 0.6 \cdot \dfrac{324.96}{200000} \end{matrix}\right\}$$

$$\Delta\varepsilon = 0.001544$$

Rebar stress:

$$\sigma_s = E_s \varepsilon_s = E_s \kappa \left(d - x_n\right) = 200000 \cdot 9.05797 \cdot 10^{-6} \left(360 - 180.62\right) = 324.96 \frac{\text{N}}{\text{mm}^2}$$

k_t depends the durability of the load, this load is long-term loading:

$$k_t = 0.4$$

Effective tensile rebar ratio:

$$\rho_{\text{eff}} = \frac{A_s}{A_{c,eff}} = \frac{\dfrac{4\phi^2\pi}{4}}{b h_{c,eff}} = \frac{\dfrac{4\phi^2\pi}{4}}{b \cdot \min\left\{\begin{matrix} 2.5\left(h-d\right) \\ \dfrac{h-x_{II}}{3} \\ \dfrac{h}{2} \end{matrix}\right\}} = \frac{\dfrac{4 \cdot 20^2\pi}{4}}{200 \cdot \min\left\{\begin{matrix} 2.5 \cdot \left(400-360\right) \\ \dfrac{400-180.62}{3} \\ \dfrac{400}{2} \end{matrix}\right\}} = 0.0859$$

First we need to check if the longitudinal bars are close to each other:

$$t_{limit} = 5\left(\left(c+\phi_s\right)+\frac{\phi}{2}\right) = 5 \cdot \left(\left(20+10\right)+\frac{20}{2}\right) = 200\,\text{mm} \quad ; \quad t_{actual} = 40\,\text{mm}$$

$t_{\text{limit}} > t_{\text{actual}}$ therefore the rebars are close to each other, so the distance between cracks calculated as follows:

$$s_{r,max} = 3.4\left(c+\phi_s\right) + 0.425\,k_1 k_2 \frac{\phi}{\rho_{\text{eff}}} = 3.4 \cdot \left(20+10\right) + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{20}{0.0859} = 141.57\,\text{mm}$$

k_1 depends the cohesion between the rebars and concrete, the reinforcements are ribbed:

$k_1 = 0.8$

k_2 depends the strains in the cross-section, in this case we have pure bending:

$k_2 = 0.5$

The difference between the hand calculations are under 3% and the results of the second hand calculation coincide with the FEM-Design results.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.4 Crack width calculation of a beam.str

9.2.5 Crack width calculation of a slab

In this example we calculate the crack width of a slab due to *elliptic* and *hyperbolic* bending conditions. First of all the applied reinforcement is *orthogonal* and then the applied reinforcement is *non-orthogonal*. We calculate the crack width with hand calculation and then compare the results with FEM-Design values. The crack width calculation is relevant in SLS combination, thus the internal forces (moments) in the examples come from a quasi-permanent serviceability combination.

The calculation of the crack directions is based on [10] and [12] according to the tensor of the reserve forces due to an arbitrary internal force and reinforcement distribution in the slab.

Fig. 9.2.5 shows the notation system of the applied angles in this chapter. The figure is valid for bottom and top reinforcement separately.

The x-y system denotes the direction of the local co-ordinate system of the slab.

The ξ and η directions are the directions of the reinforcements.

The ξ and η angles are the angles between the reinforcement directions and axis x.

The φ angle is the angle between the two directional reinforcements.

The α angle is the angle between axis x and the direction of the crack.

The α_0 angle is the angle perpendicular to the direction of the crack.

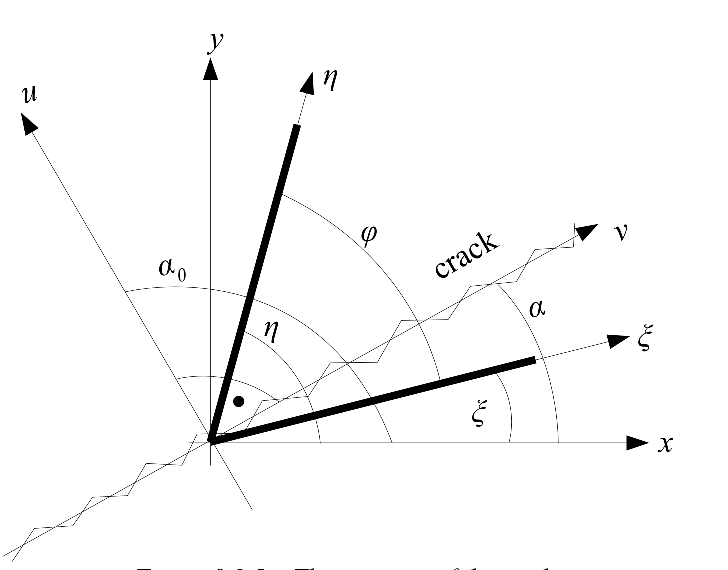


Figure 9.2.5 – The notation of the angles

The concrete covers in the different directions are valid on top and bottom equally even if there is no need for reinforcement in one direction.

Inputs:

The thickness of the slab	h = 200 mm
The elastic modulus of concrete	E_{cm} = 30 GPa, C20/25
Mean tensile strength of concrete	f_{ctm} = 2.2 N/mm^2
The Poisson's ratio of concrete	ν = 0.2
Elastic modulus of steel bars	E_s = 200 GPa
The characteristic value of yield stress of steel bars	f_{yk} = 500 MPa
Diameter of the longitudinal reinforcement (top and bot.)	ϕ_l = 10 mm
Nominal concrete cover (top and bottom as well)	c_ξ = 20 mm; c_η = 30 mm
Average concrete cover	c = 25 mm
Effective heights (top and bottom as well)	d_ξ = 175 mm; d_η = 165 mm
Effective heights (top and bottom as well)	d'_ξ = 25 mm; d'_η = 35 mm
Average effective height (bottom and top)	d = 170 mm; d' = 30 mm
Lever arm of internal forces	z = d – d' = 140 mm

9.2.5.1 Elliptic bending

The SLS moments in the slab in shell local system due to elliptic bending:

$m_x = +16\,\mathrm{kNm/m}$ the resultant of the *x* directional normal stresses.

$m_y = +8\,\mathrm{kNm/m}$ the resultant of the *y* directional normal stresses.

$m_{xy} = +6\,\mathrm{kNm/m}$ the resultant of the *x*-*y* directional shear stresses.

Orthogonal reinforcement (φ=90º between ξ and η)

The reinforcement is orthogonal and their directions concide with the local system ($x=\xi$, $y=\eta$). Fig. 9.2.5.1.1 shows the applied reinforcements and the concrete covers. Thus ξ = 0º ; η = 90º.

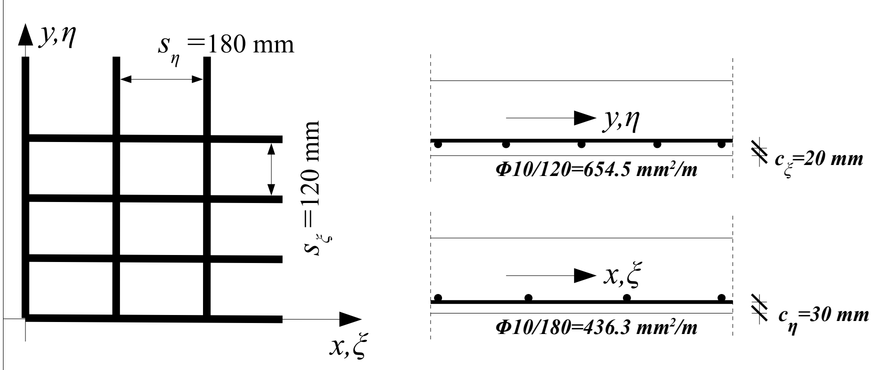


Figure 9.2.5.1.1 – The applied orthogonal reinforcement for elliptic bending

<u>The applied reinforcement in the slab</u>:

Only <u>bottom</u> reinforcement is necessary (see Chapter 9.2.2 for further information).

$$a_{s\xi}=\frac{\phi_{\xi}^{2}\pi}{4}\frac{1000}{s_{\xi}}=\frac{10^{2}\pi}{4}\frac{1000}{120}=654.5\frac{\mathrm{mm}^{2}}{\mathrm{m}}$$

$$a_{s\eta}=\frac{\phi_{\eta}^{2}\pi}{4}\frac{1000}{s_{\eta}}=\frac{10^{2}\pi}{4}\frac{1000}{180}=436.3\frac{\mathrm{mm}^{2}}{\mathrm{m}}$$

<u>Calculation of the direction of the crack based on the tensor of the reserve forces.</u>

<u>The tensor of the applied forces (effect)</u> based on the internal forces in the quasi-permanent combination:

$$\underline{\underline{E}}=\begin{bmatrix}E_1 & 0\\ 0 & E_2\end{bmatrix}\doteq\begin{bmatrix}E_x & E_{xy}\\ E_{xy} & E_y\end{bmatrix}=\begin{bmatrix}114.3 & 42.86\\ 42.86 & 57.14\end{bmatrix}\frac{\mathrm{kN}}{\mathrm{m}}\quad\text{, where}$$

$$E_x=\frac{m_x}{z}=\frac{16}{0.14}=114.3\frac{\mathrm{kN}}{\mathrm{m}}\quad,$$

$$E_y=\frac{m_y}{z}=\frac{8}{0.14}=57.14\frac{\mathrm{kN}}{\mathrm{m}}\quad,$$

$$E_{xy}=\frac{m_{xy}}{z}=\frac{6}{0.14}=42.86\frac{\mathrm{kN}}{\mathrm{m}}\quad.$$

<u>The tensor of the resisting (yield) forces</u> based on the resistance of the reinforcement:

$$\underline{\underline{R}}=\begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix}\doteq\begin{bmatrix}R_x & R_{xy}\\ R_{xy} & R_y\end{bmatrix}=\begin{bmatrix}327.3 & 0\\ 0 & 218.2\end{bmatrix}\frac{\mathrm{kN}}{\mathrm{m}}\quad\text{, where}$$

$$A_{s\xi}=a_{s\xi}f_{yk}=654.5\cdot 500=327.3\frac{\mathrm{kN}}{\mathrm{m}}\quad,$$

$$A_{s\eta}=a_{s\eta}f_{yk}=436.3\cdot 500=218.2\frac{\mathrm{kN}}{\mathrm{m}}\quad,$$

$$R_x=A_{s\xi}\cos^2(\xi)+A_{s\eta}\cos^2(\eta)=327.3\cos^2(0^\circ)+218.2\cos^2(90^\circ)=327.3\frac{\mathrm{kN}}{\mathrm{m}}\quad,$$

$$R_y=A_{s\xi}\sin^2(\xi)+A_{s\eta}\sin^2(\eta)=327.3\sin^2(0^\circ)+218.2\sin^2(90^\circ)=218.2\frac{\mathrm{kN}}{\mathrm{m}}\quad,$$

$$R_{xy}=A_{s\xi}\cos(\xi)\sin(\xi)+A_{s\eta}\cos(\eta)\sin(\eta)=327.3\cos(0^\circ)\sin(0^\circ)+218.2\cos(90^\circ)\sin(90^\circ)=$$

$$=0\frac{\mathrm{kN}}{\mathrm{m}}\quad.$$

<u>The tensor of the reserve forces:</u>

$$\underline{\underline{r}}^*=\begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} \doteq \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} - r\begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} R_x - r\,E_x & R_{xy} - r\,E_{xy} \\ R_{xy} - r\,E_{xy} & R_y - r\,E_y \end{bmatrix} ,$$

where r is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:

$$r_2^*=0$$

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

$$r_2^*=\left(\frac{r_x^*+r_y^*}{2}\right)-\sqrt{\left(\frac{r_x^*-r_y^*}{2}\right)^2+r_{xy}^{*\,2}}=0$$

$$r_2^*=\left(\frac{R_x-r\,E_x+R_y-r\,E_y}{2}\right)-\sqrt{\left(\frac{R_x-r\,E_x-R_y+r\,E_y}{2}\right)^2+(R_{xy}-r\,E_{xy})^2}=0$$

This equation gives two solutions for the r scalar internal force multiplier. The smallest positive r value has physical meaning. Without further detailed calculation the relevant r scalar load multiplier is:

$$r=2.120$$

Based on this scalar value the reserve force tensor:

$$\underline{\underline{r}}^*=\begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix}=\begin{bmatrix} 182.1 & 0 \\ 0 & 0 \end{bmatrix} \doteq \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix}=\begin{bmatrix} 84.98 & -90.86 \\ -90.86 & 97.06 \end{bmatrix}\frac{\text{kN}}{\text{m}}$$

The principal direction of the first principal reserve force gives the direction of the crack:

$$\alpha=\arctan\frac{r_I^*-r_x^*}{r_{xy}^*}=\arctan\frac{182.1-84.98}{-90.86}=-47.00^\circ$$

Now we can calculate in this direction the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

$$\alpha_0=\alpha+90^\circ=43^\circ$$

The effective reinforcement area in this direction:

$$a_{\alpha_0}=a_{s\xi}\cos^2(\alpha_0-\xi)+a_{s\eta}\cos^2(\alpha_0-\eta)=654.5\cos^2(43^\circ-0^\circ)+436.3\cos^2(43^\circ-90^\circ)=553.0\,\frac{\text{mm}^2}{\text{m}}$$

The bending moment in the direction perpendicular to the crack direction:

$$m_{\alpha_0}=m_x\cos^2\alpha_0+m_y\sin^2\alpha_0+2m_{xy}\cos\alpha_0\sin\alpha_0=$$

$$=(16)\cos^2 43^\circ+(8)\sin^2 43^\circ+2\cdot(6)\cos 43^\circ\sin 43^\circ=18.26\,\frac{\text{kNm}}{\text{m}}$$

The position of the neutral axis according to the uncracked section (Stadium I.):

$$\alpha_e=\frac{E_s}{E_{cm}}=\frac{200}{30}=6.667 \quad ; \quad x_I=\frac{\frac{h^2}{2}+\alpha_e a_{\alpha_0} d}{h+\alpha_e a_{\alpha_0}}=101.3\,\text{mm}$$

The moment of inertia (Stadium I.):

$$I_I=\frac{x_I^3}{3}+\frac{(h-x_I)^3}{3}+\alpha_e a_{\alpha_0}(d-x_I)^2=6.844\cdot 10^8\,\frac{\text{mm}^4}{\text{m}}$$

Concrete tensile stress (Stadium I.) to check if the crack exist or not:

$$\sigma_{c,\alpha_0}=\frac{m_{\alpha_0}(h-x_I)}{I_I}=\frac{18.26\cdot 10^3\cdot(200-101.3)}{6.844\cdot 10^5}=2.63\,\frac{\text{N}}{\text{mm}^2}>f_{ctm}=2.2\,\text{MPa} \quad \text{crack occurred.}$$

The position of the neutral axis according to the cracked section (Stadium II.):

$$x_{II}\frac{x_{II}}{2}=\alpha_e a_{\alpha_0}(d-x_{II}) \quad \text{thus:} \quad x_{II}=31.92\,\text{mm}$$

The moment of inertia (Stadium II.):

$$I_{II}=\frac{x_{II}^3}{3}+\alpha_e a_{\alpha_0}(d-x_{II})^2=\frac{31.92^3}{3}+6.667\cdot 0.553\cdot(170-31.92)^2=8.114\cdot 10^7\,\frac{\text{mm}^4}{\text{m}}$$

Rebar stress (Stadium II.):

$$\sigma_{s,\alpha_0}=\alpha_e\frac{m_{\alpha_0}(d-x_{II})}{I_{II}}=6.667\,\frac{18.26\cdot 10^3\cdot(170-31.92)}{8.114\cdot 10^4}=207.2\,\frac{\text{N}}{\text{mm}^2}$$

Effective tensile rebar ratio:

$$\rho_{p,eff}=\frac{a_{\alpha_0}}{a_{c,eff}}=\frac{a_{\alpha_0}}{h_{c,eff}}=\frac{0.553}{\min\begin{Bmatrix}2.5(h-d)\\ \frac{h-x_{II}}{3}\\ \frac{h}{2}\end{Bmatrix}}=\frac{0.553}{\min\begin{Bmatrix}2.5\cdot(200-170)\\ \frac{200-31.92}{3}\\ \frac{200}{2}\end{Bmatrix}}=\frac{0.553}{\min\begin{Bmatrix}75\\ 56.03\\ 100\end{Bmatrix}}=0.009870$$

The concrete and rebar strain difference:

$$\Delta\varepsilon=\varepsilon_{\mathrm{sm}}-\varepsilon_{\mathrm{cm}}=\max\begin{Bmatrix}\frac{\sigma_{s,\alpha_0}-k_t\frac{f_{ct,eff}}{\rho_{p,eff}}(1+\alpha_e\rho_{p,eff})}{E_s}\\ 0.6\frac{\sigma_{s,\alpha_0}}{E_s}\end{Bmatrix}=$$

$$=\max\begin{Bmatrix}\frac{207.2-0.4\cdot\frac{2.2}{0.00987}\cdot(1+6.667\cdot0.00987)}{200000}\\ 0.6\cdot\frac{207.2}{200000}\end{Bmatrix}=\max\begin{Bmatrix}0.0005609\\ 0.0006216\end{Bmatrix}=0.0006216$$

The criterium of the spacing of the bonded bars that they are close to each other or not:

$$sp_{cr}=5(c+\phi_l/2)=5(25+10/2)=150\,\mathrm{mm}$$

The effective spacing of the bonded bars in the direction of the crack:

$$sp=\frac{\phi_l^2\pi/4}{a_{\alpha_0}}=\frac{10^2\pi/4}{0.553}=142.0\,\mathrm{mm}$$

Now the maximum crack spacing:

$sp\leqslant sp_{cr}$ thus:

$$s_{r,max}=3.4c+0.425k_1k_2\frac{\phi_l}{\rho_{p,eff}}=3.4\cdot25+0.425\cdot0.8\cdot0.5\cdot\frac{10}{0.00987}=257.2\,\mathrm{mm}$$

Thus the crack width:

$$w_k=s_{r,max}\Delta\varepsilon=257.2\cdot0.0006216=0.1599\,\mathrm{mm}$$

Numerical results:

$$w_{k,FEM}=0.1559\,\mathrm{mm}$$

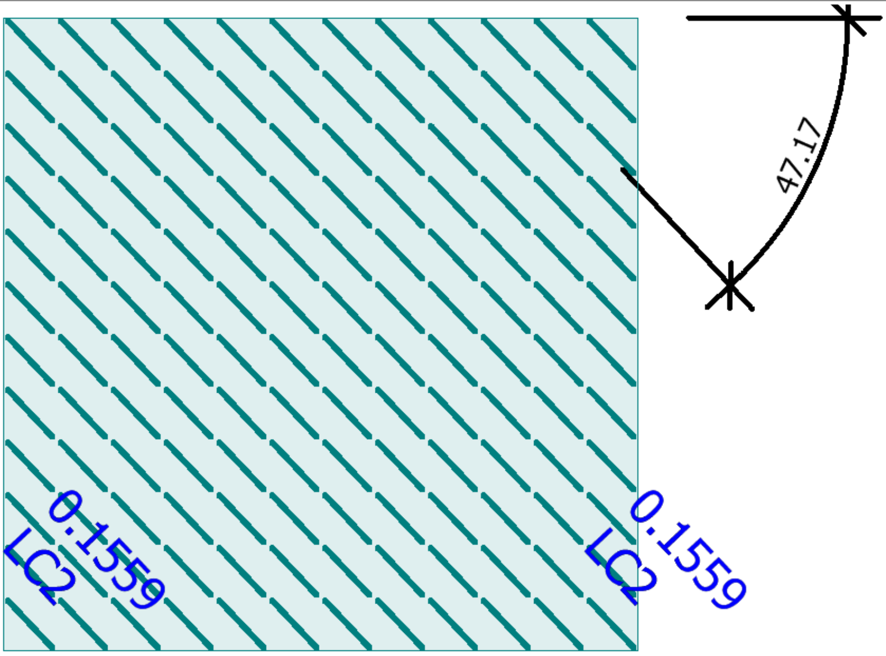


Figure 9.2.5.1.2 – The crack width [mm] *and the direction of the cracks at the bottom*

Fig. 9.2.5.1.2 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is less than 3%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.5.1 Crack width calculation in a slab with elliptic bending and orthogonal reinforcement.str

Non-orthogonal reinforcement (φ=75° between ξ and η)

The reinforcement is non-orthogonal and the ξ direction concides with the local x direction. The angle between the ξ directional reinforcement and η directional reinforcement is φ=75°. Fig. 9.2.5.1.3 shows the applied reinforcements and the concrete covers. Thus $\xi = 0°$; $\eta = 75°$.

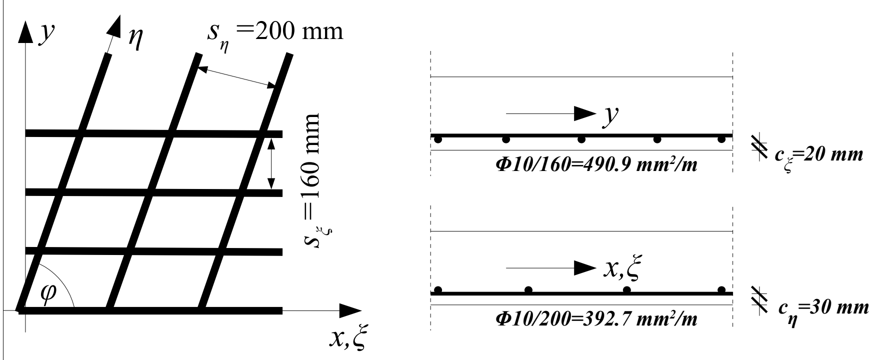


Figure 9.2.5.1.3 – The applied skew reinforcement for elliptic bending

The applied reinforcement in the slab:

Only bottom reinforcement necessary (see Chapter 9.2.2 for further information).

$$a_{s\xi} = \frac{\varphi_\xi^2 \pi}{4} \frac{1000}{s_\xi} = \frac{10^2 \pi}{4} \frac{1000}{160} = 490.9 \frac{\text{mm}^2}{\text{m}}$$

$$a_{s\eta} = \frac{\varphi_\eta^2 \pi}{4} \frac{1000}{s_\eta} = \frac{10^2 \pi}{4} \frac{1000}{200} = 392.7 \frac{\text{mm}^2}{\text{m}}$$

Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:

$$\underline{\underline{E}} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \doteq \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} 114.3 & 42.86 \\ 42.86 & 57.14 \end{bmatrix} \frac{\text{kN}}{\text{m}} \quad \text{, where}$$

$$E_x = \frac{m_x}{z} = \frac{16}{0.14} = 114.3 \frac{\text{kN}}{\text{m}} \quad ,$$

$$E_y = \frac{m_y}{z} = \frac{8}{0.14} = 57.14 \frac{\text{kN}}{\text{m}} \quad ,$$

$$E_{xy} = \frac{m_{xy}}{z} = \frac{6}{0.14} = 42.86 \frac{\text{kN}}{\text{m}} \quad .$$

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:

$$\underline{\underline{R}} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \doteq \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} = \begin{bmatrix} 258.7 & 49.1 \\ 49.1 & 183.2 \end{bmatrix} \frac{\text{kN}}{\text{m}} \quad \text{, where}$$

$$A_{s\xi}=a_{s\xi}\,f_{yk}=490.9\cdot 500=245.5\frac{\text{kN}}{\text{m}}\quad ,$$

$$A_{s\eta}=a_{s\xi}\,f_{yk}=392.7\cdot 500=196.4\frac{\text{kN}}{\text{m}}\quad ,$$

$$R_x=A_{s\xi}\cos^2(\xi)+A_{s\eta}\cos^2(\eta)=245.5\cos^2(0^{\circ})+196.4\cos^2(75^{\circ})=258.7\frac{\text{kN}}{\text{m}}\quad ,$$

$$R_y=A_{s\xi}\sin^2(\xi)+A_{s\eta}\sin^2(\eta)=245.5\sin^2(0^{\circ})+196.4\sin^2(75^{\circ})=183.2\frac{\text{kN}}{\text{m}}\quad ,$$

$$R_{xy}=A_{s\xi}\cos(\xi)\sin(\xi)+A_{s\eta}\cos(\eta)\sin(\eta)=245.5\cos(0^{\circ})\sin(0^{\circ})+196.4\cos(75^{\circ})\sin(75^{\circ})=$$
$$=49.1\frac{\text{kN}}{\text{m}}\quad .$$

<u>The tensor of the reserve forces:</u>

$$\underline{\underline{r}}^*=\begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix}\doteq\begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix}=\begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix}-r\begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix}=\begin{bmatrix} R_x-r\,E_x & R_{xy}-r\,E_{xy} \\ R_{xy}-r\,E_{xy} & R_y-r\,E_y \end{bmatrix}\quad ,$$

where r is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:

$$r_2^*=0$$

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

$$r_2^*=\left(\frac{r_x^*+r_y^*}{2}\right)-\sqrt{\left(\frac{r_x^*-r_y^*}{2}\right)^2+r_{xy}^{*\,2}}=0$$

$$r_2^*=\left(\frac{R_x-r\,E_x+R_y-r\,E_y}{2}\right)-\sqrt{\left(\frac{R_x-r\,E_x-R_y+r\,E_y}{2}\right)^2+(R_{xy}-r\,E_{xy})^2}=0$$

This equation gives two solutions for the r scalar internal force multiplier. The smallest positive r value has physical meaning. Without further detailed calculation the relevant r scalar load multiplier is:

$$r=2.058$$

Based on this scalar value the reserve force tensor:

$$\underline{\underline{r}}^*=\begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix}=\begin{bmatrix} 89.08 & 0 \\ 0 & 0 \end{bmatrix}\doteq\begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix}=\begin{bmatrix} 23.47 & -39.11 \\ -39.11 & 65.61 \end{bmatrix}\frac{\text{kN}}{\text{m}}$$

The principal direction of the first principal reserve force gives the direction of the crack:

$$\alpha=\arctan\frac{r_I^*-r_x^*}{r_{xy}^*}=\arctan\frac{89.08-23.47}{-39.11}=-59.20^{\circ}$$

Now we can calculate in this direction the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

$$\alpha_0=\alpha+90^{\circ}=30.8^{\circ}$$

The effective reinforcement area in this direction:

$$a_{\alpha_0}=a_{s\xi}\cos^2(\alpha_0-\xi)+a_{s\eta}\cos^2(\alpha_0-\eta)=490.9\cos^2(30.8^{\circ}-0^{\circ})+392.7\cos^2(30.8^{\circ}-75^{\circ})=$$

$$=564.0\frac{\text{mm}^2}{\text{m}}$$

The bending moment in the direction perpendicular to the crack direction:

$$m_{\alpha_0}=m_x\cos^2\alpha_0+m_y\sin^2\alpha_0+2m_{xy}\cos\alpha_0\sin\alpha_0=$$

$$=(16)\cos^2 30.8^{\circ}+(8)\sin^2 30.8^{\circ}+2\cdot(6)\cos 30.8^{\circ}\sin 30.8^{\circ}=19.18\frac{\text{kNm}}{\text{m}}$$

The position of the neutral axis according to the uncracked section (Stadium I.):

$$\alpha_e=\frac{E_s}{E_{cm}}=\frac{200}{30}=6.667\quad;\quad x_I=\frac{\frac{h^2}{2}+\alpha_e a_{\alpha_0}d}{h+\alpha_e a_{\alpha_0}}=101.3\,\text{mm}$$

The moment of inertia (Stadium I.):

$$I_I=\frac{x_I^3}{3}+\frac{(h-x_I)^3}{3}+\alpha_e a_{\alpha_0}(d-x_I)^2=6.848\cdot 10^8\frac{\text{mm}^4}{\text{m}}$$

Concrete tensile stress (Stadium I.) to check if the crack exist or not:

$$\sigma_{c,\alpha_0}=\frac{m_{\alpha_0}(h-x_I)}{I_I}=\frac{19.18\cdot 10^3\cdot(200-101.3)}{6.848\cdot 10^5}=2.76\frac{\text{N}}{\text{mm}^2}>f_{ctm}=2.2\,\text{MPa}\quad\text{crack occurred.}$$

The position of the neutral axis according to the cracked section (Stadium II.):

$$x_{II}\frac{x_{II}}{2}=\alpha_e a_{\alpha_0}(d-x_{II}) \quad \text{thus:} \quad x_{II}=32.19\,\mathrm{mm}$$

The moment of inertia (Stadium II.):

$$I_{II}=\frac{x_{II}^3}{3}+\alpha_e a_{\alpha_0}(d-x_{II})^2=\frac{32.19^3}{3}+6.667\cdot 0.564\cdot(170-32.19)^2=8.253\cdot 10^7\,\frac{\mathrm{mm}^4}{\mathrm{m}}$$

Rebar stress:

$$\sigma_{s,\alpha_0}=\alpha_e\frac{m_{\alpha_0}(d-x_{II})}{I_{II}}=6.667\frac{19.18\cdot 10^3\cdot(170-32.19)}{8.253\cdot 10^4}=213.5\frac{\mathrm{N}}{\mathrm{mm}^2}$$

Effective tensile rebar ratio:

$$\rho_{p,eff}=\frac{a_{\alpha_0}}{a_{c,eff}}=\frac{a_{\alpha_0}}{h_{c,eff}}=\frac{0.564}{\min\begin{Bmatrix}2.5(h-d)\\ \frac{h-x_{II}}{3}\\ \frac{h}{2}\end{Bmatrix}}=\frac{0.564}{\min\begin{Bmatrix}2.5\cdot(200-170)\\ \frac{200-32.19}{3}\\ \frac{200}{2}\end{Bmatrix}}=\frac{0.564}{\min\begin{Bmatrix}75\\ 55.94\\ 100\end{Bmatrix}}=0.01008$$

The concrete and rebar strain difference:

$$\Delta\varepsilon=\varepsilon_{\mathrm{sm}}-\varepsilon_{\mathrm{cm}}=\max\begin{Bmatrix}\dfrac{\sigma_{s,\alpha_0}-k_t\dfrac{f_{ct,eff}}{\rho_{p,eff}}(1+\alpha_e\rho_{p,eff})}{E_s}\\ 0.6\dfrac{\sigma_{s,\alpha_0}}{E_s}\end{Bmatrix}=$$

$$=\max\begin{Bmatrix}\dfrac{213.5-0.4\cdot\dfrac{2.2}{0.01008}\cdot(1+6.667\cdot 0.01008)}{200000}\\ 0.6\cdot\dfrac{213.5}{200000}\end{Bmatrix}=\max\begin{Bmatrix}0.0006017\\ 0.0006405\end{Bmatrix}=0.0006405$$

The criterium of the spacing of the bonded bars that they are close to each other or not:

$$sp_{cr}=5\left(c+\phi_l/2\right)=5\left(25+10/2\right)=150\,\mathrm{mm}$$

The effective spacing of the bonded bars in the direction of the crack:

$$sp=\frac{\phi_l^2\pi/4}{a_{\alpha_0}}=\frac{10^2\pi/4}{0.564}=139.3\,\mathrm{mm}$$

Now the maximum crack spacing:

$sp\leqslant sp_{cr}$ thus:

$$s_{r,max}=3.4c+0.425\,k_1k_2\frac{\phi_l}{\rho_{p,eff}}=3.4\cdot25+0.425\cdot0.8\cdot0.5\cdot\frac{10}{0.01008}=253.7\,\mathrm{mm}$$

Thus the crack width:

$$w_k=s_{r,max}\Delta\varepsilon=253.7\cdot0.0006405=0.1625\,\mathrm{mm}$$

Numerical results:

$$w_{k,FEM}=0.1579\,\mathrm{mm}$$

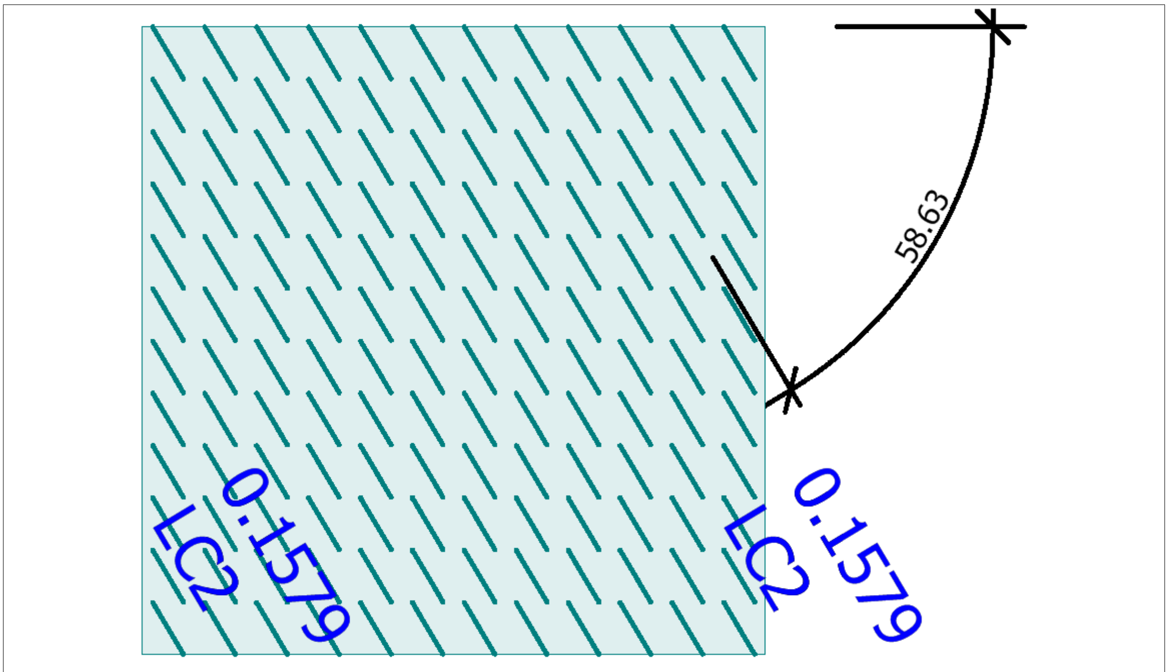


Figure 9.2.5.1.4 – The crack width [mm] *and the direction of the cracks at the bottom*

Fig. 9.2.5.1.4 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is less than 3%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.5.1 Crack width calculation in a slab with elliptic bending and skew reinforcement.str

9.2.5.2 Hyperbolic bending

The SLS moments in the slab in shell local system due to hyperbolic bending:

$m_x = +32\,\mathrm{kNm/m}$ the resultant of the *x* directional normal stresses.

$m_y = -16\,\mathrm{kNm/m}$ the resultant of the *y* directional normal stresses.

$m_{xy} = +12\,\mathrm{kNm/m}$ the resultant of the *x*-*y* directional shear stresses.

Orthogonal reinforcement (φ=90º between ξ and η)

The reinforcement is orthogonal and their directions concide with the local system ($x=\xi$, $y=\eta$). Fig. 9.2.5.2.1 shows the applied reinforcements and the concrete covers. Thus $\xi = 0°$; $\eta = 90°$.

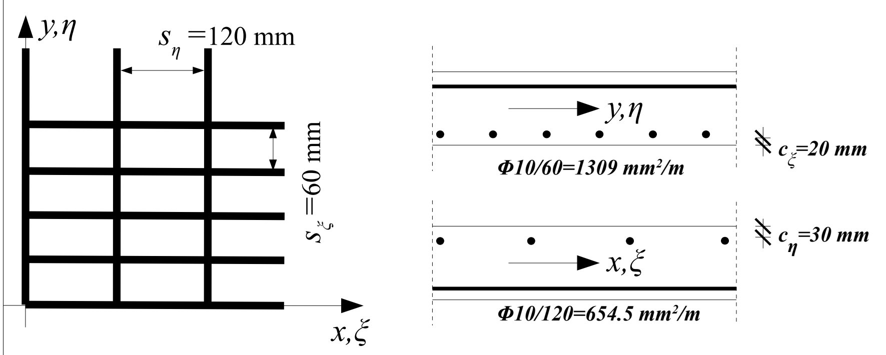


Figure 9.2.5.2.1 – The applied orthogonal reinforcement for hyperbolic bending

The applied reinforcement in the slab:

Top and bottom reinforcement are also necessary but in this case only in one direction (see Chapter 9.2.2 for further information).

Bottom:

$$a_{s\xi}^{bot} = \frac{\phi_\xi^2 \pi}{4} \frac{1000}{s_\xi} = \frac{10^2 \pi}{4} \frac{1000}{60} = 1309 \frac{\mathrm{mm}^2}{\mathrm{m}}$$

$$a_{s\eta}^{bot} = 0$$

Top:

$$a_{s\xi}^{top} = 0$$

$$a_{s\eta}^{top} = \frac{\phi_\eta^2 \pi}{4} \frac{1000}{s_\eta} = \frac{10^2 \pi}{4} \frac{1000}{120} = 654.5 \frac{\mathrm{mm}^2}{\mathrm{m}}$$

Crack width on bottom:

Calculation of the direction of the crack based on the tensor of the reserve forces.

<u>The tensor of the applied forces (effect)</u> based on the internal forces in the quasi-permanent combination:

$$\underline{\underline{E}}=\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \doteq \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} 228.5 & 85.71 \\ 85.71 & -114.3 \end{bmatrix} \frac{\text{kN}}{\text{m}} \quad \text{, where}$$

$$E_x=\frac{m_x}{z}=\frac{32}{0.14}=228.6\frac{\text{kN}}{\text{m}} \quad ,$$

$$E_y=\frac{m_y}{z}=\frac{-16}{0.14}=-114.3\frac{\text{kN}}{\text{m}} \quad ,$$

$$E_{xy}=\frac{m_{xy}}{z}=\frac{12}{0.14}=85.71\frac{\text{kN}}{\text{m}} \quad .$$

<u>The tensor of the resisting (yield) forces</u> based on the resistance on the reinforcement:

$$\underline{\underline{R}}=\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \doteq \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} = \begin{bmatrix} 654.5 & 0 \\ 0 & 62.5 \end{bmatrix} \frac{\text{kN}}{\text{m}} \quad \text{, where}$$

$$A_{s\xi}=a_{s\xi}^{bot} f_{yk}=1309\cdot 500=654.5\frac{\text{kN}}{\text{m}} \quad .$$

If there is no reinforcement on bottom in the other direction we need to consider somehow the tensile resistance of the concrete, because this may effect the relevant direction of the crack.

$$A_{s\eta}=c\cdot f_{ctm}=25\cdot 2.2=62.5\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_x=A_{s\xi}\cos^2(\xi)+A_{s\eta}\cos^2(\eta)=654.5\cos^2(0^\circ)+62.5\cos^2(90^\circ)=654.5\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_y=A_{s\xi}\sin^2(\xi)+A_{s\eta}\sin^2(\eta)=654.5\sin^2(0^\circ)+62.5\sin^2(90^\circ)=62.5\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_{xy}=A_{s\xi}\cos(\xi)\sin(\xi)+A_{s\eta}\cos(\eta)\sin(\eta)=654.5\cos(0^\circ)\sin(0^\circ)+62.5\cos(90^\circ)\sin(90^\circ)=$$
$$=0\frac{\text{kN}}{\text{m}} \quad .$$

<u>The tensor of the reserve forces:</u>

$$\underline{\underline{r}}^*=\begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} \doteq \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} - r\begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} R_x-r\,E_x & R_{xy}-r\,E_{xy} \\ R_{xy}-r\,E_{xy} & R_y-r\,E_y \end{bmatrix} \quad ,$$

where r is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:

$$r_2^*=0$$

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

$$r_2^* = \left(\frac{r_x^* + r_y^*}{2}\right) - \sqrt{\left(\frac{r_x^* - r_y^*}{2}\right)^2 + r_{xy}^{*\,2}} = 0$$

$$r_2^* = \left(\frac{R_x - r\,E_x + R_y - r\,E_y}{2}\right) - \sqrt{\left(\frac{R_x - r\,E_x - R_y + r\,E_y}{2}\right)^2 + (R_{xy} - r\,E_{xy})^2} = 0$$

This equation gives two solutions for the r scalar internal force multiplier. The smallest positive r value has physical meaning. Without further detailed calculation the relevant r scalar load multiplier is:

$$r = 2.337$$

Based on this scalar value the reserve force tensor:

$$\underline{\underline{r}}^* = \begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} = \begin{bmatrix} 450.1 & 0 \\ 0 & 0 \end{bmatrix} \doteq \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} 120.5 & -200.3 \\ -200.3 & 329.6 \end{bmatrix} \frac{\mathrm{kN}}{\mathrm{m}}$$

The principal direction of the first principal reserve force gives the direction of the crack:

$$\alpha = \arctan\frac{r_1^* - r_x^*}{r_{xy}^*} = \arctan\frac{450.1 - 120.5}{-200.3} = -58.71^\circ$$

Now we can calculate in this direction at the bottom the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

$$\alpha_0 = \alpha + 90^\circ = 31.29^\circ$$

The effective reinforcement area in this direction:

$$a_{\alpha_0}^{bot} = a_{sx}^{bot}\cos^2(\alpha_0 - \xi) + a_{sy}^{bot}\cos^2(\alpha_0 - \eta) =$$

$$= 1309\cos^2(31.29^\circ - 0^\circ) + 0\cos^2(31.29^\circ - 90^\circ) = 955.9\,\frac{\mathrm{mm}^2}{\mathrm{m}}$$

$$a_{\alpha_0}^{top} = a_{sx}^{top}\cos^2(\alpha_0 - \xi) + a_{sy}^{top}\cos^2(\alpha_0 - \eta) =$$

$$= 0\cos^2(31.29^\circ - 0^\circ) + 654.5\cos^2(31.29^\circ - 90^\circ) = 186.8\,\frac{\mathrm{mm}^2}{\mathrm{m}}$$

The bending moment in the direction perpendicular to the crack direction:

$$m_{\alpha_0} = m_x \cos^2\alpha_0 + m_y \sin^2\alpha_0 + 2m_{xy}\cos\alpha_0 \sin\alpha_0 =$$

$$= (32)\cos^2 31.29° + (-16)\sin^2 31.29° + 2\cdot 12\cos 31.29° \sin 31.29° = 29.70\,\frac{\text{kNm}}{\text{m}}$$

The position of the neutral axis according to the uncracked section (Stadium I.):

$$\alpha_e = \frac{E_s}{E_{cm}} = \frac{200}{30} = 6.667 \quad ; \quad x_I = \frac{\frac{h^2}{2} + \alpha_e a_{\alpha_0}^{bot} d_\xi + \alpha_e a_{\alpha_0}^{top} d'_\eta}{h + \alpha_e a_{\alpha_0}^{bot} + \alpha_e a_{\alpha_0}^{top}} = 101.9\,\text{mm}$$

The moment of inertia (Stadium I.):

$$I_I = \frac{x_I^3}{3} + \frac{(h-x_I)^3}{3} + \alpha_e a_{\alpha_0}^{bot}(d_\xi - x_I)^2 + \alpha_e a_{\alpha_0}^{top}(x_I - d'_\eta)^2 = 7.070\cdot 10^8\,\frac{\text{mm}^4}{\text{m}}$$

Concrete tensile stress (Stadium I.) to check the crack exist or not:

$$\sigma_{c,\alpha_0} = \frac{m_{\alpha_0}(h - x_I)}{I_I} = \frac{29.70\cdot 10^3\cdot(200-101.9)}{7.070\cdot 10^5} = 4.12\,\frac{\text{N}}{\text{mm}^2} > f_{ctm} = 2.2\,\text{MPa} \quad \text{crack occurred.}$$

The position of the neutral axis according to the cracked section (Stadium II.):

In case of this hyperbolic bending the applied reinforcements are only in one direction on one side, thus we considered the real effective height of the tensile and compressed bars instead of the average values which ones were introduced at the beginning of this chapter.

$$x_{II}\frac{x_{II}}{2} + \alpha_e a_{\alpha_0}^{top}(x_{II} - d'_\eta) = \alpha_e a_{\alpha_0}^{bot}(d_\xi - x_{II}) \quad \text{thus:} \quad x_{II} = 41.13\,\text{mm}$$

The moment of inertia (Stadium II.):

$$I_{II} = \frac{x_{II}^3}{3} + \alpha_e a_{\alpha_0}^{bot}(d_\xi - x_{II})^2 + \alpha_e a_{\alpha_0}^{top}(x_{II} - d'_\eta)^2 =$$

$$= \frac{41.13^3}{3} + 6.667\cdot 0.9559\cdot(175-41.13)^2 + 6.667\cdot 0.1868\cdot(41.13-35)^2 = 1.3745\cdot 10^8\,\frac{\text{mm}^4}{\text{m}}$$

Rebar stress:

$$\sigma_{s,\alpha_0} = \alpha_e\frac{m_{\alpha_0}(d_\xi - x_{II})}{I_{II}} = 6.667\,\frac{29.70\cdot 10^3\cdot(175-41.13)}{1.3745\cdot 10^5} = 192.9\,\frac{\text{N}}{\text{mm}^2}$$

Effective tensile rebar ratio:

$$\rho_{p,eff}=\frac{a_{\alpha_0}^{bot}}{a_{c,eff}}=\frac{a_{\alpha_0}^{bot}}{h_{c,eff}}=\frac{0.9559}{\min\begin{Bmatrix}2.5(h-d_{\xi})\\ \frac{h-x_{II}}{3}\\ \frac{h}{2}\end{Bmatrix}}=\frac{0.9559}{\min\begin{Bmatrix}2.5\cdot(200-175)\\ \frac{200-41.13}{3}\\ \frac{200}{2}\end{Bmatrix}}=\frac{0.9559}{\min\begin{Bmatrix}62.5\\ 52.96\\ 100\end{Bmatrix}}=0.01805$$

The concrete and rebar strain difference:

$$\Delta\varepsilon=\varepsilon_{\mathrm{sm}}-\varepsilon_{\mathrm{cm}}=\max\begin{Bmatrix}\frac{\sigma_{s,\alpha_0}-k_t\frac{f_{ct,eff}}{\rho_{p,eff}}(1+\alpha_e\rho_{p,eff})}{E_s}\\ 0.6\frac{\sigma_{s,\alpha_0}}{E_s}\end{Bmatrix}=$$

$$=\max\begin{Bmatrix}\frac{192.9-0.4\cdot\frac{2.2}{0.01805}\cdot(1+6.667\cdot0.01805)}{200000}\\ 0.6\cdot\frac{192.9}{200000}\end{Bmatrix}=\max\begin{Bmatrix}0.0006914\\ 0.0005787\end{Bmatrix}=0.0006914$$

The criterium of the spacing of the bonded bars that they are close to each other or not:

$$sp_{cr}=5(c_{\xi}+\phi_l/2)=5(20+10/2)=125\,\mathrm{mm}$$

The effective spacing of the bonded bars in the direction of the crack:

$$sp=\frac{\phi_l^2\pi/4}{a_{\alpha_0}^{bot}}=\frac{10^2\pi/4}{0.9559}=82.16\,\mathrm{mm}$$

Now the maximum crack spacing:

$sp\leqslant sp_{cr}$ thus:

$$s_{r,max}=3.4c_{\xi}+0.425k_1k_2\frac{\phi_l}{\rho_{p,eff}}=3.4\cdot20+0.425\cdot0.8\cdot0.5\cdot\frac{10}{0.01805}=162.2\,\mathrm{mm}$$

Thus the crack width:

$$w_k^{bot}=s_{r,max}\Delta\varepsilon=162.2\cdot0.0006914=0.1121\,\mathrm{mm}$$

Numerical results:

$w_{k,FEM}^{bot} = 0.1119\,\mathrm{mm}$

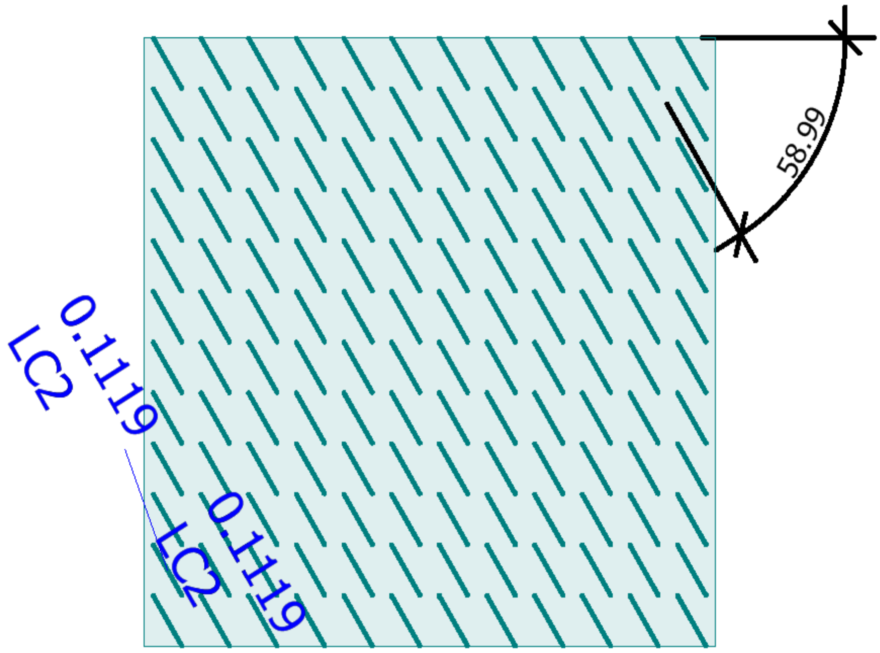


Figure 9.2.5.2.2 – The crack width [mm] *and the direction of the cracks at the bottom*

Fig. 9.2.5.2.2 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is less than 1%.

Crack width on top:

Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:

$$\underline{\underline{E}}=\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}=\begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix}=\begin{bmatrix} -228.5 & -85.71 \\ -85.71 & 114.3 \end{bmatrix}\frac{\text{kN}}{\text{m}} \quad \text{, where}$$

$$E_x=\frac{-m_x}{z}=\frac{-32}{0.14}=-228.6\frac{\text{kN}}{\text{m}} \quad ,$$

$$E_y=\frac{-m_y}{z}=\frac{+16}{0.14}=+114.3\frac{\text{kN}}{\text{m}} \quad ,$$

$$E_{xy}=\frac{-m_{xy}}{z}=\frac{-12}{0.14}=-85.71\frac{\text{kN}}{\text{m}} \quad .$$

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:

$$\underline{\underline{R}}=\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}\doteq\begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix}=\begin{bmatrix} 62.5 & 0 \\ 0 & 327.25 \end{bmatrix}\frac{\text{kN}}{\text{m}} \quad \text{, where}$$

$$A_{s\eta}=a_{s\eta}^{top} f_{yk}=654.5\cdot 500=327.25\frac{\text{kN}}{\text{m}} \quad .$$

If there is no reinforcement on top in the other direction we need to consider somehow the tensile resistance of the concrete, because this may effect the relevant direction of the crack.

$$A_{s\xi}=c\cdot f_{ctm}=25\cdot 2.2=62.5\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_x=A_{s\xi}\cos^2(\xi)+A_{s\eta}\cos^2(\eta)=62.5\cos^2(0°)+327.25\cos^2(90°)=62.5\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_y=A_{s\xi}\sin^2(\xi)+A_{s\eta}\sin^2(\eta)=62.5\sin^2(0°)+327.25\sin^2(90°)=327.25\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_{xy}=A_{s\xi}\cos(\xi)\sin(\xi)+A_{s\eta}\cos(\eta)\sin(\eta)=62.5\cos(0°)\sin(0°)+327.25\cos(90°)\sin(90°)=$$
$$=0\frac{\text{kN}}{\text{m}} \quad .$$

The tensor of the reserve forces:

$$\underline{\underline{r}}^*=\begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix}\doteq\begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix}=\begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix}-r\begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix}=\begin{bmatrix} R_x-r\,E_x & R_{xy}-r\,E_{xy} \\ R_{xy}-r\,E_{xy} & R_y-r\,E_y \end{bmatrix} \quad ,$$

where r is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:

$$r_2^* = 0$$

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

$$r_2^* = \left(\frac{r_x^* + r_y^*}{2}\right) - \sqrt{\left(\frac{r_x^* - r_y^*}{2}\right)^2 + r_{xy}^{*\,2}} = 0$$

$$r_2^* = \left(\frac{R_x - r E_x + R_y - r E_y}{2}\right) - \sqrt{\left(\frac{R_x - r E_x - R_y + r E_y}{2}\right)^2 + (R_{xy} - r E_{xy})^2} = 0$$

This equation gives two solutions for the r scalar internal force multiplier. The smallest positive r value has physical meaning. Without further detailed calculation the relevant r scalar load multiplier is:

$$r = 2.291$$

Based on this scalar value the reserve force tensor:

$$\underline{\underline{r}}^* = \begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} = \begin{bmatrix} 651.4 & 0 \\ 0 & 0 \end{bmatrix} \doteq \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} 586.0 & 196.4 \\ 196.4 & 65.39 \end{bmatrix} \frac{\mathrm{kN}}{\mathrm{m}}$$

The principal direction of the first principal reserve force gives the direction of the crack:

$$\alpha = \arctan\frac{r_1^* - r_x^*}{r_{xy}^*} = \arctan\frac{651.4 - 586.0}{196.4} = 18.42^\circ$$

Now we can calculate in this direction at the bottom the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

$$\alpha_0 = \alpha + 90^\circ = 108.4^\circ$$

The effective reinforcement area in this direction:

$$a_{\alpha_0}^{bot} = a_{sx}^{bot}\cos^2(\alpha_0 - \xi) + a_{sy}^{bot}\cos^2(\alpha_0 - \eta) =$$

$$= 1309\cos^2(108.4^\circ - 0^\circ) + 0\cos^2(108.4^\circ - 90^\circ) = 130.4\frac{\mathrm{mm}^2}{\mathrm{m}}$$

$$a_{\alpha_0}^{top} = a_{sx}^{top}\cos^2(\alpha_0 - \xi) + a_{sy}^{top}\cos^2(\alpha_0 - \eta) =$$

$$= 0\cos^2(108.4^\circ - 0^\circ) + 654.5\cos^2(108.4^\circ - 90^\circ) = 589.3\frac{\mathrm{mm}^2}{\mathrm{m}}$$

The bending moment in the direction perpendicular to the crack direction:

$$m_{\alpha_0}=m_x\cos^2\alpha_0+m_y\sin^2\alpha_0+2m_{xy}\cos\alpha_0\sin\alpha_0=$$

$$=(32)\cos^2 108.4°+(-16)\sin^2 108.4°+2\cdot 12\cos 108.4°\sin 108.4°=-18.41\,\frac{\text{kNm}}{\text{m}}$$

The position of the neutral axis according to the uncracked section (Stadium I.):

$$\alpha_e=\frac{E_s}{E_{cm}}=\frac{200}{30}=6.667 \quad ; \quad x_I=\frac{\frac{h^2}{2}+\alpha_e a_{\alpha_0}^{bot}d'_\xi+\alpha_e a_{\alpha_0}^{top}d_\eta}{h+\alpha_e a_{\alpha_0}^{bot}+\alpha_e a_{\alpha_0}^{top}}=100.9\,\text{mm}$$

The moment of inertia (Stadium I.):

$$I_I=\frac{x_I^3}{3}+\frac{(h-x_I)^3}{3}+\alpha_e a_{\alpha_0}^{top}(d_\eta-x_I)^2+\alpha_e a_{\alpha_0}^{bot}(x_I-d'_\xi)^2=6.880\cdot 10^8\,\frac{\text{mm}^4}{\text{m}}$$

Concrete tensile stress (Stadium I.) to check the crack exist or not:

$$\sigma_{c,\alpha_0}=\frac{-m_{\alpha_0}(h-x_I)}{I_I}=\frac{18.41\cdot 10^3\cdot(200-100.9)}{6.880\cdot 10^5}=2.65\,\frac{\text{N}}{\text{mm}^2}>f_{ctm}=2.2\,\text{MPa} \quad \text{crack occurred.}$$

The position of the neutral axis according to the cracked section (Stadium II.):

In case of this hyperbolic bending the applied reinforcements are only in one direction on one side, thus we considered the real effective height of the tensile and compressed bars instead of the average values which ones were introduced at the beginning of this chapter.

$$x_{II}\frac{x_{II}}{2}+\alpha_e a_{\alpha_0}^{bot}(x_{II}-d'_\xi)=\alpha_e a_{\alpha_0}^{top}(d_\eta-x_{II}) \quad \text{thus:} \quad x_{II}=32.12\,\text{mm}$$

The moment of inertia (Stadium II.):

$$I_{II}=\frac{x_{II}^3}{3}+\alpha_e a_{\alpha_0}^{top}(d_\eta-x_{II})^2+\alpha_e a_{\alpha_0}^{bot}(x_{II}-d'_\xi)^2=$$

$$=\frac{32.12^3}{3}+6.667\cdot 0.5893\cdot(165-32.12)^2+6.667\cdot 0.1304\cdot(32.12-25)^2=8.046\cdot 10^7\,\frac{\text{mm}^4}{\text{m}}$$

Rebar stress:

$$\sigma_{s,\alpha_0}=\alpha_e\frac{-m_{\alpha_0}(d_\eta-x_{II})}{I_{II}}=6.667\,\frac{18.41\cdot 10^3\cdot(165-32.12)}{8.046\cdot 10^4}=202.7\,\frac{\text{N}}{\text{mm}^2}$$

Effective tensile rebar ratio:

$$\rho_{p,eff}=\frac{a_{\alpha_0}^{top}}{a_{c,eff}}=\frac{a_{\alpha_0}^{top}}{h_{c,eff}}=\frac{0.5893}{\min\begin{Bmatrix}2.5(h-d_\eta)\\ \frac{h-x_{II}}{3}\\ \frac{h}{2}\end{Bmatrix}}=\frac{0.5893}{\min\begin{Bmatrix}2.5\cdot(200-165)\\ \frac{200-32.12}{3}\\ \frac{200}{2}\end{Bmatrix}}=\frac{0.5893}{\min\begin{Bmatrix}87.5\\ 55.96\\ 100\end{Bmatrix}}=0.01053$$

The concrete and rebar strain difference:

$$\Delta\varepsilon=\varepsilon_{\mathrm{sm}}-\varepsilon_{\mathrm{cm}}=\max\begin{Bmatrix}\frac{\sigma_{s,\alpha_0}-k_t\frac{f_{ct,eff}}{\rho_{p,eff}}(1+\alpha_e\rho_{p,eff})}{E_s}\\ 0.6\frac{\sigma_{s,\alpha_0}}{E_s}\end{Bmatrix}=$$

$$=\max\begin{Bmatrix}\frac{202.7-0.4\cdot\frac{2.2}{0.01053}\cdot(1+6.667\cdot 0.01053)}{200000}\\ 0.6\cdot\frac{202.7}{200000}\end{Bmatrix}=\max\begin{Bmatrix}0.0005663\\ 0.0006081\end{Bmatrix}=0.0006081$$

The criterium of the spacing of the bonded bars that they are close to each other or not:

$$sp_{cr}=5(c_\eta+\phi_l/2)=5(30+10/2)=175\,\mathrm{mm}$$

The effective spacing of the bonded bars in the direction of the crack:

$$sp=\frac{\phi_l^2\pi/4}{a_{\alpha_0}^{bot}}=\frac{10^2\pi/4}{0.5893}=133.3\,\mathrm{mm}$$

Now the maximum crack spacing:

$sp\leqslant sp_{cr}$ thus:

$$s_{r,max}=3.4c_\eta+0.425k_1k_2\frac{\phi_l}{\rho_{p,eff}}=3.4\cdot 30+0.425\cdot 0.8\cdot 0.5\cdot\frac{10}{0.01053}=263.4\,\mathrm{mm}$$

Thus the crack width:

$$w_k^{top}=s_{r,max}\Delta\varepsilon=263.4\cdot 0.0006081=0.1602\,\mathrm{mm}$$

Numerical results:

$w_{k,FEM}^{top} = 0.1602\,\text{mm}$

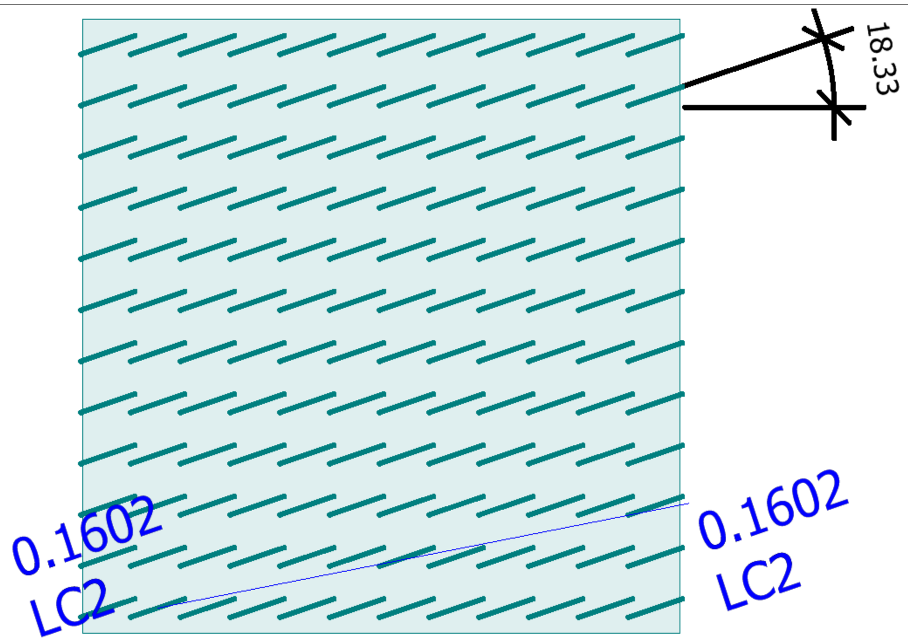


Figure 9.2.5.2.3 – The crack width [mm] *and the direction of the cracks at the top*

Fig. 9.2.5.2.3 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is 0%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.5.2 Crack width calculation in a slab with hyperbolic bending and orthogonal reinforcement.str

Non-orthogonal reinforcement (φ=75° between ξ and η)

The reinforcement is non-orthogonal and the ξ direction concides with the local x direction. The angle between the ξ directional reinforcement and η directional reinforcement is φ=75°. Fig. 9.2.5.2.4 shows the applied reinforcements and the concrete covers. Thus $\xi = 0°$; $\eta = 75°$.

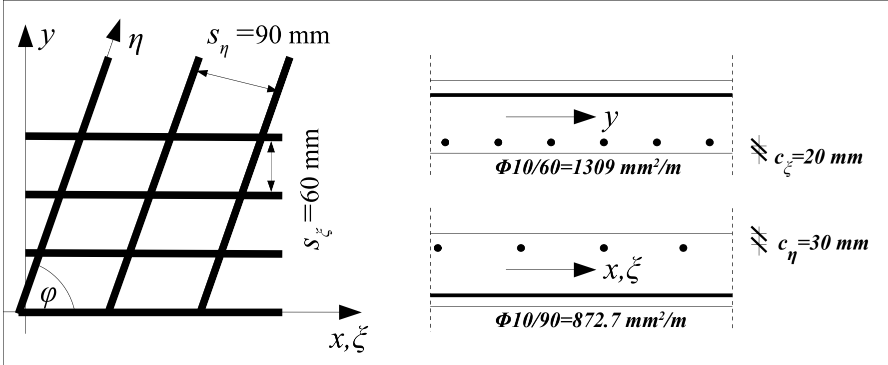


Figure 9.2.5.2.4 – The applied skew reinforcement for elliptic bending

The applied reinforcement in the slab:

Top and bottom reinforcement are also necessary but in this case only in one direction (see Chapter 9.2.2 for further information).

Bottom:

$$a_{s\xi}^{bot} = \frac{\phi_\xi^2 \pi}{4}\frac{1000}{s_\xi} = \frac{10^2 \pi}{4}\frac{1000}{60} = 1309\frac{\mathrm{mm}^2}{\mathrm{m}}$$

$$a_{s\eta}^{bot} = 0$$

Top:

$$a_{s\xi}^{top} = 0$$

$$a_{s\eta}^{top} = \frac{\phi_\eta^2 \pi}{4}\frac{1000}{s_\eta} = \frac{10^2 \pi}{4}\frac{1000}{90} = 872.7\frac{\mathrm{mm}^2}{\mathrm{m}}$$

Crack width on bottom:

Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:

$$\underline{\underline{E}} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} 228.5 & 85.71 \\ 85.71 & -114.3 \end{bmatrix}\frac{\mathrm{kN}}{\mathrm{m}}$$, where

$$E_x = \frac{m_x}{z} = \frac{32}{0.14} = 228.6\frac{\mathrm{kN}}{\mathrm{m}}$$,

$$E_y=\frac{m_y}{z}=\frac{-16}{0.14}=-114.3\,\frac{\mathrm{kN}}{\mathrm{m}} \quad ,$$

$$E_{xy}=\frac{m_{xy}}{z}=\frac{12}{0.14}=85.71\,\frac{\mathrm{kN}}{\mathrm{m}} \quad .$$

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:

$$\underline{\underline{R}}=\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}=\begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix}=\begin{bmatrix} 654.5 & 0 \\ 0 & 62.5 \end{bmatrix}\frac{\mathrm{kN}}{\mathrm{m}} \quad \text{, where}$$

$$A_{s\xi}=a_{s\xi}^{bot} f_{yk}=1309\cdot 500=654.5\,\frac{\mathrm{kN}}{\mathrm{m}} \quad .$$

If there is no reinforcement on top in the other direction we need to consider somehow the tensile resistance of the concrete perpendicular to the rebars, because this may effect the relevant direction of the crack.

$$A_{s(\xi+90^o)}=c\cdot f_{ctm}=25\cdot 2.2=62.5\,\frac{\mathrm{kN}}{\mathrm{m}} \quad ,$$

$$R_x=A_{s\xi}\cos^2(\xi)+A_{s(\xi+90^o)}\cos^2(\xi+90^\circ)=654.5\cos^2(0^\circ)+62.5\cos^2(90^\circ)=654.5\,\frac{\mathrm{kN}}{\mathrm{m}} \quad ,$$

$$R_y=A_{s\xi}\sin^2(\xi)+A_{s(\xi+90^o)}\sin^2(\xi+90^\circ)=654.5\sin^2(0^\circ)+62.5\sin^2(90^\circ)=62.5\,\frac{\mathrm{kN}}{\mathrm{m}} \quad ,$$

$$R_{xy}=A_{s\xi}\cos(\xi)\sin(\xi)+A_{s(\xi+90^o)}\cos(\xi+90^\circ)\sin(\xi+90^\circ)=$$

$$=654.5\cos(0^\circ)\sin(0^\circ)+62.5\cos(90^\circ)\sin(90^\circ)=0\,\frac{\mathrm{kN}}{\mathrm{m}} \quad .$$

The tensor of the reserve forces:

$$\underline{\underline{r}}^*=\begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix}\doteq\begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix}=\begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix}-r\begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix}=\begin{bmatrix} R_x-rE_x & R_{xy}-rE_{xy} \\ R_{xy}-rE_{xy} & R_y-rE_y \end{bmatrix} \quad ,$$

where r is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:

$$r_2^*=0$$

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

$$r_2^*=\left(\frac{r_x^*+r_y^*}{2}\right)-\sqrt{\left(\frac{r_x^*-r_y^*}{2}\right)^2+r_{xy}^{*\,2}}=0$$

$$r_2^* = \left(\frac{R_x - r\,E_x + R_y - r\,E_y}{2} \right) - \sqrt{\left(\frac{R_x - r\,E_x - R_y + r\,E_y}{2} \right)^2 + (R_{xy} - r\,E_{xy})^2} = 0$$

This equation gives two solutions for the r scalar internal force multiplier. The smallest positive r value has physical meaning. Without further detailed calculation the relevant r scalar load multiplier is:

$$r = 2.337$$

Based on this scalar value the reserve force tensor:

$$\underline{\underline{r^*}} = \begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} = \begin{bmatrix} 450.1 & 0 \\ 0 & 0 \end{bmatrix} \doteq \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} 120.5 & -200.3 \\ -200.3 & 329.6 \end{bmatrix} \frac{\text{kN}}{\text{m}}$$

The principal direction of the first principal reserve force gives the direction of the crack:

$$\alpha = \arctan \frac{r_1^* - r_x^*}{r_{xy}^*} = \arctan \frac{450.1 - 120.5}{-200.3} = -58.71°$$

Now we can calculate in this direction at the bottom the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

$$\alpha_0 = \alpha + 90° = 31.29°$$

The effective reinforcement area in this direction:

$$a_{\alpha_0}^{bot} = a_{s\xi}^{bot} \cos^2(\alpha_0 - \xi) + a_{s\eta}^{bot} \cos^2(\alpha_0 - \eta) =$$

$$= 1309 \cos^2(31.29° - 0°) + 0 \cos^2(31.29° - 75°) = 955.9 \frac{\text{mm}^2}{\text{m}}$$

$$a_{\alpha_0}^{top} = a_{s\xi}^{top} \cos^2(\alpha_0 - \xi) + a_{s\eta}^{top} \cos^2(\alpha_0 - \eta) =$$

$$= 0 \cos^2(31.29° - 0°) + 872.7 \cos^2(31.29° - 75°) = 456.0 \frac{\text{mm}^2}{\text{m}}$$

The effective internal forces in the direction perpendicular to the crack direction:

$$m_{\alpha_0} = m_x \cos^2 \alpha_0 + m_y \sin^2 \alpha_0 + 2\,m_{xy} \cos \alpha_0 \sin \alpha_0 =$$

$$= (32) \cos^2 31.29° + (-16) \sin^2 31.29° + 2 \cdot 12 \cos 31.29° \sin 31.29° = 29.70 \frac{\text{kNm}}{\text{m}}$$

The position of the neutral axis according to the uncracked section (Stadium I.):

$$\alpha_e = \frac{E_s}{E_{cm}} = \frac{200}{30} = 6.667 \quad ; \quad x_I = \frac{\frac{h^2}{2} + \alpha_e a_{\alpha_0}^{bot} d_\xi + \alpha_e a_{\alpha_0}^{top} d'_\eta}{h + \alpha_e a_{\alpha_0}^{bot} + \alpha_e a_{\alpha_0}^{top}} = 101.3\,\mathrm{mm}$$

The moment of inertia (Stadium I.):

$$I_I = \frac{x_I^3}{3} + \frac{(h - x_I)^3}{3} + \alpha_e a_{\alpha_0}^{bot} (d_\eta - x_I)^2 + \alpha_e a_{\alpha_0}^{top} (x_I - d'_\eta)^2 = 7.150 \cdot 10^8 \frac{\mathrm{mm}^4}{\mathrm{m}}$$

Concrete tensile stress (Stadium I.) to check the crack exist or not:

$$\sigma_{c,\alpha_0} = \frac{m_{\alpha_0}(h - x_I)}{I_I} = \frac{29.70 \cdot 10^3 \cdot (200 - 101.3)}{7.150 \cdot 10^5} = 4.10 \frac{\mathrm{N}}{\mathrm{mm}^2} > f_{ctm} = 2.2\,\mathrm{MPa} \quad \text{crack occurred.}$$

The position of the neutral axis according to the cracked section (Stadium II.):

In case this of hyperbolic bending the applied reinforcements are only in one direction on one side, thus we considered the real effective height of the tensile and compressed bars instead of the average values which ones were introduced at the beginning of this chapter.

$$x_{II} \frac{x_{II}}{2} + \alpha_e a_{\alpha_0}^{top} (x_{II} - d'_\eta) = \alpha_e a_{\alpha_0}^{bot} (d_\xi - x_{II}) \quad \text{thus:} \quad x_{II} = 40.91\,\mathrm{mm}$$

The moment of inertia (Stadium II.):

$$I_{II} = \frac{x_{II}^3}{3} + \alpha_e a_{\alpha_0}^{bot} (d_\xi - x_{II})^2 + \alpha_e a_{\alpha_0}^{top} (x_{II} - d'_\eta)^2 =$$

$$= \frac{40.91^3}{3} + 6.667 \cdot 0.9559 \cdot (175 - 40.91)^2 + 6.667 \cdot 0.456 \cdot (40.91 - 35)^2 = 1.3752 \cdot 10^8 \frac{\mathrm{mm}^4}{\mathrm{m}}$$

Rebar stress:

$$\sigma_{s,\alpha_0} = \alpha_e \frac{m_{\alpha_0}(d_\xi - x_{II})}{I_{II}} = 6.667 \frac{29.70 \cdot 10^3 \cdot (175 - 40.91)}{1.3752 \cdot 10^5} = 193.1 \frac{\mathrm{N}}{\mathrm{mm}^2}$$

Effective tensile rebar ratio:

$$\rho_{p,eff}=\frac{a_{\alpha_0}^{bot}}{a_{c,eff}}=\frac{a_{\alpha_0}^{bot}}{h_{c,eff}}=\frac{0.9559}{\min\begin{Bmatrix}2.5(h-d_\xi)\\ \dfrac{h-x_{II}}{3}\\ \dfrac{h}{2}\end{Bmatrix}}=\frac{0.9559}{\min\begin{Bmatrix}2.5\cdot(200-175)\\ \dfrac{200-40.91}{3}\\ \dfrac{200}{2}\end{Bmatrix}}=\frac{0.9559}{\min\begin{Bmatrix}62.5\\ 53.03\\ 100\end{Bmatrix}}=0.01803$$

The concrete and rebar strain difference:

$$\Delta\varepsilon=\varepsilon_{\mathrm{sm}}-\varepsilon_{\mathrm{cm}}=\max\begin{Bmatrix}\dfrac{\sigma_{s,\alpha_0}-k_t\dfrac{f_{ct,eff}}{\rho_{p,eff}}(1+\alpha_e\rho_{p,eff})}{E_s}\\ 0.6\dfrac{\sigma_{s,\alpha_0}}{E_s}\end{Bmatrix}=$$

$$=\max\begin{Bmatrix}\dfrac{193.1-0.4\cdot\dfrac{2.2}{0.01803}\cdot(1+6.667\cdot0.01803)}{200000}\\ 0.6\cdot\dfrac{193.1}{200000}\end{Bmatrix}=\max\begin{Bmatrix}0.0006921\\ 0.0005793\end{Bmatrix}=0.0006921$$

The criterium of the spacing of the bonded bars that they are close to each other or not:

$$sp_{cr}=5(c_\xi+\phi_l/2)=5(20+10/2)=125\,\mathrm{mm}$$

The effective spacing of the bonded bars in the direction of the crack:

$$sp=\frac{\phi_l^2\pi/4}{a_{\alpha_0}^{bot}}=\frac{10^2\pi/4}{0.9559}=82.16\,\mathrm{mm}$$

Now the maximum crack spacing:

$sp\leqslant sp_{cr}$ thus:

$$s_{r,max}=3.4c_\xi+0.425k_1k_2\frac{\phi_l}{\rho_{p,eff}}=3.4\cdot20+0.425\cdot0.8\cdot0.5\cdot\frac{10}{0.01805}=162.2\,\mathrm{mm}$$

Thus the crack width:

$$w_k^{bot}=s_{r,max}\Delta\varepsilon=162.2\cdot0.0006921=0.1123\,\mathrm{mm}$$

Numerical results:

$w_{k,FEM}^{bot} = 0.1121\,\text{mm}$

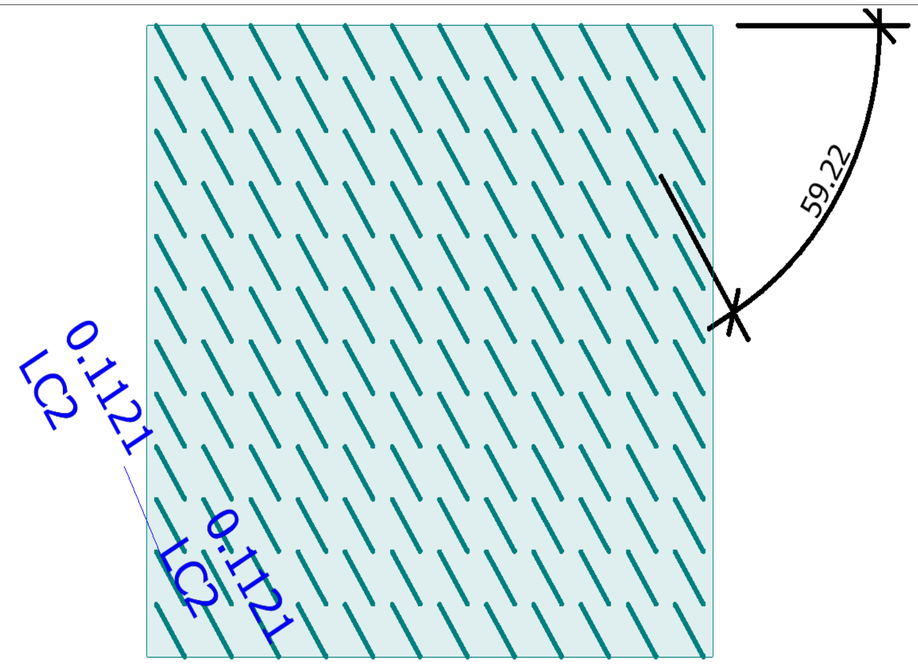


Figure 9.2.5.2.5 – The crack width [mm] *and the direction of the cracks at the bottom*

Fig. 9.2.5.2.5 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is less than 1%.

Crack width on top:

Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:

$$\underline{\underline{E}}=\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}=\begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix}=\begin{bmatrix} -228.5 & -85.71 \\ -85.71 & 114.3 \end{bmatrix}\frac{\text{kN}}{\text{m}} \text{, where}$$

$$E_x=\frac{-m_x}{z}=\frac{-32}{0.14}=-228.6\frac{\text{kN}}{\text{m}} \quad ,$$

$$E_y=\frac{-m_y}{z}=\frac{+16}{0.14}=+114.3\frac{\text{kN}}{\text{m}} \quad ,$$

$$E_{xy}=\frac{-m_{xy}}{z}=\frac{-12}{0.14}=-85.71\frac{\text{kN}}{\text{m}} \quad .$$

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:

$$\underline{\underline{R}}=\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}=\begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix}=\begin{bmatrix} 87.55 & 93.48 \\ 93.48 & 411.4 \end{bmatrix}\frac{\text{kN}}{\text{m}} \text{, where}$$

$$A_{s\eta}=a_{s\eta}^{top} f_{yk}=872.7\cdot 500=436.4\frac{\text{kN}}{\text{m}} \quad .$$

If there is no reinforcement in the other direction on top we need to consider somehow the tensile resistance of the concrete perpendicular to the rebars, because this may effect the relevant direction of the crack.

$$A_{s(\eta+90^o)}=c\cdot f_{ctm}=25\cdot 2.2=62.5\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_x=A_{s(\eta+90^o)}\cos^2(\eta+90^o)+A_{s\eta}\cos^2(\eta)=62.5\cos^2(165^\circ)+436.4\cos^2(75^\circ)=87.55\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_y=A_{s(\eta+90^o)}\sin^2(\eta+90^o)+A_{s\eta}\sin^2(\eta)=62.5\sin^2(165^\circ)+436.4\sin^2(75^\circ)=411.4\frac{\text{kN}}{\text{m}} \quad ,$$

$$R_{xy}=A_{s(\eta+90^o)}\cos(\eta+90^o)\sin(\eta+90^o)+A_{s\eta}\cos(\eta)\sin(\eta)=$$

$$=62.5\cos(165^\circ)\sin(165^\circ)+436.4\cos(75^\circ)\sin(75^\circ)=$$

$$=93.48\frac{\text{kN}}{\text{m}} \quad .$$

The tensor of the reserve forces:

$$\underline{\underline{r}}^*=\begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix}=\begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix}=\begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix}-r\begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix}=\begin{bmatrix} R_x-r\,E_x & R_{xy}-r\,E_{xy} \\ R_{xy}-r\,E_{xy} & R_y-r\,E_y \end{bmatrix} \quad ,$$

where r is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:

$$r_2^* = 0$$

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

$$r_2^* = \left(\frac{r_x^* + r_y^*}{2}\right) - \sqrt{\left(\frac{r_x^* - r_y^*}{2}\right)^2 + r_{xy}^{*\,2}} = 0$$

$$r_2^* = \left(\frac{R_x - r E_x + R_y - r E_y}{2}\right) - \sqrt{\left(\frac{R_x - r E_x - R_y + r E_y}{2}\right)^2 + (R_{xy} - r E_{xy})^2} = 0$$

This equation gives two solutions for the r scalar internal force multiplier. The smallest positive r value has physical meaning. Without further detailed calculation the relevant r scalar load multiplier is:

$$r = 2.375$$

Based on this scalar value the reserve force tensor:

$$\underline{\underline{r}}^* = \begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} = \begin{bmatrix} 770.2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} 630.2 & 297.0 \\ 297.0 & 140.0 \end{bmatrix} \frac{\text{kN}}{\text{m}}$$

The principal direction of the first principal reserve force gives the direction of the crack:

$$\alpha = \arctan\frac{r_1^* - r_x^*}{r_{xy}^*} = \arctan\frac{770.2 - 630.2}{297} = 25.24^\circ$$

Now we can calculate in this direction at the bottom the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

$$\alpha_0 = \alpha + 90^\circ = 115.2^\circ$$

The effective reinforcement area in this direction:

$$a_{\alpha_0}^{bot} = a_{s\xi}^{bot}\cos^2(\alpha_0 - \xi) + a_{s\eta}^{bot}\cos^2(\alpha_0 - \eta) =$$

$$= 1309\cos^2(115.2^\circ - 0^\circ) + 0\cos^2(115.2^\circ - 75^\circ) = 237.3\,\frac{\text{mm}^2}{\text{m}}$$

$$a_{\alpha_0}^{top} = a_{s\xi}^{top}\cos^2(\alpha_0 - \xi) + a_{s\eta}^{top}\cos^2(\alpha_0 - \eta) =$$

$$=0\cos^2(115.2^\circ-0^\circ)+872.7\cos^2(115.2^\circ-75^\circ)=509.1\,\frac{\mathrm{mm}^2}{\mathrm{m}}$$

The effective internal forces in the direction perpendicular to the crack direction:

$$m_{\alpha_0}=m_x\cos^2\alpha_0+m_y\sin^2\alpha_0+2m_{xy}\cos\alpha_0\sin\alpha_0=$$

$$=(32)\cos^2 115.2^\circ+(-16)\sin^2 115.2^\circ+2\cdot 12\cos 115.2^\circ\sin 115.2^\circ=-16.54\,\frac{\mathrm{kNm}}{\mathrm{m}}$$

The position of the neutral axis according to the uncracked section (Stadium I.):

$$\alpha_e=\frac{E_s}{E_{cm}}=\frac{200}{30}=6.667\quad;\quad x_I=\frac{\frac{h^2}{2}+\alpha_e a_{\alpha_0}^{bot}d'_\xi+\alpha_e a_{\alpha_0}^{top}d_\eta}{h+\alpha_e a_{\alpha_0}^{bot}+\alpha_e a_{\alpha_0}^{top}}=100.5\,\mathrm{mm}$$

The moment of inertia (Stadium I.):

$$I_I=\frac{x_I^3}{3}+\frac{(h-x_I)^3}{3}+\alpha_e a_{\alpha_0}^{top}(d_\eta-x_I)^2+\alpha_e a_{\alpha_0}^{bot}(x_I-d'_\xi)^2=6.899\cdot 10^8\,\frac{\mathrm{mm}^4}{\mathrm{m}}$$

Concrete tensile stress (Stadium I.) to check the crack exist or not:

$$\sigma_{c,\alpha_0}=\frac{-m_{\alpha_0}(h-x_I)}{I_I}=\frac{16.54\cdot 10^3\cdot(200-100.5)}{6.899\cdot 10^5}=2.39\,\frac{\mathrm{N}}{\mathrm{mm}^2}>f_{ctm}=2.2\,\mathrm{MPa}\quad \text{crack occurred.}$$

The position of the neutral axis according to the cracked section (Stadium II.):

In case of this hyperbolic bending the applied reinforcements are only in one direction on top side, thus we considered the real effective height of the tensile and compressed bars instead of the average values which ones were introduced at the beginning of this chapter.

$$x_{II}\frac{x_{II}}{2}+\alpha_e a_{\alpha_0}^{bot}(x_{II}-d'_\xi)=\alpha_e a_{\alpha_0}^{top}(d_\eta-x_{II})\quad \text{thus:}\quad x_{II}=30.02\,\mathrm{mm}$$

The moment of inertia (Stadium II.):

$$I_{II}=\frac{x_{II}^3}{3}+\alpha_e a_{\alpha_0}^{top}(d_\eta-x_{II})^2+\alpha_e a_{\alpha_0}^{bot}(x_{II}-d'_\xi)^2=$$

$$=\frac{30.02^3}{3}+6.667\cdot 0.5091\cdot(165-30.02)^2+6.667\cdot 0.2373\cdot(30.02-25)^2=7.090\cdot 10^7\,\frac{\mathrm{mm}^4}{\mathrm{m}}$$

Rebar stress:

$$\sigma_{s,\alpha_0}=\alpha_e\frac{-m_{\alpha_0}(d_\eta-x_{II})}{I_{II}}=6.667\frac{16.54\cdot10^3\cdot(165-30.02)}{7.090\cdot10^4}=209.9\frac{\mathrm{N}}{\mathrm{mm}^2}\quad;$$

Effective tensile rebar ratio:

$$\rho_{p,eff}=\frac{a_{\alpha_0}^{top}}{a_{c,eff}}=\frac{a_{\alpha_0}^{top}}{h_{c,eff}}=\frac{0.5091}{\min\begin{Bmatrix}2.5(h-d_\eta)\\ \dfrac{h-x_{II}}{3}\\ \dfrac{h}{2}\end{Bmatrix}}=\frac{0.5091}{\min\begin{Bmatrix}2.5\cdot(200-165)\\ \dfrac{200-30.02}{3}\\ \dfrac{200}{2}\end{Bmatrix}}=\frac{0.5091}{\min\begin{Bmatrix}87.5\\56.66\\100\end{Bmatrix}}=0.008985$$

The concrete and rebar strain difference:

$$\Delta\varepsilon=\varepsilon_{\mathrm{sm}}-\varepsilon_{\mathrm{cm}}=\max\begin{Bmatrix}\dfrac{\sigma_{s,\alpha_0}-k_t\dfrac{f_{ct,eff}}{\rho_{p,eff}}(1+\alpha_e\rho_{p,eff})}{E_s}\\ 0.6\dfrac{\sigma_{s,\alpha_0}}{E_s}\end{Bmatrix}=$$

$$=\max\begin{Bmatrix}\dfrac{209.9-0.4\cdot\dfrac{2.2}{0.008985}\cdot(1+6.667\cdot0.008985)}{200000}\\ 0.6\cdot\dfrac{209.9}{200000}\end{Bmatrix}=\max\begin{Bmatrix}0.0005305\\0.0006297\end{Bmatrix}=0.0006297$$

The criterium of the spacing of the bonded bars that they are close to each other or not:

$$sp_{cr}=5(c_\eta+\phi_l/2)=5(30+10/2)=175\ \mathrm{mm}$$

The effective spacing of the bonded bars in the direction of the crack:

$$sp=\frac{\phi_l^2\pi/4}{a_{\alpha_0}^{bot}}=\frac{10^2\pi/4}{0.5091}=154.3\ \mathrm{mm}$$

Now the maximum crack spacing:

$sp\leqslant sp_{cr}$ thus:

$$s_{r,max}=3.4c_\eta+0.425k_1k_2\frac{\phi_l}{\rho_{p,eff}}=3.4\cdot30+0.425\cdot0.8\cdot0.5\cdot\frac{10}{0.008985}=291.2\ \mathrm{mm}$$

Thus the crack width:

$$w_k^{top} = s_{r,max} \Delta\varepsilon = 263.4 \cdot 0.0006297 = 0.1834 \text{ mm}$$

Numerical results:

$$w_{k,FEM}^{top} = 0.1827 \text{ mm}$$

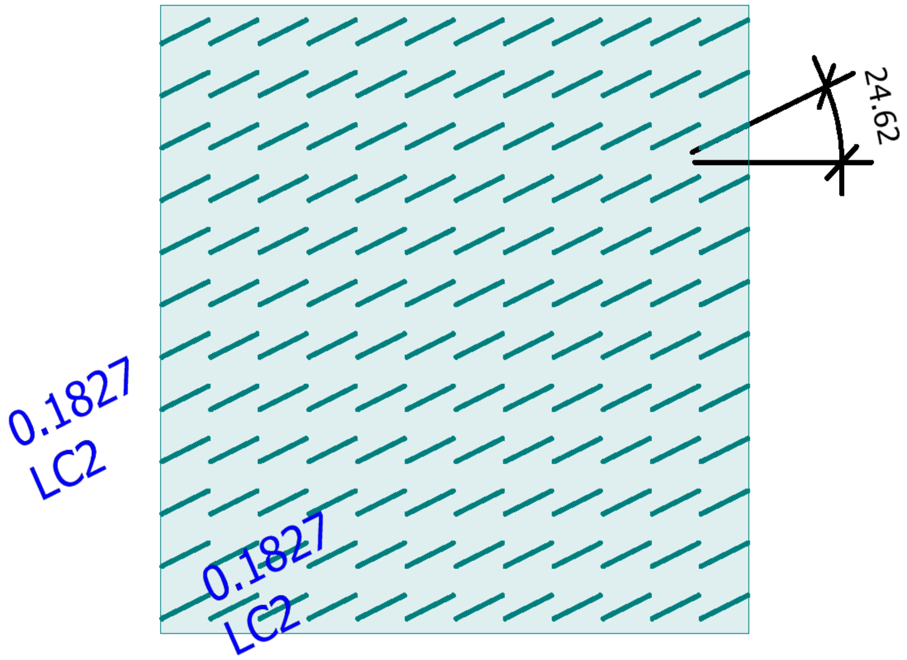


Figure 9.2.5.2.6 – The crack width [mm] *and the direction of the cracks at the top*

Fig. 9.2.5.2.6 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is less than 1%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.5.2 Crack width calculation in a slab with hyperbolic bending and skew reinforcement.str

9.2.6 Punching calculation of a slab

In this section we will check five different types of punching reinforcement:
bended bar, circular stirrups, open stirrups, stud rail, PSB stud rail.

Inputs:

Concrete	
Concrete characteristic compressive strength	$f_{ck} = 25\ N/mm^2$
Plate height	h = 200 mm
Cover	c = 20 mm
Partial factor of concrete	$\gamma_c = 1.50$
Reinforcement	
x' direction	
Reinforcing steel characteristic yield strength	$f_{yk} = 500\ N/mm^2$
Rebars Young modulus	$E_s = 200$ GPa
Partial factor of reinforcing steel	$\gamma_s = 1.15$
Longitudinal rebar diameter	$\phi_x = 20$ mm
Distance between longitudinal reinforcements	$s_x = 150$ mm
Longitudinal bars effective distance	$d_{l,x} = 170$ mm
y' direction	
Reinforcing steel characteristic yield strength	$f_{yk} = 500\ N/mm^2$
Rebars Young modulus	$E_s = 200$ GPa
Longitudinal rebar diameter	$\phi_y = 20$ mm
Distance between longitudinal reinforcements	$s_y = 150$ mm
Longitudinal bars effective distance	$d_{l,y} = 150$ mm
Geometry	
Plate width in x direction	$L_x = 4.0$ m
Plate width in y direction	$L_y = 4.0$ m
Circle column diameter	$d_{column} = 25$ cm
Specific normal force in x direction	$n_{x,Ed} = 40$ kN/m
Specific normal froce in y direction	$n_{y,Ed} = 40$ kN/m
Vertical surface total load	$q_{z,Ed} = 25\ kN/m^2$

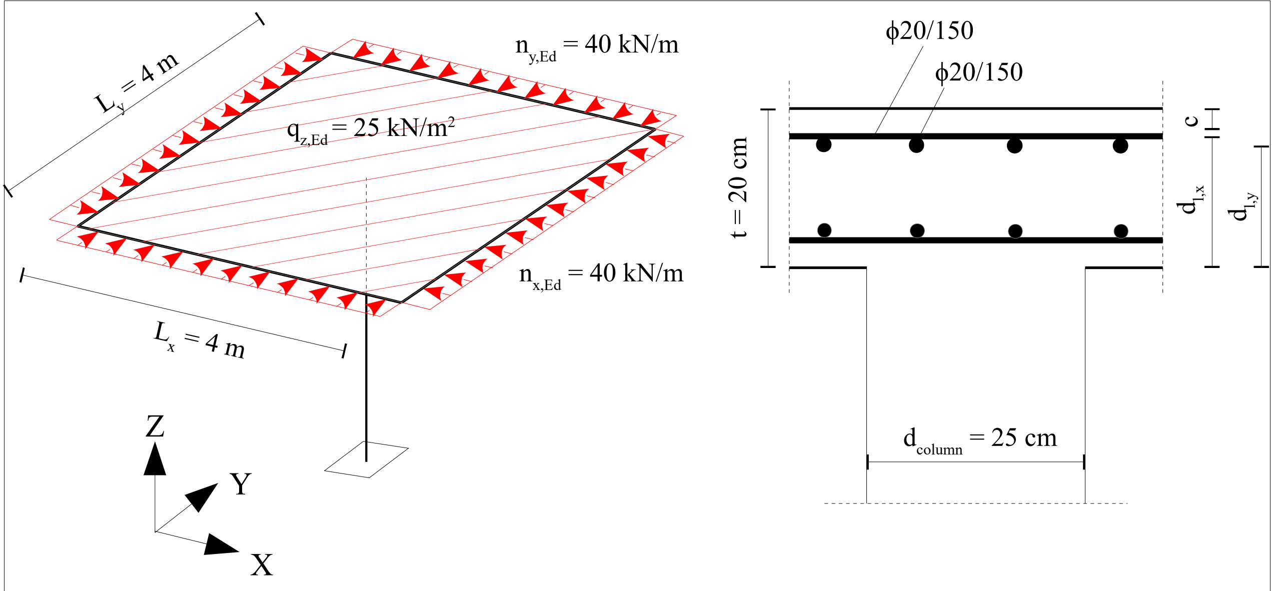


Figure 9.2.6.1 – The geometry, loads and the reinforcements

Fig. 9.2.6.1 shows the analyzed punching problem.

9.2.6.1 Bended bars

Inputs (see Fig. 9.2.6.1.1):

Reinforcement	
Reinforcing steel characteristic yield strength	$f_{ywk} = 500$ N/mm^2
Bended bar diameter	$\phi_{bb} = 14$ mm
Partial factor of reinforcing steel	$\gamma_s = 1.15$
Bended bar distance	$s_r = 120$ mm

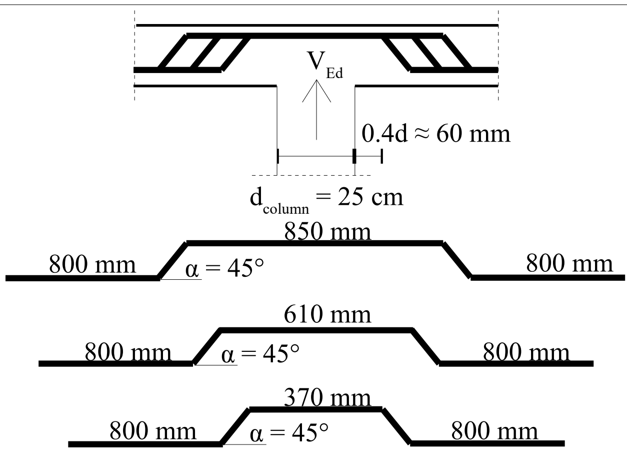


Figure 9.2.6.1.1 – The applied punching reinforcement in x direction (the same valid in y direction)

The effective longitudinal reinforcement ratios:

$$\rho_{l,x}=\frac{\frac{\phi_x^2\pi}{4}\frac{1000}{s_x}}{d_{l,x}\cdot 1000}=\frac{\frac{20^2\pi}{4}\frac{1000}{150}}{170\cdot 1000}=0.01232 \quad ; \quad \rho_{l,y}=\frac{\frac{\phi_y^2\pi}{4}\frac{1000}{s_y}}{d_{l,y}\cdot 1000}=\frac{\frac{20^2\pi}{4}\frac{1000}{150}}{150\cdot 1000}=0.01396$$

$$\rho_l=\min\begin{Bmatrix}\sqrt{\rho_{l,x}\cdot\rho_{l,y}}\\ 0.02\end{Bmatrix}=\min\begin{Bmatrix}\sqrt{0.01232\cdot 0.01396}\\ 0.02\end{Bmatrix}=\min\begin{Bmatrix}0.01312\\ 0.02\end{Bmatrix}=0.01312$$

The average effective height of the longitudinal reinforcements:

$$d=\frac{d_{l,x}+d_{l,y}}{2}=\frac{170+150}{2}=160\,\text{mm}$$

Concrete compression resistance

The column reaction:

$$V_{Ed}=L_x L_y q_{z,Ed}=4\cdot 4\cdot 25=400\,\text{kN}$$

The perimeter of the column:

$$u_0=d_{column}\pi=250\cdot\pi=785.40\,\text{mm}$$

Because this column is an inner column the β is:

$$\beta=1.15$$

The specific shear force at u_0:

$$v_{Ed}=\beta\frac{V_{Ed}}{d\,u_0}=1.15\frac{400000}{160\cdot 785.40}=3.66\frac{\text{N}}{\text{mm}^2}$$

The concrete compression resistance:

$$v_{Rd,max}=0.5\nu f_{cd}=0.5\cdot 0.54\frac{25}{1.5}=4.50\frac{\text{N}}{\text{mm}^2}$$ where: $$\nu=0.6\left(1-\frac{f_{ck}}{250}\right)=0.6\left(1-\frac{25}{250}\right)=0.54$$

Concrete compression utilization:

$$\frac{v_{Ed}}{v_{Rd,max}}=\frac{3.66}{4.5}=81.35\%$$

Utilization is less than 100% thus the slab can bear the acting shear force. Fig. 9.2.6.1.2 shows the relevant results with FEM-Design. The results are identical with the hand calculation.

Shear reinforcement resistance

Punching perimeter at *2d* distance from the edge of the column:

$$u_1=2\left(\frac{d_{column}}{2}+2d\right)\pi=2\left(\frac{250}{2}+2\cdot 160\right)\pi=2796\,mm$$

The shear force at u_1 perimeter:

$$v_{Ed}=\beta\frac{V_{Ed}}{d\,u_1}=1.15\frac{400000}{160\cdot 2796}=1.028\frac{\mathrm{N}}{\mathrm{mm}^2}$$

Modifying factors for concrete shear resistance:

$$C_{Rd,c}=\frac{0.18}{\gamma_c}=\frac{0.18}{1.5}=0.12\quad;\quad k_1=0.1$$

Average normal stress in the concrete:

$$\sigma_{cp}=\min\left\{\begin{matrix}\dfrac{\sigma_{cx}+\sigma_{cy}}{2}=\dfrac{\dfrac{n_{x,Ed}}{1000h}+\dfrac{n_{y,Ed}}{1000h}}{2}\\ 0.2f_{cd}\end{matrix}\right\}=\min\left\{\begin{matrix}\dfrac{\dfrac{40000}{1000\cdot 200}+\dfrac{40000}{1000\cdot 200}}{2}\\ 0.2\dfrac{25}{1.5}\end{matrix}\right\}=0.2\frac{\mathrm{N}}{\mathrm{mm}^2}$$

$$k=\min\left\{\begin{matrix}1+\sqrt{\dfrac{200}{d}}\\ 2\end{matrix}\right\}=\min\left\{\begin{matrix}1+\sqrt{\dfrac{200}{160}}\\ 2\end{matrix}\right\}=\min\left\{\begin{matrix}2.12\\ 2\end{matrix}\right\}=2$$

$$v_{min}=0.035k^{1.5}f_{ck}^{0.5}=0.035\cdot 2^{1.5}\cdot 25^{0.5}=0.495\frac{\mathrm{N}}{\mathrm{mm}^2}$$

The shear resistance of concrete:

$$v_{Rd,c}=\max\left\{\begin{matrix}C_{Rd,c}k(100\rho_l f_{ck})^{\frac{1}{3}}\\ v_{min}\end{matrix}\right\}+k_1\sigma_{cp}=\max\left\{\begin{matrix}0.12\cdot 2\cdot(100\cdot 0.01312\cdot 25)^{\frac{1}{3}}\\ 0.495\end{matrix}\right\}+0.1\cdot 0.2=$$
$$=\max\left\{\begin{matrix}0.768\\ 0.495\end{matrix}\right\}+0.1\cdot 0.2=0.788\frac{\mathrm{N}}{\mathrm{mm}^2}$$

We need to check if we need any punching reinforcement or not:

$$\frac{v_{Ed}}{v_{Rd,c}}=\frac{1.028}{0.788}=1.305>1.0$$

Thus we need punching reinforcement.

Bended bars area in the u_1 perimeter:

$$A_{sw}=4\frac{\phi_{bb}^2\pi}{4}=4\frac{14^2\pi}{4}=615.75\,\mathrm{mm}^2$$

Effective tensile strength of the bended bars:

$$f_{y,sw,eff}=\min\left\{\begin{matrix}f_{ywd}\\ 250+\dfrac{d}{4}\end{matrix}\right\}=\min\left\{\begin{matrix}\dfrac{f_{ywk}}{\gamma_s}\\ 250+\dfrac{d}{4}\end{matrix}\right\}=\min\left\{\begin{matrix}\dfrac{500}{1.15}\\ 250+\dfrac{160}{4}\end{matrix}\right\}=\min\left\{\begin{matrix}434.78\\ 290\end{matrix}\right\}=290\frac{\mathrm{N}}{\mathrm{mm}^2}$$

The punching reinforcement resistance:

$$v_{Rd,sw} = 1.5\frac{d}{s_r} A_{sw} f_{y,sw,eff} \frac{1}{u_1 d} \sin\alpha = 1.5\frac{160}{120} 615.75 \cdot 290 \frac{1}{2796 \cdot 160} \sin 45° = 0.564 \frac{\mathrm{N}}{\mathrm{mm}^2}$$

Punching shear resistance:

$$v_{Rd,cs} = \min\begin{bmatrix} 0.75\, v_{Rd,c} + v_{Rd,sw} \\ k_{max}\, v_{Rd,c} \end{bmatrix} = \min\begin{bmatrix} 0.75 \cdot 0.788 + 0.564 \\ 1.5 \cdot 0.788 \end{bmatrix} = 1.155 \frac{\mathrm{N}}{\mathrm{mm}^2}$$

Punching shear resistance utilization:

$$\frac{v_{Ed}}{v_{Rd,cs}} = \frac{1.028}{1.155} = 88.98\,\%$$

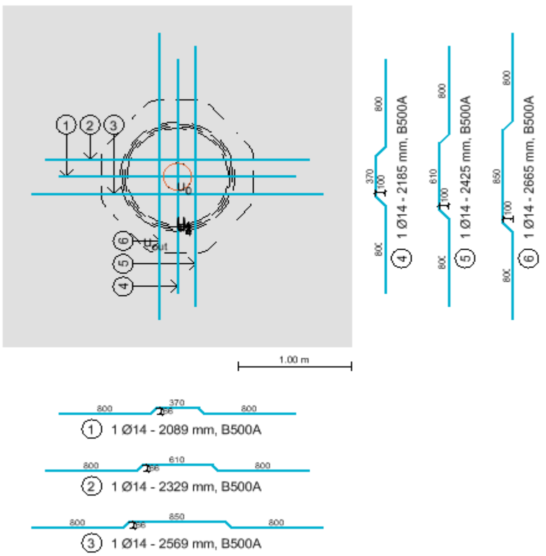
Fig. 9.2.6.1.2 and the table below shows the relevant FEM-Design results.

	Concrete compression	Shear reinforcement	Concrete shear
Hand calculation	4.5 N/mm^2	1.155 N/mm^2	0.788 N/mm^2
FEM-Design calculation	4.5 N/mm^2	1.18 N/mm^2	0.79 N/mm^2
Difference	0.00%	2.10 %	0.20 %

The differences between the numerical and hand calculations are less than 3%.

PU.(C.1).1
Maximum of load combinations

1:50



C25/30

t	=	200.00 mm
$f_{cd,ult.}$	=	16.67 N/mm²
$f_{cd,acc./seis.}$	=	20.83 N/mm²
f_{ck}	=	25.00 N/mm²
k_{max}	=	1.50

Concrete compression resistance - Part 1.1: 6.4.3
LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed,0}}{u_0 \cdot d} = \frac{1.15 \cdot 399691.34}{784 \cdot 160} = 3.66\ \text{N/mm}^2 \qquad (6.38)$$

$$v_{Rd,max} = 0.5 \cdot v \cdot f_{cd} = 0.5 \cdot 0.54 \cdot 16.67 = 4.50\ \text{N/mm}^2$$

$v_{Ed} = 3.66\ \text{N/mm}^2 \le v_{Rd,max} = 4.50\ \text{N/mm}^2$ (6.53) - OK

Shear reinforcement resistance - Part 1.1: 6.4.3

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_1 \cdot d} \qquad (6.38)$$

$$v_{Rd,sw} = 1.5 \frac{d}{s_r} A_{sw} f_{yws,ef} \frac{1}{u_1 d} \sin(\alpha)$$

$$v_{Rd,cs} = \min(0.75\, v_{Rd,c} + v_{Rd,sw}, k_{max}\, v_{Rd,c})$$

Perimeter index	1	2	3	4
Dist [mm]	320	340	360	380
V_{Ed} [kN]	399.69	399.69	399.69	399.69
β [-]	1.15	1.15	1.15	1.15
$\overline{d}$ [mm]	160	160	160	160
u [m]	2.79	2.92	3.04	3.17
v_{Ed} [N/mm²]	1.03	0.99	0.94	0.91
$v_{Rd,c}$ [N/mm²]	0.79	0.79	0.79	0.79
$v_{Rd,sw}$ [N/mm²]	0.60	0.57	0.55	0.53
$v_{Rd,cs}$ [N/mm²]	1.18	1.16	1.14	1.12
Utilization [%]	87	85	83	81

Concrete shear resistance - Part 1.1: 6.4.3
LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_{Out} \cdot d} = \frac{1.15 \cdot 399691.34}{3726 \cdot 160} = 0.77\ \text{N/mm}^2 \qquad (6.38)$$

$$v_{Rd,c} = \max\left(C_{Rd,c} \cdot k\,(100 \cdot \rho_l \cdot f_{ck})^{1/3}, v_{min}\right) + k_1 \cdot \sigma_{cp} =$$
$$= \max\left(0.12 \cdot 2.00\,(100 \cdot 0.0131 \cdot 25.00)^{1/3}, 0.49\right) + 0.10 \cdot 0.20 = 0.79\ \text{N/mm}^2 \qquad (6.47)$$

$v_{Ed} = 0.77\ \text{N/mm}^2 \le v_{Rd,c} = 0.79\ \text{N/mm}^2$ - OK

Figure 9.2.6.1.2 – The punching detailed results in FEM-Design

9.2.6.2 Circular stirrups

Inputs (see Fig. 9.2.6.2.1):

Reinforcement	
Reinforcing steel characteristic yield strength	$f_{ywk} = 500\ N/mm^2$
Circular stirrup diameter	$\phi_{stirrup} = 10\ mm$
Stirrup height	$h_s = 120\ mm$
Stirrup width	$w_s = s_r = 120\ mm$
Partial factor of reinforcing steel	$\gamma_s = 1.15$

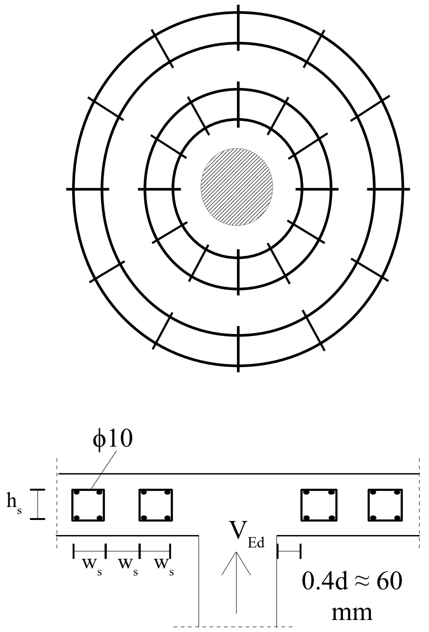


Figure 9.2.6.2.1 – The applied punching reinforcement

Concrete compression resistance

The calculation and the results are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Shear reinforcement resistance

The results of the shear resistance of the conrete without punching reinforcement are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Thus it means that we need punching reinforcement.

Punching perimeter at *2d* distance from the edge of the column:

$$u_1 = 2\left(\frac{d_{column}}{2} + 2d\right)\pi = 2\left(\frac{250}{2} + 2\cdot 160\right)\pi = 2796\,mm$$

The area of the 12 stirrups at the u_1 perimeter (see Fig. 9.2.6.2.1):

$$A_{sw} = 12\frac{\phi_{\text{stirrup}}^2\pi}{4} = 12\frac{10^2\pi}{4} = 942.48\,\text{mm}^2$$

Effective tensile strength of the stirrups:

$$f_{y,sw,eff} = \min\begin{Bmatrix} f_{ywd} \\ 250 + \frac{d}{4} \end{Bmatrix} = \min\begin{Bmatrix} \frac{f_{ywk}}{\gamma_s} \\ 250 + \frac{d}{4} \end{Bmatrix} = \min\begin{Bmatrix} \frac{500}{1.15} \\ 250 + \frac{160}{4} \end{Bmatrix} = \min\begin{Bmatrix} 434.78 \\ 290 \end{Bmatrix} = 290\frac{\text{N}}{\text{mm}^2}$$

The punching reinforcement resistance:

$$v_{Rd,sw} = 1.5\frac{d}{s_r}A_{sw}f_{y,sw,eff}\frac{1}{u_1 d}\sin\alpha = 1.5\frac{160}{120}942.48\cdot 290\frac{1}{2796\cdot 160}\sin 90° = 1.222\frac{\text{N}}{\text{mm}^2}$$

Punching shear resistance:

$$v_{Rd,cs} = \min\begin{bmatrix} 0.75v_{Rd,c} + v_{Rd,sw} \\ k_{max}v_{Rd,c} \end{bmatrix} = \min\begin{bmatrix} 0.75\cdot 0.788 + 1.222 \\ 1.5\cdot 0.788 \end{bmatrix} = 1.182\frac{\text{N}}{\text{mm}^2}$$

Shear reinforcement utilization:

$$\frac{v_{Ed}}{v_{Rd,cs}} = \frac{1.028}{1.182} = 86.97\,\%$$

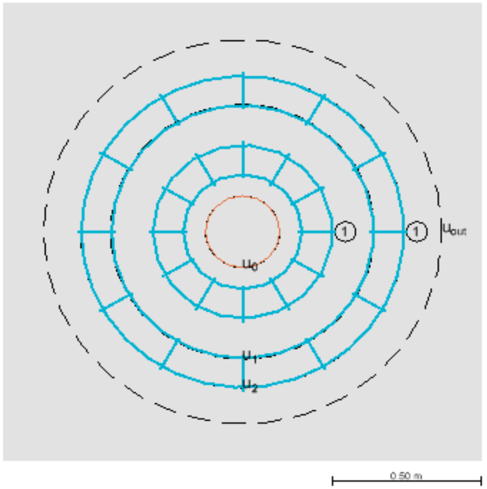
Fig. 9.2.6.2.2 shows the relevant FEM-Design results.

	Concrete compression	Shear reinforcement	Concrete shear
Hand calculation	4.5 N/mm^2	1.182 N/mm^2	0.788 N/mm^2
FEM-Design calculation	4.5 N/mm^2	1.18 N/mm^2	0.79 N/mm^2
Difference	0.00%	0.00 %	0.20 %

The differences between the numerical and hand calculations are under 0.5%.

PU.(C.2).1
Maximum of load combinations

1:20



C25/30		
t	=	200.00 mm
$f_{cd,ult.}$	=	16.67 N/mm^2
$f_{cd,acc./seis.}$	=	20.83 N/mm^2
f_{ck}	=	25.00 N/mm^2
k_{max}	=	1.50

(1) 24 Ø10 - 500 mm, B500B

Ø8 - 18.67 m, B500B, radiuses: 0.20, 0.30, 0.44, 0.54 m

Concrete compression resistance - Part 1.1: 6.4.3
LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed,0}}{u_0 \cdot d} = \frac{1.15 \cdot 399691.34}{784 \cdot 160} = 3.66\ \text{N/mm}^2 \quad (6.38)$$

$$v_{Rd,max} = 0.5 \cdot v \cdot f_{cd} = 0.5 \cdot 0.54 \cdot 16.67 = 4.50\ \text{N/mm}^2$$

$v_{Ed} = 3.66\ \text{N/mm}^2 \le v_{Rd,max} = 4.50\ \text{N/mm}^2$ (6.53) - OK

Shear reinforcement resistance - Part 1.1: 6.4.3

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_1 \cdot d} \quad (6.38)$$

$$v_{Rd,sw} = 1.5 \frac{d}{s_r} A_{sw} f_{yws,ef} \frac{1}{u_1 d} \sin(\alpha)$$

$$v_{Rd,cs} = \min(0.75\ v_{Rd,c} + v_{Rd,sw},\ k_{max}\ v_{Rd,c})$$

Perimeter index	1	2
Dist [mm]	320	420
V_{Ed} [kN]	399.69	399.69
β [-]	1.15	1.15
d [mm]	160	160
u [m]	2.79	3.42
v_{Ed} [N/mm^2]	1.03	0.84
$v_{Rd,c}$ [N/mm^2]	0.79	0.79
$v_{Rd,sw}$ [N/mm^2]	1.26	0.99
$v_{Rd,cs}$ [N/mm^2]	1.18	1.18
Utilization [%]	87	71

Concrete shear resistance - Part 1.1: 6.4.3
LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_{Out} \cdot d} = \frac{1.15 \cdot 399691.34}{4207 \cdot 160} = 0.68\ \text{N/mm}^2 \quad (6.38)$$

$$v_{Rd,c} = \max\left(C_{Rd,c} \cdot k\ (100 \cdot \rho_l \cdot f_{ck})^{1/3},\ v_{min}\right) + k_1 \cdot \sigma_{cp} =$$
$$= \max\left(0.12 \cdot 2.00\ (100 \cdot 0.0131 \cdot 25.00)^{1/3},\ 0.49\right) + 0.10 \cdot 0.20 = 0.79\ \text{N/mm}^2 \quad (6.47)$$

$v_{Ed} = 0.68\ \text{N/mm}^2 \le v_{Rd,c} = 0.79\ \text{N/mm}^2$ - OK

Figure 9.2.6.2.2 – The punching detailed results in FEM-Design

9.2.6.3 Open stirrups

Inputs:

Reinforcement	
Reinforcing steel characteristic yield strength	f_{ywk} = 500 N/mm^2
Open stirrup diameter	ϕ_{os} = 4 mm
Stirrup height	s_x = 160 mm
Stirrup width	s_y = 80 mm
Partial factor of reinforcing steel	γ_s = 1.15

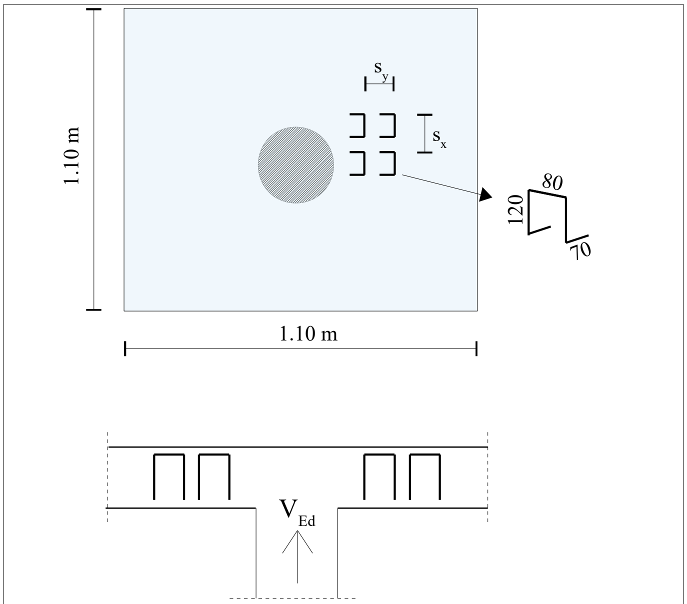


Figure 9.2.6.3.1 – The applied punching reinforcement

Concrete compression resistance

The calculation and the results are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Shear reinforcement resistance

The results of the shear resistance of the conrete without punching reinforcement are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Thus it means that we need punching reinforcement.

Effective stirrup area:

$$a_{sw} = \frac{2\frac{\phi_{os}^2 \pi}{4}}{s_x s_y} = \frac{2\frac{4^2 \pi}{4}}{160 \cdot 80} = 0.001963$$

Effective tensile strength of the open stirrups:

$$f_{y,sw,eff}=\min\begin{Bmatrix} f_{ywd} \\ 250+\frac{d}{4} \end{Bmatrix}=\min\begin{Bmatrix} \frac{f_{ywk}}{\gamma_s} \\ 250+\frac{d}{4} \end{Bmatrix}=\min\begin{Bmatrix} \frac{500}{1.15} \\ 250+\frac{160}{4} \end{Bmatrix}=\min\begin{Bmatrix} 434.78 \\ 290 \end{Bmatrix}=290\frac{\mathrm{N}}{\mathrm{mm}^2}$$

The punching reinforcement resistance:

$$v_{Rd,sw}=1.5\,d\,a_{sw}\,f_{y,sw,eff}\frac{1}{d}\sin\alpha=1.5\cdot160\cdot0.001963\cdot290\frac{1}{160}\sin 90°=0.8539\frac{\mathrm{N}}{\mathrm{mm}^2}$$

Punching shear resistance:

$$v_{Rd,cs}=\min\begin{bmatrix} 0.75\,v_{Rd,c}+v_{Rd,sw} \\ k_{max}\,v_{Rd,c} \end{bmatrix}=\min\begin{bmatrix} 0.75\cdot0.788+0.8539 \\ 1.5\cdot0.788 \end{bmatrix}=1.182\frac{\mathrm{N}}{\mathrm{mm}^2}$$

Open stirrups utilization:

$$\frac{v_{Ed}}{v_{Rd,cs}}=\frac{1.028}{1.182}=86.97$$

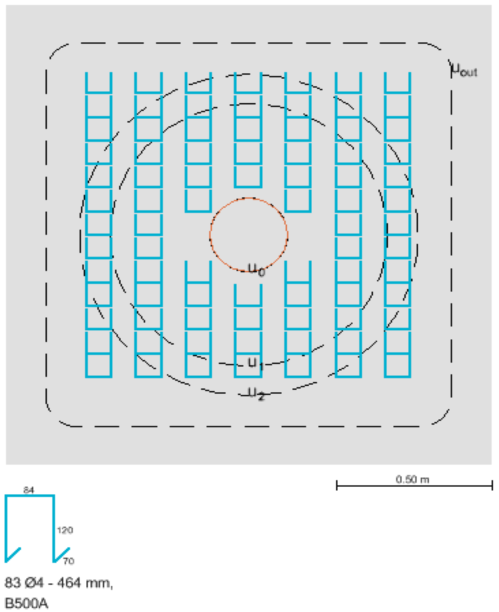
Fig. 9.2.6.3.2 shows the relevant FEM-Design results.

	Concrete compression	Shear reinforcement	Concrete shear
Hand calculation	4.5 N/mm^2	1.182 N/mm^2	0.788 N/mm^2
FEM-Design calculation	4.5 N/mm^2	1.18 N/mm^2	0.79 N/mm^2
Difference	0.00%	0.00%	0.20 %

The differences between the numerical and hand calculations are under 0.5%.

PU.(C.3).1
Maximum of load combinations

1:20



C25/30

t	=	200.00 mm
$f_{cd,ult.}$	=	16.67 N/mm^2
$f_{cd,acc./seis.}$	=	20.83 N/mm^2
f_{ck}	=	25.00 N/mm^2
k_{max}	=	1.50

Concrete compression resistance - Part 1.1: 6.4.3

LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed,0}}{u_0 \cdot d} = \frac{1.15 \cdot 399691.34}{784 \cdot 160} = 3.66\ \text{N/mm}^2 \qquad (6.38)$$

$$v_{Rd,max} = 0.5 \cdot v \cdot f_{cd} = 0.5 \cdot 0.54 \cdot 16.67 = 4.50\ \text{N/mm}^2$$

$$v_{Ed} = 3.66\ \text{N/mm}^2 \le v_{Rd,max} = 4.50\ \text{N/mm}^2 \qquad (6.53)\ \text{- OK}$$

Shear reinforcement resistance - Part 1.1: 6.4.3

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_1 \cdot d} \qquad (6.38)$$

$$v_{Rd,sw} = 1.5 \frac{d}{s_r} A_{sw} f_{yws,ef} \frac{1}{u_1\, d} \sin(\alpha)$$

$$v_{Rd,cs} = \min(0.75\, v_{Rd,c} + v_{Rd,sw},\ k_{max}\, v_{Rd,c})$$

Perimeter index	1	2
Dist [mm]	320	420
V_{Ed} [kN]	399.69	399.69
β [-]	1.15	1.15
d [mm]	160	160
u [m]	2.79	3.42
v_{Ed} [N/mm^2]	1.03	0.84
$v_{Rd,c}$ [N/mm^2]	0.79	0.79
$v_{Rd,sw}$ [N/mm^2]	0.85	0.85
$v_{Rd,cs}$ [N/mm^2]	1.18	1.18
Utilization [%]	87	71

Concrete shear resistance - Part 1.1: 6.4.3

LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_{Out} \cdot d} = \frac{1.15 \cdot 399691.34}{5027 \cdot 160} = 0.57\ \text{N/mm}^2 \qquad (6.38)$$

$$v_{Rd,c} = \max\left(C_{Rd,c} \cdot k\,(100 \cdot \rho_l \cdot f_{ck})^{1/3},\ v_{min}\right) + k_1 \cdot \sigma_{cp} =$$
$$= \max\left(0.12 \cdot 2.00\,(100 \cdot 0.0131 \cdot 25.00)^{1/3},\ 0.49\right) + 0.10 \cdot 0.20 = 0.79\ \text{N/mm}^2 \qquad (6.47)$$

$$v_{Ed} = 0.57\ \text{N/mm}^2 \le v_{Rd,c} = 0.79\ \text{N/mm}^2\ \text{- OK}$$

Figure 9.2.6.3.2 – The punching detailed results in FEM-Design

9.2.6.4 Stud rail general product

Inputs:

Reinforcement	
Reinforcing steel characteristic yield strength	f_{ywk} = 500 N/mm^2
Stud rail diameter	ϕ_s = 10 mm
Stud rail distances	s_r = 120 mm
Stud rail number on one circle	n = 16 pcs
Partial factor of reinforcing steel	γ_s = 1.15

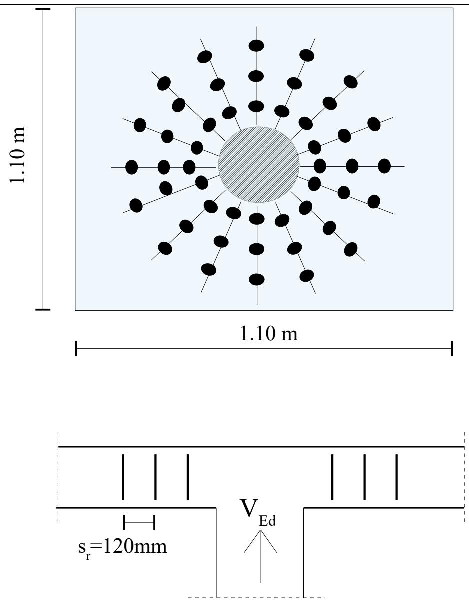


Figure 9.2.6.4.1 – The applied punching reinforcement, first stud from the column face: 60 mm

Concrete compression resistance

The calculation and the results are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Shear reinforcement resistance

The results of the shear resistance of the conrete without punching reinforcement are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Thus it means that we need punching reinforcement.

Punching perimeter at *2d* distance from the edge of the column:

$$u_1 = 2\left(\frac{d_{column}}{2} + 2d\right)\pi = 2\left(\frac{250}{2} + 2\cdot 160\right)\pi = 2796\ mm$$

The area of the 16 stirrups at the u_1 perimeter (see Fig. 9.2.6.4.1):

$$A_{sw}=16\frac{\phi_s^2\pi}{4}=16\frac{10^2\pi}{4}=1256\,\text{mm}^2$$

Effective tensile strength of the open stirrups:

$$f_{y,sw,eff}=\min\begin{Bmatrix} f_{ywd} \\ 250+\frac{d}{4}\end{Bmatrix}=\min\begin{Bmatrix} \frac{f_{ywk}}{\gamma_s} \\ 250+\frac{d}{4}\end{Bmatrix}=\min\begin{Bmatrix} \frac{500}{1.15} \\ 250+\frac{160}{4}\end{Bmatrix}=\min\begin{Bmatrix} 434.78 \\ 290\end{Bmatrix}=290\frac{\text{N}}{\text{mm}^2}$$

The punching reinforcement resistance:

$$v_{Rd,sw}=1.5\frac{d}{s_r}A_{sw}f_{y,sw,eff}\frac{1}{u_1 d}\sin\alpha=1.5\frac{160}{120}1256\cdot 290\frac{1}{2796\cdot 160}\sin 90°=1.628\frac{\text{N}}{\text{mm}^2}$$

Punching shear resistance:

$$v_{Rd,cs}=\min\begin{bmatrix} 0.75v_{Rd,c}+v_{Rd,sw} \\ k_{max}v_{Rd,c}\end{bmatrix}=\min\begin{bmatrix} 0.75\cdot 0.788+1.628 \\ 1.5\cdot 0.788\end{bmatrix}=1.182\frac{\text{N}}{\text{mm}^2}$$

Stud rail utilization:

$$\frac{v_{Ed}}{v_{Rd,cs}}=\frac{1.028}{1.182}=86.97$$

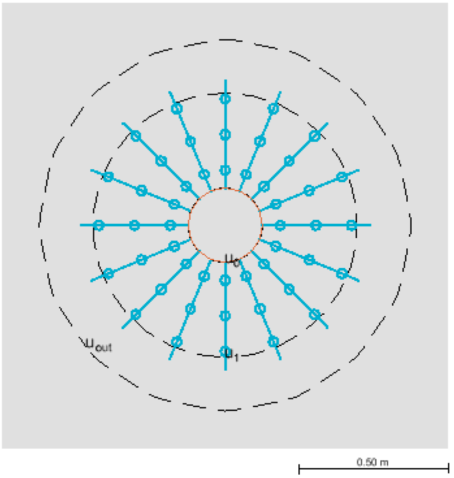
Fig. 9.2.6.4.2 shows the relevant FEM-Design results.

	Concrete compression	Shear reinforcement	Concrete shear
Hand calculation	4.5 N/mm^2	1.182 N/mm^2	0.788 N/mm^2
FEM-Design calculation	4.5 N/mm^2	1.18 N/mm^2	0.79 N/mm^2
Difference	0.00%	0.00%	0.20 %

The differences between the numerical and hand calculations are under 0.5%.

PU.(C.5).1
Maximum of load combinations

1:20

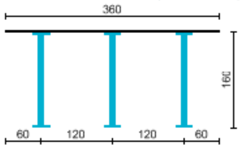


C25/30

t	=	200.00 mm
$f_{cd,ult.}$	=	16.67 N/mm^2
$f_{cd,acc./seis.}$	=	20.83 N/mm^2
f_{ck}	=	25.00 N/mm^2
k_{max}	=	1.50

1:10

16xB500B-10/160-3/360(60/2x120/60)



Perimeter index	1
Dist [mm]	320
V_{Ed} [kN]	399.91
β [-]	1.15
d [mm]	160
u [m]	2.79
v_{Ed} [N/mm^2]	1.03
$v_{Rd,c}$ [N/mm^2]	0.79
$v_{Rd,sw}$ [N/mm^2]	1.63
$v_{Rd,cs}$ [N/mm^2]	1.18
Utilization [%]	87

Concrete compression resistance - Part 1.1: 6.4.3
LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed,0}}{u_0 \cdot d} = \frac{1.15 \cdot 399907.42}{784 \cdot 160} = 3.67\ \mathrm{N/mm^2} \qquad (6.38)$$

$$v_{Rd,max} = 0.5 \cdot v \cdot f_{cd} = 0.5 \cdot 0.54 \cdot 16.67 = 4.50\ \mathrm{N/mm^2}$$

$v_{Ed} = 3.67\ \mathrm{N/mm^2} \le v_{Rd,max} = 4.50\ \mathrm{N/mm^2}$ (6.53) - OK

Shear reinforcement resistance - Part 1.1: 6.4.3

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_1 \cdot d} \qquad (6.38)$$

$$v_{Rd,sw} = 1.5 \frac{d}{s_r} A_{sw} f_{yws,ef} \frac{1}{u_1\, d} \sin(\alpha)$$

$$v_{Rd,cs} = \min(0.75\, v_{Rd,c} + v_{Rd,sw},\ k_{max}\, v_{Rd,c})$$

Concrete shear resistance - Part 1.1: 6.4.3
LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_{Out} \cdot d} = \frac{1.15 \cdot 399907.42}{3908 \cdot 160} = 0.74\ \mathrm{N/mm^2} \qquad (6.38)$$

$$v_{Rd,c} = \max\left(C_{Rd,c} \cdot k\,(100 \cdot \rho_l \cdot f_{ck})^{1/3},\ v_{min}\right) + k_1 \cdot \sigma_{cp} =$$
$$= \max\left(0.12 \cdot 2.00\,(100 \cdot 0.0131 \cdot 25.00)^{1/3},\ 0.49\right) + 0.10 \cdot 0.20 = 0.79\ \mathrm{N/mm^2} \qquad (6.47)$$

$v_{Ed} = 0.74\ \mathrm{N/mm^2} \le v_{Rd,c} = 0.79\ \mathrm{N/mm^2}$ - OK

Figure 9.2.6.4.2 – The punching detailed results in FEM-Design

9.2.6.5 Stud rail PSB product according to ETA-13/0151

Inputs:

Reinforcement	
Reinforcing steel characteristic yield strength	f_{yk} = 500 N/mm^2
Stud rail diameter	ϕ_s = 10 mm
Stud rail distances	s_r = 120 mm
Stud rail number on one circle	n = 8 pcs
Partial factor of reinforcing steel	γ_s = 1.15

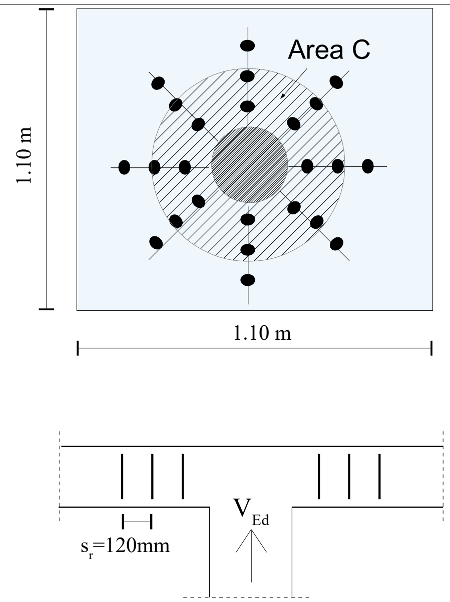


Figure 9.2.6.5.1 – The applied punching reinforcement, first stud from the column face: 60 mm

Concrete shear resistance

The perimeter of the column:

$$u_0 = d_{column}\pi = 250\cdot\pi = 785.40\,\text{mm}$$

Because this column is an inner column the β is:

$$\beta = 1.15$$

The u_1 perimeter:

$$u_1 = 2\left(\frac{d_{column}}{2} + 2d\right)\pi = 2\left(\frac{250}{2} + 2\cdot160\right)\pi = 2796\,mm$$

The effective longitudinal reinforcement ratios:

$$\rho_{l,x}=\frac{\frac{\phi_x^2\pi}{4}\frac{1000}{s_x}}{d_{l,x}\cdot 1000}=\frac{\frac{20^2\pi}{4}\frac{1000}{150}}{170\cdot 1000}=0.01232 \quad ; \quad \rho_{l,y}=\frac{\frac{\phi_y^2\pi}{4}\frac{1000}{s_y}}{d_{l,y}\cdot 1000}=\frac{\frac{20^2\pi}{4}\frac{1000}{150}}{150\cdot 1000}=0.01396$$

$$\rho_l=\min\begin{Bmatrix}\sqrt{\rho_{l,x}\cdot\rho_{l,y}}\\ 0.02\\ 0.5 f_{cd}/f_{yd}\end{Bmatrix}=\min\begin{Bmatrix}\sqrt{0.01232\cdot 0.01396}\\ 0.02\\ 0.5\cdot 16.67/435\end{Bmatrix}=\min\begin{Bmatrix}0.0131\\ 0.02\\ 0.019\end{Bmatrix}=0.0131$$

The average effective height of the longitudinal reinforcements:

$$d=\frac{d_{l,x}+d_{l,y}}{2}=\frac{170+150}{2}=160\,\mathrm{mm}$$

The specific shear force at u_1:

$$v_{Ed}=\beta\frac{V_{Ed}}{d\,u_1}=1.15\frac{400000}{160\cdot 2796}=1.028\frac{\mathrm{N}}{\mathrm{mm}^2}$$

Calculation of $C_{rd,c}$:

$$\frac{u_0}{d}=\frac{785.4}{160}=4.91>4.0 \quad \text{therefore:}$$

$$C_{Rd,c}=\frac{0.18}{\gamma_c}=0.12 \quad ; \quad k_1=0.1$$

Average normal stress in the concrete:

$$\sigma_{cp}=\min\begin{Bmatrix}\frac{\sigma_{cx}+\sigma_{cy}}{2}=\frac{\frac{n_{x,Ed}}{1000h}+\frac{n_{y,Ed}}{1000h}}{2}\\ 0.2 f_{cd}\end{Bmatrix}=\min\begin{Bmatrix}\frac{\frac{40000}{1000\cdot 200}+\frac{40000}{1000\cdot 200}}{2}\\ 0.2\frac{25}{1.5}\end{Bmatrix}=0.2\frac{\mathrm{N}}{\mathrm{mm}^2}$$

$$k=\min\begin{Bmatrix}1+\sqrt{\frac{200}{d}}\\ 2\end{Bmatrix}=\min\begin{Bmatrix}1+\sqrt{\frac{200}{160}}\\ 2\end{Bmatrix}=\min\begin{Bmatrix}2.12\\ 2\end{Bmatrix}=2$$

$$d\le 600\,mm \quad \text{therefore:}$$

$$v_{min}=0.035k^{1.5}f_{ck}^{0.5}=0.035\cdot 2^{1.5}\cdot 25^{0.5}=0.495\frac{\mathrm{N}}{\mathrm{mm}^2}$$

$$v_{Rd,c}=\max\begin{Bmatrix}C_{Rd,c}k(100\rho_l f_{ck})^{\frac{1}{3}}\\ v_{min}\end{Bmatrix}+k_1\sigma_{cp}=\max\begin{Bmatrix}0.12\cdot 2\cdot(100\cdot 0.01312\cdot 25)^{\frac{1}{3}}\\ 0.495\end{Bmatrix}+0.1\cdot 0.2=$$
$$=\max\begin{Bmatrix}0.768\\ 0.495\end{Bmatrix}+0.1\cdot 0.2=0.788\frac{\mathrm{N}}{\mathrm{mm}^2}$$

$v_{Ed} \geq v_{Rd,c}$, therefore punching reinforcement is needed.

The maximum of punching resistance with punching reinforcement (the example is a slab):

$$v_{Rd,max} = 1.96 \cdot v_{Rd,c} = 1.96 \cdot 0.788 = 1.544 \frac{\mathrm{N}}{\mathrm{mm}^2}$$

$v_{Rd,max} \geq v_{Ed} \geq v_{Rd,c}$, thus the PSB reinforcement is applicable.

Shear reinforcement resistance

The C area is the area closer than 1.125d from the column face (see Fig. 9.2.6.5.1).

$m_c = 8$ is the number of elements (rows) in the area C.

$n_c = 2$ is the studs of each element (row) in the area C.

$d_A = 10\,\mathrm{mm}$ is the shaft diameter of double headed stud.

$\eta = 1.0$ because d < 200 mm.

The total PSB resistance:

$$V_{Rd,sy} = m_c \cdot n_c \frac{d_A^2 \cdot \pi \cdot f_{yk}}{4 \cdot \gamma_s \cdot \eta} = 8 \cdot 2 \cdot \frac{10^2 \cdot \pi \cdot 500}{4 \cdot 1.15 \cdot 1.0} = 546.36\,\mathrm{kN}$$

$\frac{\beta V_{Ed}}{V_{Rd,y}} = \frac{1.15 \cdot 400}{546.36} = 84.2$ the PSB resistance is adequate and the detailings are correct.

Fig. 9.2.6.5.2 shows the relevant FEM-Design results.

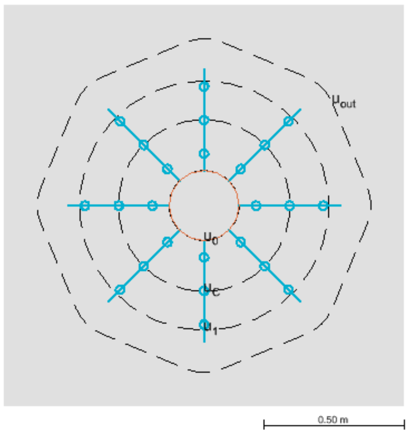
	Concrete shear resistance	Shear reinforcement
Hand calculation	0.788 N/mm^2	546.36 kN
FEM-Design calculation	0.79 N/mm^2	546.36 kN
Difference	0.20%	0.00%

The differences between the numerical and hand calculations are under 0.5%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.2.6 Punching calculation of a slab.str

PU.(C.4).1
Maximum of load combinations

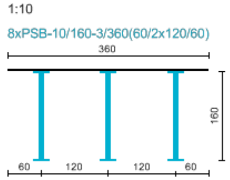
1:20



C25/30

t	=	200.00 mm
$f_{cd,ult.}$	=	16.67 N/mm^2
$f_{cd,acc./seis.}$	=	20.83 N/mm^2
f_{ck}	=	25.00 N/mm^2
k_{max}	=	1.50

1:10



Concrete compression resistance - Part 1.1: 6.4.3
Not relevant

Concrete shear resistance (u1) - ETA-13/0151
LC: 'Load'

$$v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_1 \cdot d} = \frac{1.15 \cdot 399907.42}{2791 \cdot 160} = 1.03\ N/mm^2 \quad (A1)$$

$$k = \min\left(1 + \left(\frac{200}{d}\right)^{0.5}, 2.0\right) = \min\left(1 + \left(\frac{200}{160}\right)^{0.5}, 2.0\right) = 2.00$$

$$\rho_l = \min\left((\rho_{l1} \cdot \rho_{l2})^{0.5}, 0.5 \cdot f_{ck} / f_{yd}, 0.02\right) =$$
$$= \min\left((0.0131 \cdot 0.0131)^{0.5}, 0.5 \cdot 16.67 / 434.78, 0.02\right) = 0.0131$$

$$v_{Rd,c} = \max\left(C_{Rd,c} \cdot k\,(100 \cdot \rho_l \cdot f_{ck})^{1/3}, v_{min}\right) + k_1 \cdot \sigma_{cp} =$$
$$= \max\left(0.12 \cdot 2.00\,(100 \cdot 0.0131 \cdot 25.00)^{1/3}, 0.49\right) + 0.10 \cdot 0.20 = 0.79\ N/mm^2 \quad (A3)$$

$$v_{Rd,max} = 1.96 \cdot v_{Rd,c} = 1.96 \cdot 0.79 = 1.54\ N/mm^2 \quad (A8)$$

$v_{Ed} = 1.03\ N/mm^2 > v_{Rd,c} = 0.79\ N/mm^2 \rightarrow$ Punching shear reinforcement is neccessary.

$v_{Ed} = 1.03\ N/mm^2 \leq v_{Rd,max} = 1.54\ N/mm^2$ - OK
Punching shear reinforcement is applicable.

Reinforcement shear resistance (Area C) - ETA-13/0151
LC: 'Load'

$$\eta = \max\left(1.0, \min\left(1.6, 1.0 + (d - 200) \cdot \frac{1.6 - 1.0}{800 - 200}\right)\right) =$$
$$= \max\left(1.0, \min\left(1.6, 1.0 + (160 - 200) \cdot \frac{1.6 - 1.0}{800 - 200}\right)\right) = 1.00$$

$$V_{Rd,sy} = n_{stud,C} \cdot A_{si} \cdot f_{yd} / \eta = 16 \cdot 78.54 \cdot 434.78 / 1.00 = 546.36\ kN \quad (A7)$$

$\beta \cdot V_{Ed} = 1.15 \cdot 399.91 = 459.89\ kN \leq V_{Rd,sy} = 546.36\ kN$ - OK

Concrete shear resistance (uOut) - ETA-13/0151
LC: 'Load'

$$v_{Ed} = \frac{\beta_{red} \cdot V_{Ed}}{u_{Out} \cdot d} = \frac{1.15 \cdot 399907.42}{3669 \cdot 160} = 0.78\ N/mm^2 \quad (A4)$$

$$k = \min\left(1 + \left(\frac{200}{d}\right)^{0.5}, 2.0\right) = \min\left(1 + \left(\frac{200}{160}\right)^{0.5}, 2.0\right) = 2.00$$

$$\rho_l = \min\left((\rho_{l1} \cdot \rho_{l2})^{0.5}, 0.5 \cdot f_{ck} / f_{yd}, 0.02\right) =$$
$$= \min\left((0.0131 \cdot 0.0131)^{0.5}, 0.5 \cdot 16.67 / 434.78, 0.02\right) = 0.0131$$

$$v_{Rd,c} = \max\left(C_{Rd,c} \cdot k\,(100 \cdot \rho_l \cdot f_{ck})^{1/3}, v_{min}\right) + k_1 \cdot \sigma_{cp} =$$
$$= \max\left(0.12 \cdot 2.00\,(100 \cdot 0.0131 \cdot 25.00)^{1/3}, 0.49\right) + 0.10 \cdot 0.20 = 0.79\ N/mm^2 \quad (A3)$$

$v_{Ed} = 0.78\ N/mm^2 \leq v_{Rd,c} = 0.79\ N/mm^2$ - OK

Figure 9.2.6.5.2 – The punching detailed results in FEM-Design

9.2.7 Interaction of normal force and biaxial bending in a column

In this section we will calculate the utilization and the load-bearing capacity of a cantilever column under biaxial bending and normal force (see Fig. 9.2.7.1) according to non-linear concrete and reinforcing steel material model.

Inputs:

Concrete characteristic compressive strength	$f_{ck} = 20$ N/mm^2
Partial factor of concrete	$\gamma_c = 1.50$
Effective creep factor	$\varphi_{ef} = 2$
Partial factor of concrete Young's modulus	$\gamma_{cE} = 1.2$
Reinforcing steel characteristic yield strength	$f_{yk} = 500$ N/mm^2
Partial factor of reinforcing steel	$\gamma_s = 1.15$
Ultimate limit strain of reinforcing steel	$\varepsilon_{uk} = 2.5$ %, $\varepsilon_{ud} = 2.25$ %
Slope of plastic part (material model)	k = 1.05
Longitudinal bar diameter	$\phi = 20$ mm
Stirrup diameter	$\phi_s = 8$ mm
Concrete cover (on stirrups)	c = 20 mm
Column height	L = 3.0 m
Column width (square section)	b = 300 mm; h = 300 mm
The effective depth	d = 300 – 20 – 8 – 20/2 = 262 mm
Normal force	$N_{Ed} = 500$ kN
Bending moment in y' direction	$M_{Ed,y}{}^{I} = 10$ kNm
Bending moment in z' direction	$M_{Ed,z}{}^{I} = 10$ kNm

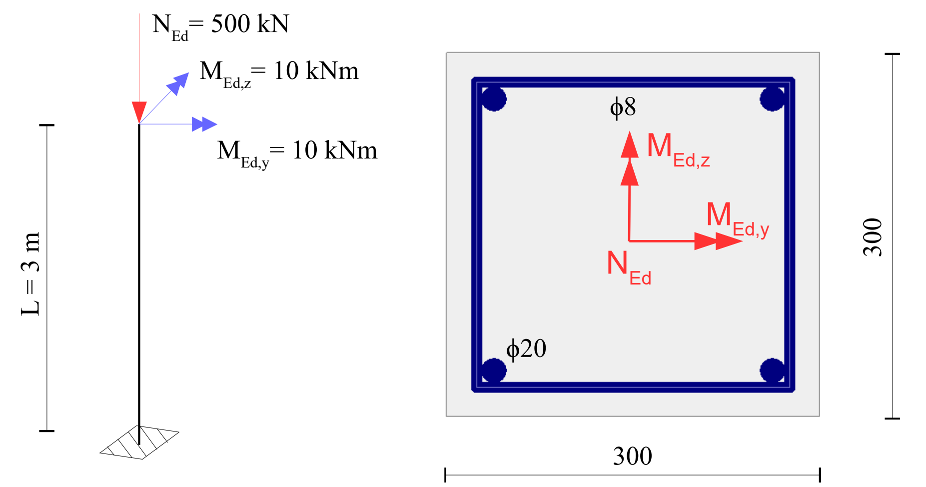


Figure 9.2.7.1 – Cantilever column with the external forces and moments and the RC section - the x direction is the axis of the column, y and z are respect to the moment directions

In Eurocode 2 there are different methods to calculate the load-bearing capacity of the column. One of them is the *Nominal stiffness method* and an other one is the *Nominal curvature method.* We will calculate the utilizations and the load-bearing capacities with independent "hand" calculations respect to both mentioned methods then we will compare the results with FEM-Design calculations.

9.2.7.1 Nominal stiffness method

Here we will calculate the nominal bending stiffness of the column and then increase the acting moments to consider second order effects. The Eurocode 2 suggests to increase the moment only in the unfavorable principal direction. In FEM-Design to ensure the most unfavourable result the first order moments will be increased with the effect of the imperfections and second order effects in both principal directions to be on the safe side and get the most unfavourable condition.

The nominal bending stiffness of the column:

$$EI = K_c E_{cd} I_c + K_s E_s I_s = 0.0566 \cdot 25000 \cdot 6.75 \cdot 10^8 + 1 \cdot 200000 \cdot 1.576 \cdot 10^7 = 4.107 \cdot 10^{12}\ \mathrm{Nmm}^2 \quad ,$$

where:

$K_c = \dfrac{k_1 k_2}{1+\varphi_{ef}} = \dfrac{1 \cdot 0.1698}{1+2} = 0.0566$ is a factor for effects of cracking, creep etc.

$k_1 = \sqrt{\dfrac{f_{ck}}{20}} = \sqrt{\dfrac{20}{20}} = 1$ is a factor which depends on concrete strength.

$k_2 = \min\begin{Bmatrix} n\dfrac{\lambda}{170} \\ 0.2 \end{Bmatrix} = \min\begin{Bmatrix} 0.4167\dfrac{69.282}{170} \\ 0.2 \end{Bmatrix} = 0.1698$ is a factor which depends on axial force and slenderness.

$n = \dfrac{N_{Ed}}{A_c f_{cd}} = \dfrac{N_{Ed}}{b^2 \dfrac{f_{ck}}{\gamma_c}} = \dfrac{500000}{300^2 \dfrac{20}{1.5}} = 0.4167$ is the relative axial force.

$\lambda = \dfrac{l_0}{i} = \dfrac{2L}{i} = \dfrac{2 \cdot 3000}{86.603} = 69.282$ is the slenderness ratio.

$i = \sqrt{\dfrac{I_c}{A_c}} = \sqrt{\dfrac{\dfrac{b^4}{12}}{b^2}} = \sqrt{\dfrac{\dfrac{300^4}{12}}{300^2}} = 86.603\ \mathrm{mm}$ is the radius of gyration of the uncracked concrete section.

$I_c = \frac{b^4}{12} = \frac{300^4}{12} = 6.75 \cdot 10^8 \, \mathrm{mm}^4$ is the inertia of the uncraked concrete section.

$E_{cd} = E_{cm} / \gamma_{cE} = 30000/1.2 = 25000 \, \mathrm{MPa}$ is the design value of the modulus of elasticity of concrete.

$K_s = 1$ is a factor for contribution of reinforcement.

$$I_s = A_s \left(d - \frac{b}{2} \right)^2 = 4 \frac{\phi^2 \pi}{4} \left[d - \frac{b}{2} \right]^2 = 4 \frac{20^2 \pi}{4} \left[262 - \frac{300}{2} \right]^2 = 1.576 \cdot 10^7 \, \mathrm{mm}^4$$

is the second moment of area of reinforcement, about the centre of area of the concrete.

After this we need to increase the first order moments in both direction due to imperfection and second order effects.

Considering the effect of the imperfection (see the former underlined comment also):

$$M_{0,Ed,y} = M^I_{Ed,y} + e_i N_{Ed} = 10000000 + 15 \cdot 500000 = 17.5 \, \mathrm{kNm}$$

$$M_{0,Ed,z} = M^I_{Ed,z} + e_i N_{Ed} = 10000000 + 15 \cdot 500000 = 17.5 \, \mathrm{kNm}$$

The eccentricity according to the imperfection in both direction:

$$e_i = \frac{l_0}{400} = \frac{2L}{400} = \frac{2 \cdot 3000}{400} = 15 \, \mathrm{mm}$$

NOTE: In Eurocode 2 the $\frac{l_0}{400}$ value may be reduced in the function of the column height, but FEM-Design do not consider this effect thus we are on the safe side.

The increased design moment values according to the second order effects (see the former underlined comment also):

$$M_{Ed,y} = \frac{M_{0,Ed,y}}{1 - \frac{N_{Ed}}{N_B}} = \frac{17.5}{1 - \frac{500}{1126.2}} = 31.47 \, \mathrm{kNm} \quad \text{and}$$

$$M_{Ed,z} = \frac{M_{0,Ed,z}}{1 - \frac{N_{Ed}}{N_B}} = \frac{17.5}{1 - \frac{500}{1126.2}} = 31.47 \, \mathrm{kNm} \quad ,$$

where N_B is the buckling load based on nominal stiffness of the column:

$$N_B = \frac{\pi^2 EI}{l_0^2} = \frac{\pi^2 4.107 \cdot 10^{12}}{(2 \cdot 3000)^2} = 1126000\,\text{N} = 1126\,\text{kN}$$

Now we need to calculate the resistance (load-bearing capacity). For this the M-N interaction curve (bending moment – normal force interaction curve)in the principal directions or the M-N interaction surface is necessary. In Eurocode 2 there is a simplified method to check the utilization with the aid of the M-N interaction curves in the principal directions but there is a more accurate solution for the problem with the help of the M-N interaction surface. Let's see what is the different between these calculation methods and then we show you what is the FEM-Design solution for this problem.

First of all with independent numerical calculation we provided the M-N interaction curve in the principal directions. According to the square and double-symmetric cross section these two curves will be identical each other (see Fig. 9.2.7.2).

At the given design normal force value (N_{Ed} = *500 kN*) the moment resistance with numerical calculation in both *y* and *z* direction is: $M_{Rd,y} = M_{Rd,z} = 104.1\,\text{kNm}$ (see Fig. 9.2.7.2).

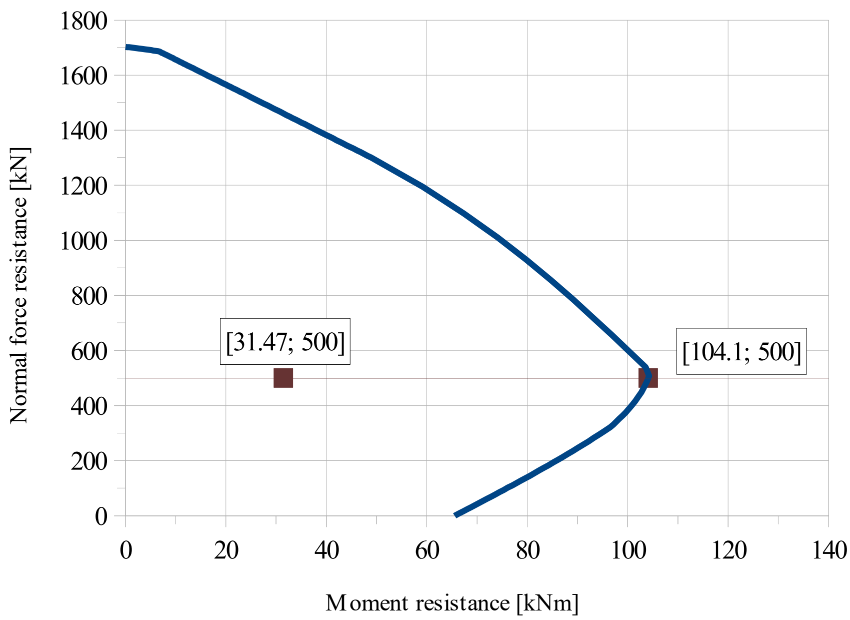


Figure 9.2.7.2 – The M-N interaction curve in the x-y or in the x-z plain

The utilization based the approximation formula of Eurocode 2 is the following:

$$\left(\frac{M_{Ed,y}}{M_{Rd,y}}\right)^a + \left(\frac{M_{Ed,z}}{M_{Rd,z}}\right)^a = \left(\frac{31.47}{104.1}\right)^{1.155} + \left(\frac{31.47}{104.1}\right)^{1.155} = 50.23\,\%$$

The normal force utilization is:

$$\frac{N_{Ed}}{N_{Rd}}=\frac{N_{Ed}}{A_c f_{cd}+A_s f_{yd}}\approx\frac{N_{Ed}}{b^2 f_{cd}+4\frac{\phi^2\pi}{4}f_{yd}}=\frac{500000}{300^2 13.33+4\frac{20^2\pi}{4}435}=0.2863$$

The "*a*" value can be linearly interpolated based on the following table:

$$a=1.155$$

$\frac{N_{Ed}}{N_{Rd}}$	0.1	0.7	1.0
a	1.0	1.5	2.0

The M-N curves were calculated based on the material models what you can see in Fig. 9.2.7.3. The strain values for concrete coincide with the values what were mentioned in Chapter 9.2.4.

NOTE: Due to numerical stability FEM-Design uses a bit modified concrete material model according to Fig. 9.2.4.3 but by ultimate limit state with the design stress values.

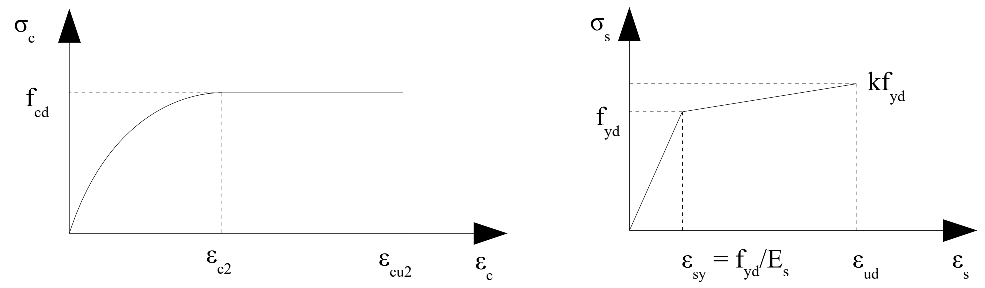


Figure 9.2.6.3 – The material models for the M-N curve calculations
left: concrete (only compression), right: rebars (both tension and compression)

Concrete:

$$\sigma_c(\varepsilon_c)=f_{cd}\left[1-\left(1-\frac{\varepsilon_c}{\varepsilon_{c2}}\right)^2\right] \quad if\ 0\leq\varepsilon_c\leq\varepsilon_{c2}$$

$$\sigma_c(\varepsilon_c)=f_{cd} \quad if\ \varepsilon_{c2}<\varepsilon_c\leq\varepsilon_{cu2}$$

Rebars:

$$\sigma_s(\varepsilon_s)=\varepsilon_s E_s \quad if\ \ \varepsilon_s\leq\frac{f_{yd}}{E_s}$$

$$\sigma_s(\varepsilon_s)=f_{yd}+(k-1)f_{yd}\frac{\varepsilon_s-\frac{f_{yd}}{E_s}}{\varepsilon_{ud}-\frac{f_{yd}}{E_s}} \quad if\ \ \frac{f_{yd}}{E_s}<\varepsilon_s\leq\varepsilon_{ud}$$

Fig. 9.2.7.4 shows a slice of the M-N surface in the direction of 45° of angle between y and z axes. Based on this figure we can calculate the utilization by hand in a more accurate way. At the N_{Ed} = *500 kN* level the maximum moment capacity is *81.96 kNm* for the 45° direction (see Fig. 9.2.7.4). Based on this moment capacity we can calculate the maximum moment capacity in the principal directions according to the equal design moments in y and z directions:

$$M_{Rd,y}^{precise}(N_{Ed})=M_{Rd,z}^{precise}(N_{Ed})=\frac{81.96}{\sqrt{2}}=57.95\,\text{kNm}$$

Due to this value we can calculate the “a” value more precisely based on tha fact that in this example the increased moment values are the same in the two directions. The following expression must be true:

$2\left(\frac{57.95}{104.1}\right)^{a^{precise}}=1.00$ and based on this: $a^{precise}=1.183$. Thus the more precise utilization is:

$$\left(\frac{M_{Ed,y}}{M_{Rd,y}}\right)^{a^{precise}}+\left(\frac{M_{Ed,z}}{M_{Rd,z}}\right)^{a^{precise}}=\left(\frac{31.47}{104.1}\right)^{1.183}+\left(\frac{31.47}{104.1}\right)^{1.183}=48.57\%$$

This means that the Eurocode formula is on the safe side based on this utilization type.

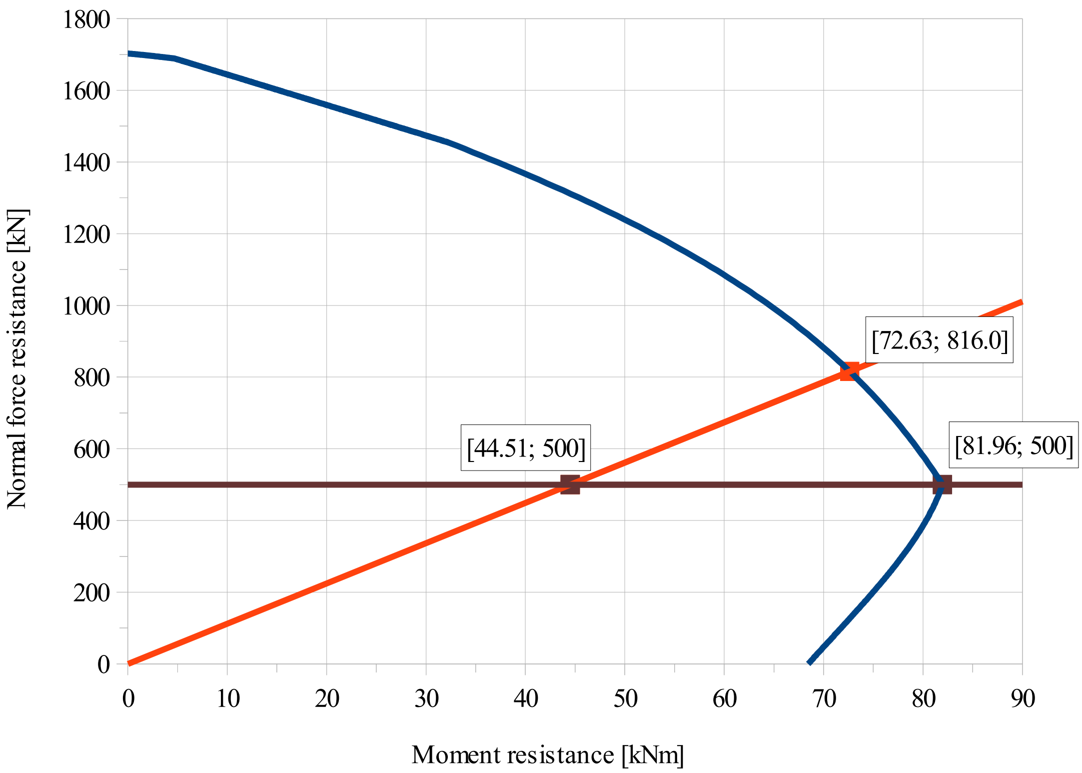


Figure 9.2.7.4 – A slice of the M-N surface between y and z axes with 45° of angle

Fig. 9.2.7.4 shows a slice of the M-N surface in the 45° of angle direction between y and z axes (which is the relevant direction in this example, see the increased design moment values in the two directions). In FEM-Design the utilization is the following:

$$N_{Ed} = \eta\, N_{Rd} \quad ; \quad M_{Ed,y} = \eta\, M_{Rd,y} \quad ; \quad M_{Ed,z} = \eta\, M_{Rd,z}$$

where η is the utilization, it means that if we increase the normal force and the two bending moments linearly at the same time we will reach the failure surface of the cross-section (see the red line in Fig. 9.2.7.4). With other words the eccentricity is constant.

According to this scenario the utilization with the independent "hand" calculation is the following:

The red line breaks through the M-N curve at (see Fig. 9.2.7.4):

$N_{Rd} = 816.0\,\text{kN}$, $M_{Rd} = 72.63\,\text{kNm}$

The utilization of the column based on the red line geometrical point of view according to Fig. 9.2.7.4:

$$\eta = \frac{\sqrt{M_{Ed,y}^2 + M_{Ed,z}^2 + N_{Ed}^2}}{\sqrt{M_{Rd}^2 + N_{Rd}^2}} = \frac{\sqrt{31.47^2 + 31.47^2 + 500^2}}{\sqrt{72.63^2 + 816^2}} = 61.27\%$$

Utilization based on the FEM-Design detailed results is (see Fig. 9.2.7.5): $\eta = 64\,\%$

The difference between the "hand" and numerical calculations is less than 5%. This difference comes from the fact that FEM-Design uses a bit modified concrete material model (see. Fig. 9.2.4.3, but here the stress values were replaced by the design values) to ensure the numerical stability for every case.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.7 Interaction of normal force and biaxial bending in a column.str

Section utilization for axial effects (Part 1.1: 5.8, 6.1)
Consideration of second order effects

$\lambda = \frac{l_0}{i}$ (5.14)

2nd order effect is considered according to nominal stiffness method. (Part 1.1: 5.8.7)

Sections	7
λ_{lim} [-]	21.41
$l_{0,1}$ [mm]	6000
i_1 [mm]	86.6
λ_1 [-]	69.28
2nd order effect in direction 1	considered
$l_{0,2}$ [mm]	6000
i_2 [mm]	86.6
λ_2 [-]	69.28
2nd order effect in direction 2	considered

$n = N_{Ed} / (A_c f_{cd})$

$k_1 = \sqrt{f_{ck} / 20}$ (5.23)

$k_2 = \min\left\{ n \frac{\lambda}{170}, 0.20 \right\}$ (5.24)

$K_c = k_1 \cdot k_2 / (1 + \varphi_{ef})$ (5.22)

$K_s = 1.00$ (5.22)

$EI = K_c E_{cd} I_c + K_s E_s I_s$ (5.21)

$N_B = \frac{\pi^2 EI}{l_0^2}$ (5.17)

$M_{0Ed} = M^I_{Ed} + s N_{Ed} \frac{l_0}{400}$ (5.17)

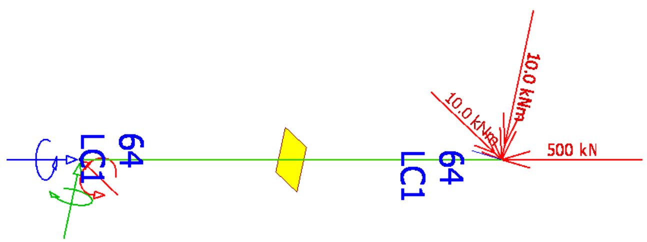
$M^{II}_{Ed} = \frac{M_{0Ed}}{1 - | N_{Ed}| / N_B}$ (5.30)

$e_{min,1}$ = max(20 mm, h_1 / 30) = max(20 mm, h_1 / 30) = 20 mm (6.1(4))

$e_{min,2}$ = max(20 mm, h_2 / 30) = max(20 mm, h_2 / 30) = 20 mm (6.1(4))

$|M^{II}_{Ed}| \le | N_{Ed}\ e_{min}|$ --> $M_{Ed} = s\ N_{Ed}\ e_{min}$

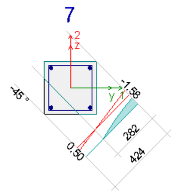
$|M^{II}_{Ed}| > | N_{Ed}\ e_{min}|$ --> $M_{Ed} = M^{II}_{Ed}$



Sections	7
LC	1
N_{Ed} [kN]	-500.00
n [-]	0.42
k_1 [-]	1.00
$k_{2,1}$ [-]	0.17
$K_{c,1}$ [-]	0.06
$(E_s I_s)_1$ [N/mm^2]	3.1527e+12
$(EI)_1$ [N mm^2]	4.1078e+12
$N_{B,1}$ [kN]	1126.18
$M^I_{Ed,1}$ [kNm]	10.00
s_1 [-]	-1.00
$M_{0Ed,1}$ [kNm]	17.50
$M^{II}_{Ed,1}$ [kNm]	31.47
$M_{Ed,1}$ [kNm]	31.47
$k_{2,2}$ [-]	0.17
$K_{c,2}$ [-]	0.06
$(E_s I_s)_2$ [N/mm^2]	3.1527e+12
$(EI)_2$ [N mm^2]	4.1078e+12
$N_{B,2}$ [kNm]	1126.18
$M^I_{Ed,2}$ [kNm]	-10.00
s_2 [-]	1.00
$M_{0Ed,2}$ [kNm]	-17.50
$M^{II}_{Ed,2}$ [kNm]	-31.47
$M_{Ed,2}$ [kNm]	-31.47

Stresses and strains (Part 1.1: 6.1(2), 6.1(8), 3.1.7)
$\varepsilon_{steel,lim} = \varepsilon_{ud}$

Ultimate limit state



Sections	7
LC	1
ε_{steel} [-]	0.00128
$\varepsilon_{steel,lim}$ [-]	0.02250
$(\varepsilon_{steel}/\varepsilon_{steel,lim})_{max}$ [-]	0.06
$\varepsilon_{c,max}$ [-]	-0.00158
$\varepsilon_{c,comp}$ [-]	-0.00056
max($\varepsilon_{c,max}/\varepsilon_{cu2}$, $\varepsilon_{c,comp}/\varepsilon_{cu2}$) [-]	0.45

Section utilization

Ultimate internal forces: $N_{ult} = v\ N_{Ed}$; $M_{ult,1} = v\ M_{Ed,1}$; $M_{ult,2} = v\ M_{Ed,2}$
Utilization: 1 / v

Sections	7
LC	1
N_{Ed} [kN]	-500.00
$M_{Ed,1}$ [kNm]	31.47
$M_{Ed,2}$ [kNm]	-31.47
Utilization [%]	64

Figure 9.2.7.5 – The detailed results of the RC column in FEM-Design with nominal stiffness method

9.2.7.2 Nominal curvature method

Here we will calculate the nominal curvature of the column and then increase the acting moments based on this curvature to consider second order effects. The Eurocode 2 suggests to increase the moment only in the unfavorable principal direction. In FEM-Design to ensure the most unfavourable result the first order moments will be increased with the effect of the imperfections and second order effects in both principal directions to be on the safe side and get the most unfavourable condition.

Considering the effect of the imperfection:

$$M_{0,Ed,y}=M^{I}_{Ed,y}+e_i N_{Ed}=10000000+15\cdot 500000=17.5\,\text{kNm}$$

$$M_{0,Ed,z}=M^{I}_{Ed,z}+e_i N_{Ed}=10000000+15\cdot 500000=17.5\,\text{kNm}$$

The eccentricity according to the imperfection in both direction:

$$e_i=\frac{l_0}{400}=\frac{2L}{400}=\frac{2\cdot 3000}{400}=15\,\text{mm}$$ (see also the relevant NOTE in Chapter 9.2.6.1).

The second order eccentricity:

$$e_2=\frac{\frac{1}{r}l_0^2}{c}=\frac{1.816\cdot 10^{-5}\cdot 6000^2}{10}=65.38\,\text{mm}\quad ,$$

where the curvature:

$$\frac{1}{r}=K_r K_\varphi \frac{1}{r_0}=0.9842\cdot 1\cdot 1.845\cdot 10^{-5}=1.816\cdot 10^{-5}\frac{1}{\text{mm}}\quad ,$$

where:

$$\frac{1}{r_0}=\frac{\varepsilon_{yd}}{0.45d}=\frac{\frac{f_{yd}}{E_s}}{0.45d}=\frac{\frac{435}{200000}}{0.45\cdot 262}=1.845\cdot 10^{-5}\frac{1}{\text{mm}}$$

$$K_r=\min\left\{\begin{matrix}\frac{n_u-n}{n_u-n_{bal}}\\ 1\end{matrix}\right\}=\frac{1.4554-0.4167}{1.4554-0.4}=0.9842$$ is a correction factor depending on axial load.

$$n_u=1+\omega=1+0.4554=1.4554\quad ;\quad \omega=\frac{A_s f_{yd}}{A_c f_{cd}}=\frac{4\frac{\phi^2\pi}{4}\frac{f_{yk}}{\gamma_s}}{b^2\frac{f_{ck}}{\gamma_c}}=\frac{4\frac{20^2\pi}{4}\frac{500}{1.15}}{300^2\frac{20}{1.5}}=0.4554$$

The relative axial normal force:

$$n=\frac{N_{Ed}}{A_c f_{cd}}=\frac{N_{Ed}}{b^2\frac{f_{ck}}{\gamma_c}}=\frac{500000}{300^2\frac{20}{1.5}}=0.4167 \quad , \quad n_{bal}=0.4$$

$$K_{\varphi}=\max\begin{Bmatrix}1+\beta\varphi_{ef}\\1\end{Bmatrix}=\max\begin{Bmatrix}0.9762\\1\end{Bmatrix}=1$$ is a factor for taking account of creep.

Where:

$$\beta=0.35+\frac{f_{ck}}{200}-\frac{\lambda}{150}=0.35+\frac{20}{200}-\frac{69.282}{150}=-0.01188$$

$$\lambda=\frac{l_0}{i}=\frac{2L}{i}=\frac{2\cdot 3000}{86.603}=69.282$$ the slenderness ration (see Chapter 9.2.7.1 also).

The increased design moment values according to the second order effects:

$$M_{Ed,y}=M_{0,Ed,y}+N_{Ed}e_2=17.5+500\cdot 0.06538=50.19\,\text{kNm}$$ and

$$M_{Ed,z}=M_{0,Ed,z}+N_{Ed}e_2=17.5+500\cdot 0.06538=50.19\,\text{kNm}$$.

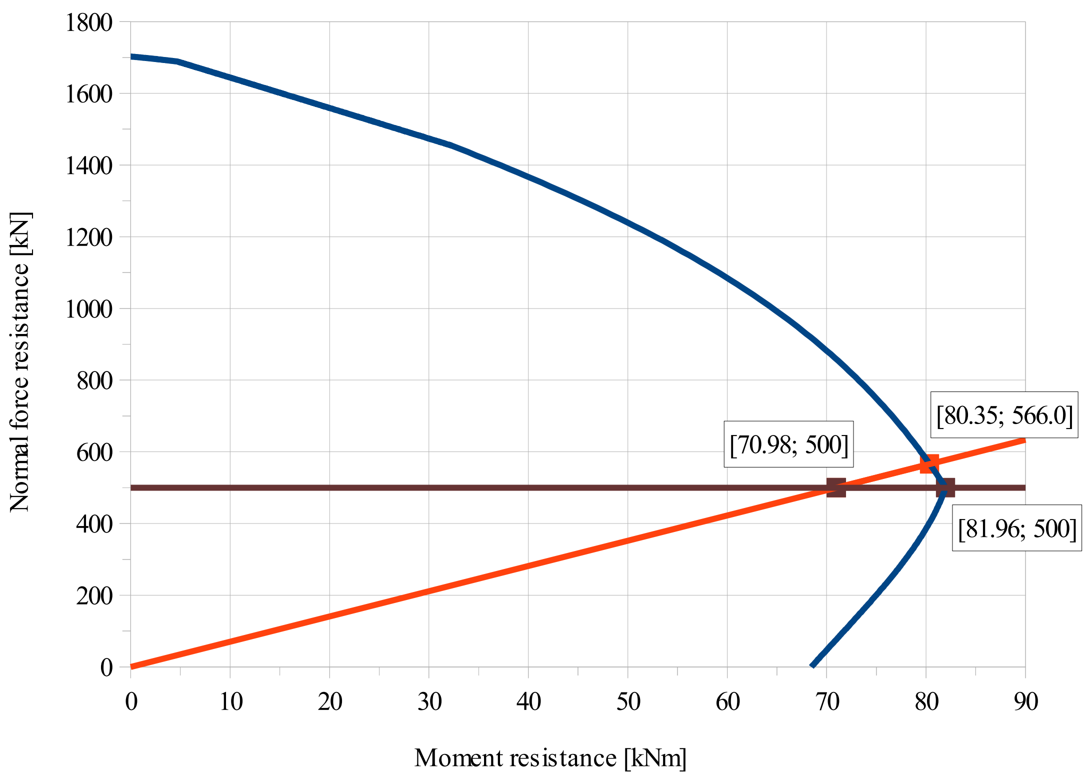


Figure 9.2.7.6 – A slice of the M-N surface between y and z axes with 45° of angle

Fig. 9.2.7.6 shows a slice of the M-N surface in the 45° of angle direction between *y* and *z* axes (which is the relevant direction in this example, see the increased design moment values in the two directions). In FEM-Design the utilization is the following:

$$N_{Ed}=\eta N_{Rd} \quad ; \quad M_{Ed,y}=\eta M_{Rd,y} \quad ; \quad M_{Ed,z}=\eta M_{Rd,z}$$

where η is the utilization, it means that if we increase the normal force and the two bending moments linearly at the same time we will reach the failure surface of the cross-section (see the red line in Fig. 9.2.7.6). With other words the eccentricity is constant.

According to this scenario the utilization with the independent "hand" calculation is the following:

The red line breaks through the M-N curve at (see Fig. 9.2.7.6):

$$N_{Rd}=566.0\,\mathrm{kN}\,,\,M_{Rd}=80.35\,\mathrm{kNm}$$

The utilization of the column based on the red line geometrical point of view according to Fig. 9.2.7.6:

$$\eta=\frac{\sqrt{M_{Ed,y}^2+M_{Ed,z}^2+N_{Ed}^2}}{\sqrt{M_{Rd}^2+N_{Rd}^2}}=\frac{\sqrt{50.19^2+50.19^2+500^2}}{\sqrt{80.35^2+566.0^2}}=88.33\,\%$$

Utilization based on the FEM-Design detailed results (see Fig. 9.2.7.7): $\eta=92.7\,\%$

The difference between the "hand" and numerical calculations is around 5%. This difference comes from the fact that FEM-Design uses a bit modified concrete material model (see. Fig. 9.2.4.3, but here the stress values were replaced by the design values) to ensure the numerical stability for every case.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.7 Interaction of normal force and biaxial bending in a column.str

Section utilization for axial effects (Part 1.1: 5.8, 6.1)
Consideration of second order effects

$\lambda = \frac{l_0}{i}$ (5.14)

2nd order effect is considered according to nominal curvature method. (Part 1.1: 5.8.8)

Sections	7
λ_{lim} [-]	21.41
$l_{0,1}$ [mm]	6000
i_1 [mm]	86.6
λ_1 [-]	69.28
2nd order effect in direction 1	considered
$l_{0,2}$ [mm]	6000
i_2 [mm]	86.6
λ_2 [-]	69.28
2nd order effect in direction 2	considered

$\omega = \frac{A_s f_{yd}}{A_c f_{cd}}$

$v = 1 + \omega$

$n = N_{Ed} / (A_c + f_{cd})$

$n_{bal} = 0.4$

$K_r = \min\left\{\frac{v - n}{v - n_{bal}}, 1.0\right\}$ (5.36)

$\beta = 0.35 + \frac{f_{ck}}{200} - \frac{\lambda}{150}$

$K_\varphi = \max(1 + \beta\,\varphi_{ef}, 1.0)$ (5.37)

$d = \frac{h}{2} + i_s$ (5.35)

rt $\frac{1}{r_0} = \frac{\varepsilon_{yd}}{0.45\,d}$

$\frac{1}{r} = K_r\,K_\varphi\,\frac{1}{r_0}$

$c = 10$

$e_2 = \frac{l_0}{400} + \frac{\frac{1}{r} l_0^2}{c}$ (5.2(7))

$M_2 = s\,N_{Ed}\,e_2$

$e_{min,1} = \max(20\text{ mm}, h_1 / 30) = \max(20\text{ mm}, h_1 / 30) = 20\text{ mm}$ (6.1(4))

$e_{min,1} = \max(20\text{ mm}, h_2 / 30) = \max(20\text{ mm}, h_2 / 30) = 20\text{ mm}$ (6.1(4))

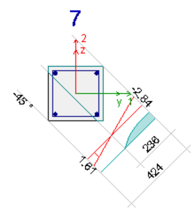
$|N_{Ed}\,e_{min}| > |M_{0Ed} + M_2|$ --> $M_{Ed} = s\,N_{Ed}\,e_{min}$

$|N_{Ed}\,e_{min}| \le |M_{0Ed} + M_2|$ --> $M_{Ed} = M_{0Ed} + M_2$

Sections	7
LC	1
A_s [mm^2]	1256.64
ω [-]	0.46
n_u [-]	1.46
N_{Ed} [kN]	-500.00
n [-]	0.42
K_r [-]	0.98
β_1 [-]	-0.01
$K_{\varphi,1}$ [-]	1.00
$i_{s,1}$ [mm]	112.00
d_1 [mm]	262.00
$1/r_{0,1}$ [1/mm]	1.8439e-05
$1/r_1$ [1/mm]	1.8147e-05
$e_{2,1}$ [mm]	80.33
s_1 [-]	-1.00
$M_{0Ed,1}$ [kNm]	10.00
$M_{Ed,1}$ [kNm]	50.17
β_2 [-]	-0.01
$K_{\varphi,2}$ [-]	1.00
$i_{s,2}$ [mm]	112.00
d_2 [mm]	262.00
$1/r_{0,2}$ [1/mm]	1.8439e-05
$1/r_2$ [1/mm]	1.8147e-05
$e_{2,2}$ [mm]	80.33
s_2 [-]	1.00
$M_{0Ed,2}$ [kNm]	-10.00
$M_{Ed,2}$ [kNm]	-50.17

Stresses and strains (Part 1.1: 6.1(2), 6.1(8), 3.1.7)
$\varepsilon_{steel,lim} = \varepsilon_{ud}$

Ultimate limit state



Sections	7
LC	1
ε_{steel} [-]	0.00219
$\varepsilon_{steel,lim}$ [-]	0.02250
$(\varepsilon_{steel}/\varepsilon_{steel,lim})_{max}$ [-]	0.10
$\varepsilon_{c,max}$ [-]	-0.00284
$\varepsilon_{c,comp}$ [-]	-0.00066
$\max(\varepsilon_{c,max}/\varepsilon_{cu2}, \varepsilon_{c,comp}/\varepsilon_{cu2})$ [-]	0.81

Section utilization

Ultimate internal forces: $N_{ult} = v\,N_{Ed}$; $M_{ult,1} = v\,M_{Ed,1}$; $M_{ult,2} = v\,M_{Ed,2}$
Utilization: 1 / v

Sections	7
LC	1
N_{Ed} [kN]	-500.00
$M_{Ed,1}$ [kNm]	50.17
$M_{Ed,2}$ [kNm]	-50.17
Utilization [%]	93

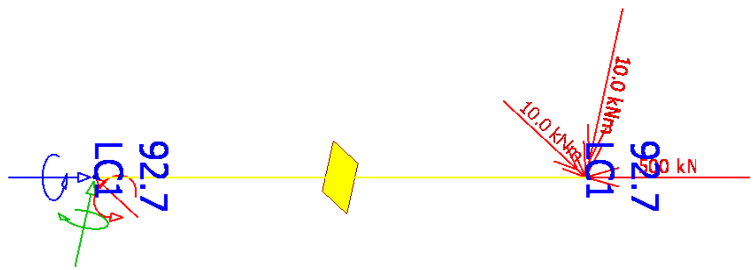


Figure 9.2.7.7 – The detailed results of the RC column in FEM-Design with nominal stiffness method

9.2.8 Calculation of a statically indeterminate beam with post tensioned cables

Inputs:

Characteristic tensile strength of steel	f_{pk} = 1860 MPa
Cross-section of one strand	A_p = 150 mm^2
Number of strands	n = 2
Curvature coefficient	μ = 0.05
Wobble coefficient	κ = 0.007 1/m
Anchorage set slip	g = 4 mm
Young's modulus of the strand	E_p = 195 MPa
Jacking side	point A
Young's modulus of concrete when post tensioning applied	$E_{cm} = E_{cm}(t)$ = 30 GPa
The final value of creep coefficient	$\varphi(t,t_0)$ = 2.00
Final value of shrinkage	ε_{cs} = 0.4 ‰
Relaxation class of the strands	Class 2
Value of relaxation loss	ρ_{1000} = 2.5 %
The cross-section of the beam is rectangle	b = 300 mm, h = 600 mm

In this example we would like to calculate the equivalent forces of a post tensioned cable system before and after the long term stress losses and compare these results with FEM-Design results. For these calculations we need the data which were indicated above in the table and we need to know the shape of the cables. The geometry of the statically indeterminate beam and the shape of the cables are shown in Fig. 9.2.8.1.

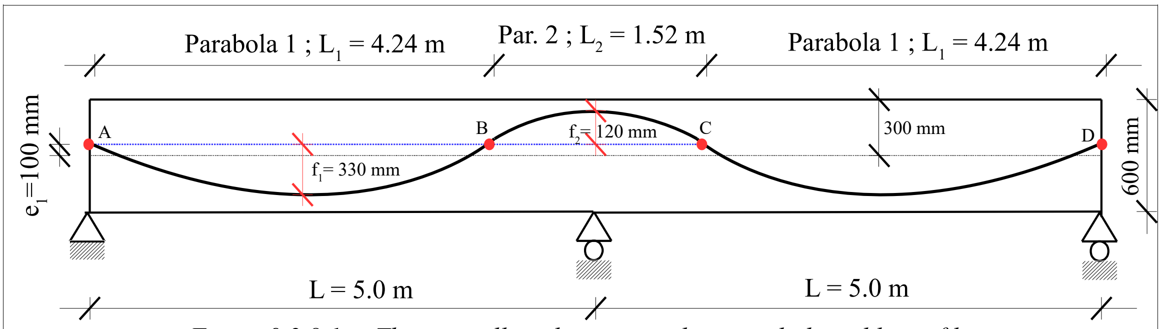


Figure 9.2.8.1 – The statically indeterminate beam with the cable profile

The angular deviation of the cables is a function starting at point A and it depends on the shape of the cables (e.g. base points and inflections). Now we have parabolic shapes on each parts (see Fig. 9.2.8.1).

The angular deviation function: $\alpha(x)$

The values of the angular deviation function at some typical points:

$$\alpha(A)=0\,\text{rad} \quad ; \quad \alpha(B)=\alpha(A)+2\,\text{atan}\frac{4f_1}{L_1}=0.6039\,\text{rad} \quad ;$$

$$\alpha(C)=\alpha(B)+2\,\text{atan}\frac{4f_2}{L_2}=1.215\,\text{rad} \quad ; \quad \alpha(D)=\alpha(C)+2\,\text{atan}\frac{4f_1}{L_1}=1.819\,\text{rad}$$

Based on these values the function of the angular deviation, see Fig. 9.2.8.2.

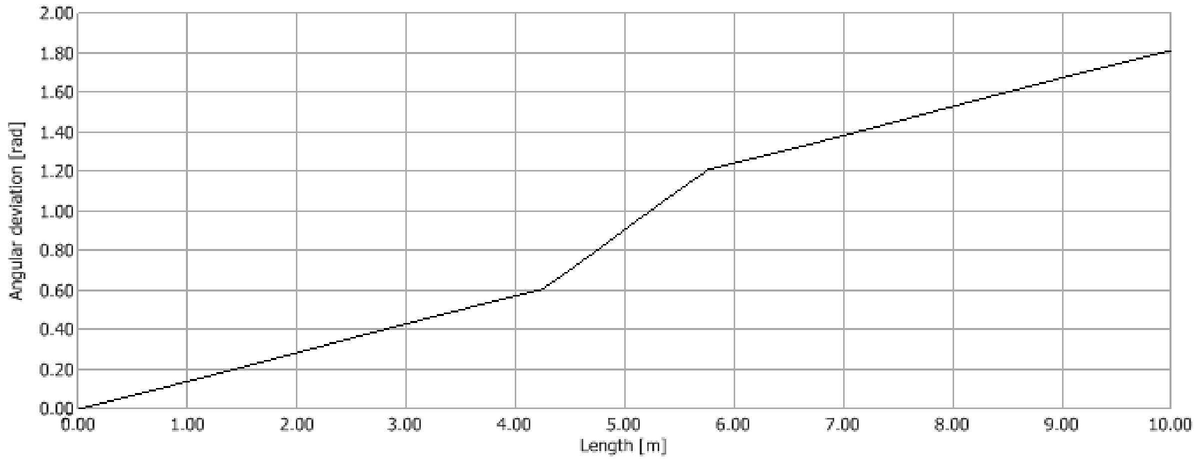


Figure 9.2.8.2 – The function of the angular deviation [*α(x)*]

Let's be the stress at the active jacking end (point A) during post tensioning:

$$\sigma_{p0}=0.8\,f_{pk}=1488\,\text{MPa}$$

NOTE:

By a real design task according to the Eurocode 2 we need to consider an upper bound for this value, e.g.: $\sigma_{p0,max}=\min(0.8\,f_{pk};0.9\,f_{p0,1k})$, but by this example we do not consider it because it does not affect the method of the following calculation. In FEM-Design the user should give the σ_{p0} value and all of the calculations will consider this input stress as jacking stress.

Before the calculation of the equivalent forces which come from the post tensioning of the beam, first of all we need to calculate the different stress losses.

Fisrtly we consider the stress losses which come from the technology of post tensioning.

The stresses in the strands are also a function considering the losses due to friction.

The function of the stresses considering losses due to friction: $\sigma_{p1}(x)$

The values of the function of the stresses considering losses due to friction at some typical points:

$$\sigma_{p1}(A)=\sigma_{p0}=1488\,\text{MPa}$$

$$\sigma_{p1}(B)=\sigma_{p0}e^{-\mu(\alpha_B+\kappa L_1)}=1488\,e^{-0.05(0.6039+0.007\cdot 4.24)}=1442\,\text{MPa}$$

$$\sigma_{p1}(C)=\sigma_{p0}e^{-\mu(\alpha_C+\kappa(L_1+L_2))}=1488\,e^{-0.05(1.215+0.007\cdot(4.24+1.52))}=1397\,\text{MPa}$$

$$\sigma_{p1}(D)=\sigma_{p0}e^{-\mu(\alpha_D+\kappa(L_1+L_2+L_1))}=1488\,e^{-0.05(1.819+0.007\cdot(4.24+1.52+4.24))}=1354\,\text{MPa}$$

Based on these values the function of the stresses considering the losses due to friction, see Fig. 9.2.8.3.

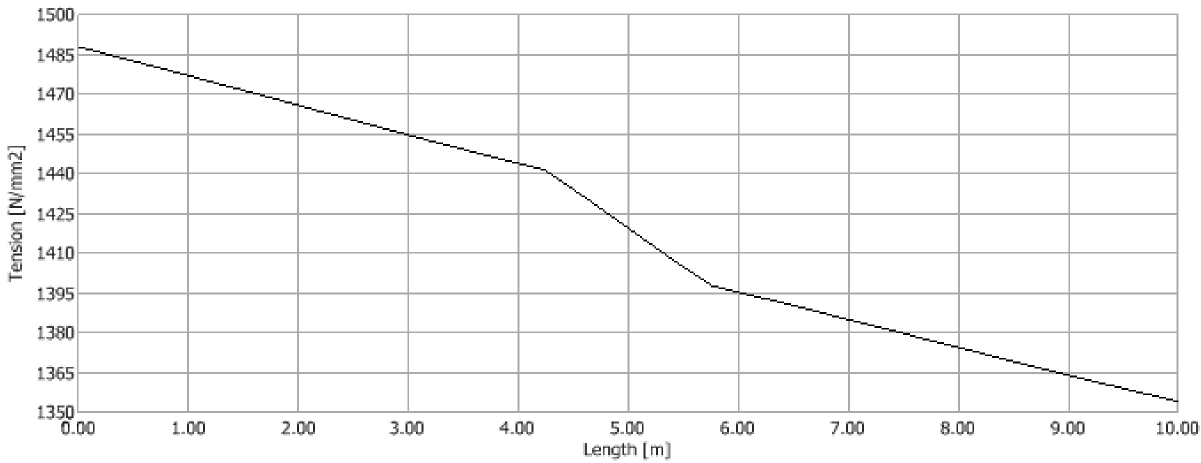


Figure 9.2.8.3 – The function of the stresses considering losses due to friction [$\sigma_{p1}(x)$]

Now we need to calculate the stresses considering the losses due to the anchorage set slip.

The jacking is applied at the start point A.

The length of the effect of the anchorage set slip (L_{s1}) comes from the following equation:

$$g\cdot E_P=2\int_0^{L_{s1}}\left(\sigma_{p1}(x)-\sigma_{pl}(L_{s1})\right)dx=2\int_0^{L_{s1}}\left(\sigma_{p1}(x)-\sigma_{p0}e^{-\mu(\alpha(L_{s1})+\kappa L_{s1})}\right)dx$$

The solution of this equation after some iterations is:

$$L_{s1}=6.667\,\text{m}$$

The stress loss at the active jacking side due to anchorage set slip:

$$\Delta\sigma_{s1}=2\left[\sigma_{p0}\left[1-e^{-\mu(\alpha(L_{s1})+\kappa L_{s1})}\right]\right]=2\left[1488\left[1-e^{-0.05(1.344+0.007\cdot 6.667)}\right]\right]=200\,\text{MPa}$$

The stresses in the strands are also a function considering the anchorage set slip.

The function of the stresses considering losses due to anchorage set slip and friction: $\sigma_{p2}(x)$

The values of this function at some typical points:

$$\sigma_{p2}(A)=-\sigma_{p1}(A)+2\sigma_{p0}-\Delta\sigma_{s1}=-1488+2\cdot1488-200=1288\,\text{MPa}$$

$$\sigma_{p2}(B)=-\sigma_{p1}(B)+2\sigma_{p0}-\Delta\sigma_{s1}=-1442+2\cdot1488-200=1334\,\text{MPa}$$

$$\sigma_{p2}(C)=-\sigma_{p1}(C)+2\sigma_{p0}-\Delta\sigma_{s1}=-1397+2\cdot1488-200=1379\,\text{MPa}$$

$$\sigma_{p2}(L_{s1})=-\sigma_{p1}(L_{s1})+2\sigma_{p0}-\Delta\sigma_{s1}=-1388+2\cdot1488-200=1388\,\text{MPa}$$

$$\sigma_{p2}(D)=\sigma_{p1}(D)=1354\,\text{MPa}$$

Based on these values the function of the stresses considering the losses due to anchorage set slip and friction, see Fig. 9.2.8.4.

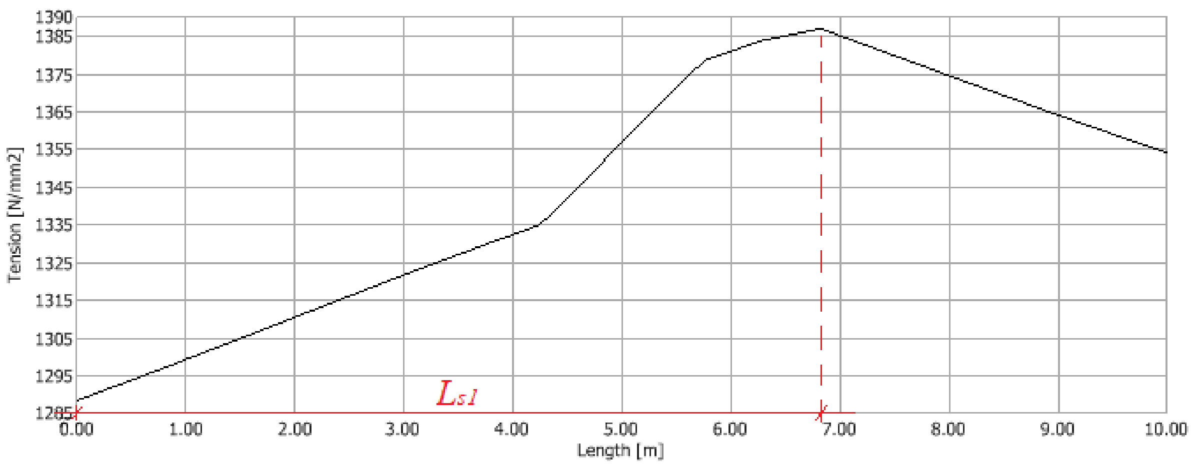


Figure 9.2.8.4 – The function of the stresses considering losses due to anchorage slip and friction [$\sigma_{p2}(x)$]

The average stress value in the strands after the losses due to anchorage set slip and friction:

$$\sigma_m=\frac{\int_0^{2L}\sigma_{p2}(x)\,dx}{2L}=\frac{13445}{10}=1344.5\,\text{MPa}$$

NOTE:

By a real design task according to the Eurocode 2 we need to consider an upper bound for this value, e.g.: $\sigma_{m,max}=\min(0.75 f_{pk}; 0.85 f_{p0,1k})$, but by this example we do not consider it because it does not affect the remaining calculation. In FEM-Design the program is also calculating an average stress after considering the losses due to anchorage set slip and friction and using this value to estimate the short and long term stress losses.

We need to consider the short and long term stress losses in the strands after the jacking.

Next to the frictional and anchorage set slip losses another short term loss is the losses due to elastic shortening.

The average normal stress in the concrete cross-section at the level of the anchorages:

$$\sigma_c = \frac{n\sigma_m A_p}{A_c} + \frac{n\sigma_m A_p e_1}{I_c} e_1 = \frac{2\cdot 1344.5\cdot 150}{300\cdot 600} + \frac{2\cdot 1344.5\cdot 150\cdot 100}{300\cdot 600^3/12} 100 = 2.988\,\text{MPa}$$

The average elastic stress losses according to Eurocode 2:

$$\Delta\sigma_{el} = \frac{n-1}{2n}\sigma_c \frac{E_p}{E_{cm}(t)} = \frac{2-1}{2\cdot 2} 2.988 \frac{195}{30} = 4.856\,\text{MPa}$$

The average stress in the strands after the elastic shortening:

$$\sigma_{ti} = \sigma_m - \Delta\sigma_{el} = 1344.5 - 4.856 = 1341.5\,\text{MPa}$$

The average stress in concrete at the level of the anchorages considering the elastic shortening:

$$\sigma_{c,ti} = \frac{n\sigma_{ti} A_p}{A_c} + \frac{n\sigma_{ti} A_p e_1}{I_c} e_1 = \frac{2\cdot 1341.5\cdot 150}{300\cdot 600} + \frac{2\cdot 1341.5\cdot 150\cdot 100}{300\cdot 600^3/12} 100 = 2.981\,\text{MPa}$$

After the short term stress losses we can calculate the long term stress losses in the strands.

The stress losses due to creep:

$$\Delta\sigma_{cr} = \frac{E_p}{E_{cm}}\varphi(t,t_0)\sigma_{c,ti} = \frac{195}{30} 2.00\cdot 2.981 = 38.75\,\text{MPa}$$

The stress losses due to shrinkage:

$$\Delta\sigma_s = E_p \varepsilon_{cs} = 195\cdot 0.4 = 78\,\text{MPa}$$

The stress losses due to relaxation:

$$\mu_p = \frac{\sigma_{ti}}{f_{pk}} = \frac{1341.5}{1860} = 0.72$$

$$\Delta\sigma_{pr} = 0.66\,\rho_{1000} e^{9.1\mu_p}\left(\frac{t}{1000}\right)^{0.75(1-\mu_p)} 10^{-5}\sigma_{ti} = 0.66\cdot 2.5\, e^{9.1\cdot 0.72}\left(\frac{500000}{1000}\right)^{0.75(1-0.72)} 10^{-5}\cdot 1341.5$$

(Relaxation class: 2.)

$$\Delta\sigma_{pr} = 57.19\,\text{MPa}$$

These three effects, namely the creep, shrinkage and the relaxation are in interaction. The stress losses due to the interaction:

$$\Delta\sigma_{p,c+s+r}=\frac{\Delta\sigma_s+0.8\Delta\sigma_{pr}+\Delta\sigma_{cr}}{1+\frac{E_p}{E_{cm}}\frac{nA_p}{A_c}\left(1+\frac{A_c}{I_c}z_{cp}^2\right)[1+0.8\varphi(t,t_0)]}$$

$$\Delta\sigma_{p,c+s+r}=\frac{78+0.8\cdot 57.19+38.75}{1+\frac{195}{30}\frac{2\cdot 150}{300\cdot 600}\left(1+\frac{300\cdot 600}{300\cdot 600^3/12}100^2\right)[1+0.8\cdot 2.00]}=156.6\,\mathrm{MPa}$$

The average stress in the strands before the long term losses (T0):

$$\sigma_{ti}=\sigma_m-\Delta\sigma_{el}=1344.5-4.856=1341.5\,\mathrm{MPa}$$

The average stress in the strands after the long term losses (T∞):

$$\sigma_t=\sigma_{ti}-\Delta\sigma_{p,c+s+r}=1344.5-156.6=1187.9\,\mathrm{MPa}$$

Based on these values we can calculate the equivalent forces which will represent the effect of the post tensioning on the statically indeterminate beam.

The equivalent forces at T0 time before the long term losses:

The concentrated forces at the ends (at the centroid of the concrete cross-section):

$$P_0=nA_p\sigma_{ti}=2\cdot 150\cdot 1341.5=402.5\,\mathrm{kN}$$

The angle of the tangent of the cable at the ends:

$$\alpha=\mathrm{atan}\frac{4f_1}{L_1}=\mathrm{atan}\frac{4\cdot 330}{4240}=17.29^{\circ}$$

The horizontal and vertical components of the concentrated forces at the ends:

$$P_{0H}=P_0\cos\alpha=402.5\cdot\cos 17.29^{\circ}=384.3\,\mathrm{kN}$$

$$P_{0V}=P_0\sin\alpha=402.5\cdot\sin 17.29^{\circ}=119.5\,\mathrm{kN}$$

The concentrated moments at the ends according to the eccentricities at the ends:

$$M_0=P_{0H}e_1=384.3\cdot 0.1=38.4\,\mathrm{kNm}$$

The intensity of the distributed load according to the different parabola shapes:

$$u_{01}=\frac{8P_0 f_1}{L_1^2}=\frac{8\cdot 402.5\cdot 0.33}{4.24^2}=59.1\frac{\text{kN}}{\text{m}}$$

$$u_{02}=\frac{8P_0 f_2}{L_2^2}=\frac{8\cdot 402.5\cdot 0.12}{1.52^2}=167.3\frac{\text{kN}}{\text{m}}$$

The equivalent forces at T∞ time after the long term losses:

The concentrated forces at the ends (at the centroid of the concrete cross-section):

$$P_\infty=n A_p \sigma_t=2\cdot 150\cdot 1187.9=356.4\,\text{kN}$$

The horizontal and vertical components of the concentrated forces at the ends:

$$P_{\infty H}=P_\infty \cos\alpha=356.4\cdot\cos 17.29^\circ=340.3\,\text{kN}$$

$$P_{\infty V}=P_\infty \sin\alpha=356.4\cdot\sin 17.29^\circ=105.9\,\text{kN}$$

The concentrated moments at the ends according to the eccentricities at the ends:

$$M_\infty=P_{\infty H} e_1=340.3\cdot 0.1=34.0\,\text{kNm}$$

The intensity of the distributed load according to the different parabola shapes:

$$u_{\infty 1}=\frac{8P_\infty f_1}{L_1^2}=\frac{8\cdot 356.4\cdot 0.33}{4.24^2}=52.3\frac{\text{kN}}{\text{m}}$$

$$u_{\infty 2}=\frac{8P_\infty f_2}{L_2^2}=\frac{8\cdot 356.4\cdot 0.12}{1.52^2}=148.1\frac{\text{kN}}{\text{m}}$$

Fig. 9.2.8.5 shows the equivalent forces on the statically indeterminate beam before and after the long term stress losses in the strands.

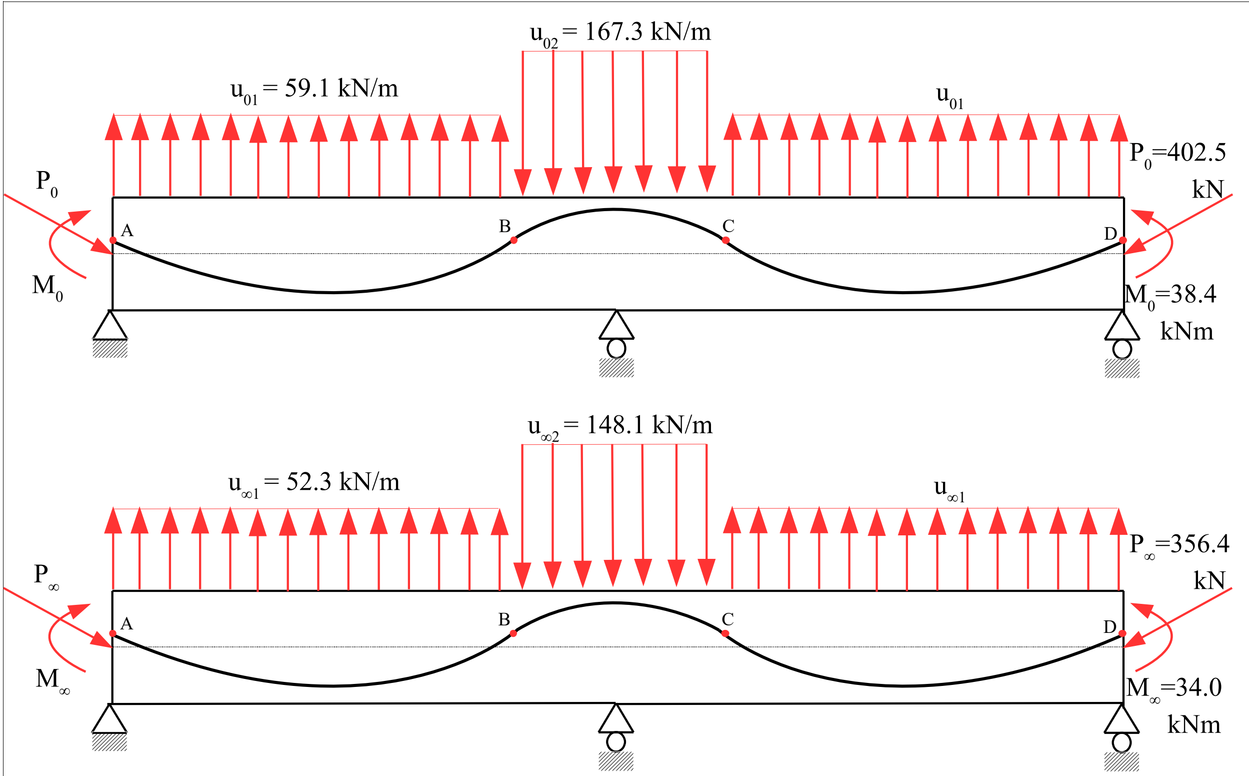


Figure 9.2.8.5 – The equivalent forces before [above] *and after* [below] *the long term stress losses*

Considering the equivalent forces after the long term stress losses ($T\infty$, Fig. 9.2.8.5 below) the camber (at the mid-span) of the statically indeterminate beam (without detailed calculation according to the theory of elasticity) is:

$e_{p,camber} = 2.86\,\mathrm{mm}$

By this camber calculation we considered only the effect of the equivalent forces (Fig. 9.2.8.5 below) and the effective modulus of elasticity of concrete: $E_{c,eff} = E_{cm}/(1+\varphi(t,t_0)) = 10$ GPa.

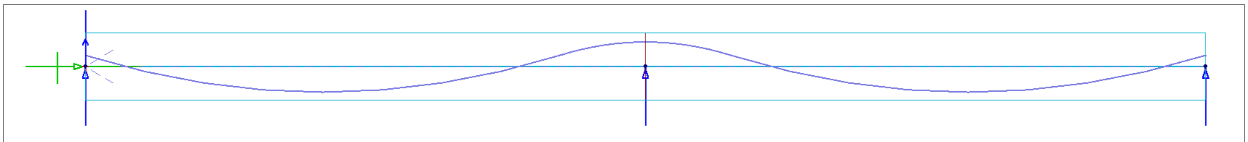


Figure 9.2.8.6 – The shape of the post tensioned cables in FEM-Design

Fig. 9.2.8.6. shows the shape of the post tensioned cable in FEM-Design.

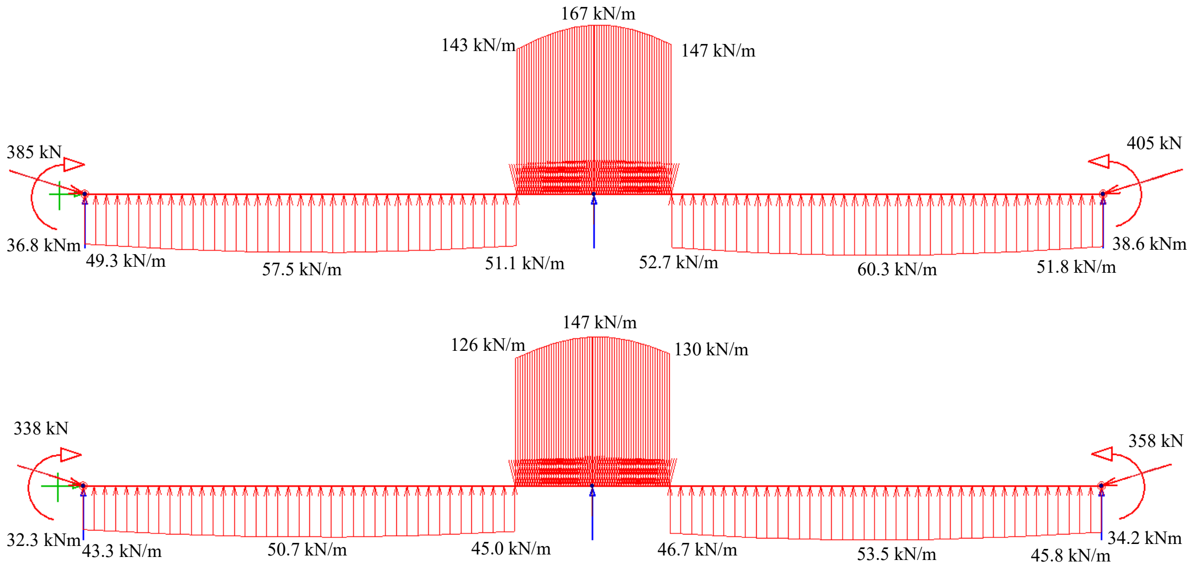


Figure 9.2.8.7 – The equivalent forces before [above, T0] *and after* [below,T∞] *the long term stress losses in FEM-Design*

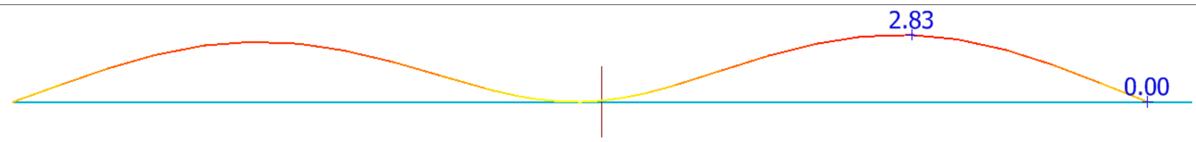


Figure 9.2.8.8 – The vertical translations [mm] *in FEM-Design from the equivalent post tensioning loads at T∞ time considering the effective modulus of eleasticity of concrete*

We can say that the results of the hand calculation and the automatic post tensioned cable calculation of FEM-Design are identical. See the FEM-Design results about the equivalent forces in Fig. 9.2.8.7 and the camber in Fig. 9.2.8.8.

Keep in mind that FEM-Design post tensioned cable modul calculates the equivalent forces in more precise way than this hand calculation and considers the curvatures of the shape of the cable in more accurate way. Theoretically the calculated equivalent forces are in equilibrium but the presented hand calculation method does not consider the friction force (and its eccentricities), but FEM-Design is checking the equilibrium of the equivalent forces and automatically applying a distributed axial force and a distributed bending moment on the beam if it is necessary. These forces and moments are also indicated in the program by the post tensioned equivalent load cases.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.8 Calculation of a statically indeterminate beam with post tensioned cables.str

9.3 Steel design

9.3.1 Interaction of normal force, bending moment and shear force

In this sub-chapter an IPE 200 beam will be investigated under the interaction of normal force, shear force and bending moment around its strong axis (see Fig. 9.3.1.1). Stability analysis will not be considered here only strength resistance calculations.

Inputs:

Yield strength of structural steel	$f_y = 235\ N/mm^2$
Cross-sectional width	b = 100 mm
Cross-sectional height	h = 200 mm
Flange thickness	$t_f = 8.5$ mm
Web thickness	$t_w = 5.6$ mm
Web height	$h_w = 159$ mm
Radius of root fillet	r = 12 mm
Cross-sectional area	$A = 2848\ mm^2$
Plastic cross-sectional modulus	$W_{pl,y} = 220638\ mm^3$
Normal force	$N_{Ed} = 70$ kN (compression)
Bending moment around strong (y') axis	$M_{Ed} = 37.5$ kNm
Shear force	$V_{Ed} = 150$ kN

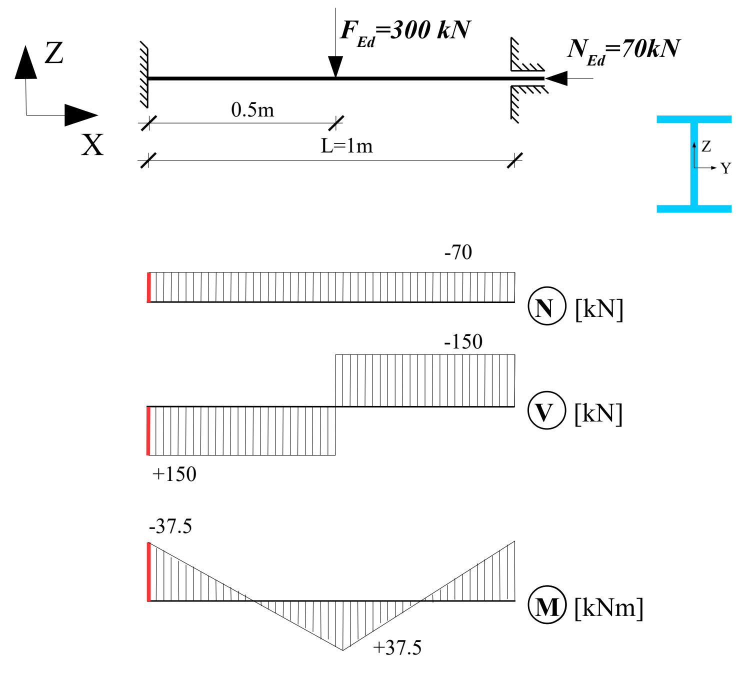


Figure 9.3.1.1 – The statical system and internal forces

First we need to make the classification of the cross section individually for every internal forces:

The coefficient depending on f_y:

$$\varepsilon=\sqrt{\frac{235}{f_y}}=\sqrt{\frac{235}{235}}=1$$

Classification due to normal force

Flanges:

$$c=\frac{b-t_w-2r}{2}=\frac{100-5.6-2\cdot 12}{2}=35.2\,\text{mm} \quad ; \quad \frac{c}{t}=\frac{c}{t_f}=\frac{35.2}{8.5}=4.14 \quad ; \quad \frac{c}{t}<9\,\varepsilon$$

Because $4.14<9$ thus the flanges are in Class 1.

Web:

$$\frac{c}{t}=\frac{h_w}{t_w}=\frac{159}{5.6}=28.39 \quad ; \quad \frac{c}{t}<33\,\varepsilon$$

Because $28.39<33$ thus the web is in Class 1.

Therefore the cross section is in Class 1 under normal force.

Classification due to bending moment around strong axis

Flange:

$$c=\frac{b-t_w-2\mathrm{r}}{2}=\frac{100-5.6-2\cdot 12}{2}=35.2\,\text{mm} \qquad \frac{c}{t}=\frac{c}{t_f}=\frac{35.2}{8{,}5}=4.14 \quad ; \quad \frac{c}{t}<9\,\varepsilon$$

because $4.14<9$ thus the flange is in Class 1.

Web:

$$\frac{c}{t}=\frac{h_w}{t_w}=\frac{159}{5.6}=28.39 \qquad \frac{c}{t}<72\,\varepsilon$$

because $28.39<72$ thus the web is in Class 1.

Therefore the cross section is in Class 1 under bending moment around strong axis.

Interaction of bending, shear and axial force

Normal force resistance under compression:

$$N_{c,Rd} = A\frac{f_y}{\gamma_{M0}} = 2848\frac{235}{1.0} = 669.3\,\text{kN}$$

Bending moment resistance around strong axis:

$$M_{c,Rd} = W_{pl,y}\frac{f_y}{\gamma_{M0}} = 220638\frac{235}{1.0} = 51.85\,\text{kNm}$$

Shear resistance:

$$A_v = A - 2bt_f + (t_w + 2\text{r})t_f = 2848 - 2\cdot100\cdot8.5 + (5.6 + 2\cdot12)8.5 = 1400\,\text{mm}^2$$

$$V_{pl,Rd} = A_v\frac{f_y/\sqrt{3}}{\gamma_{M0}} = 1400\frac{235/\sqrt{3}}{1.0} = 189.9\,\text{kN}$$

The moment resistance should be reduced if $V_{Ed} > 0.5\,V_{pl,Rd}$.

Now $150\,\text{kN} > 94.95\,\text{kN}$ therefore the reduction factor is:

$$\rho = \left(\frac{2V_{Ed}}{V_{pl,Rd}} - 1\right)^2 = \left(\frac{2\cdot150}{189.9} - 1\right)^2 = 0.3361$$

The reduced bending resistance:

$$M_{y,V,Rd} = \left(W_{pl,y} - \frac{\rho A_w^2}{4t_w}\right)\frac{f_y}{\gamma_{M0}} = \left(220638 - \frac{0.3361\cdot1400^2}{4\cdot5.6}\right)\frac{235}{1.0} = 44.94\,\text{kNm}$$

The utilization according to the interaction:

$$\frac{N_{Ed}}{N_{c,Rd}} + \frac{M_{y,Ed}}{M_{y,V,Rd}} = \frac{70}{669.3} + \frac{37.5}{44.94} = 0.939 < 1.0$$ thus the resistance is adequate.

Fig. 9.3.1.2 shows the statical system and the internal forces in FEM-Design.

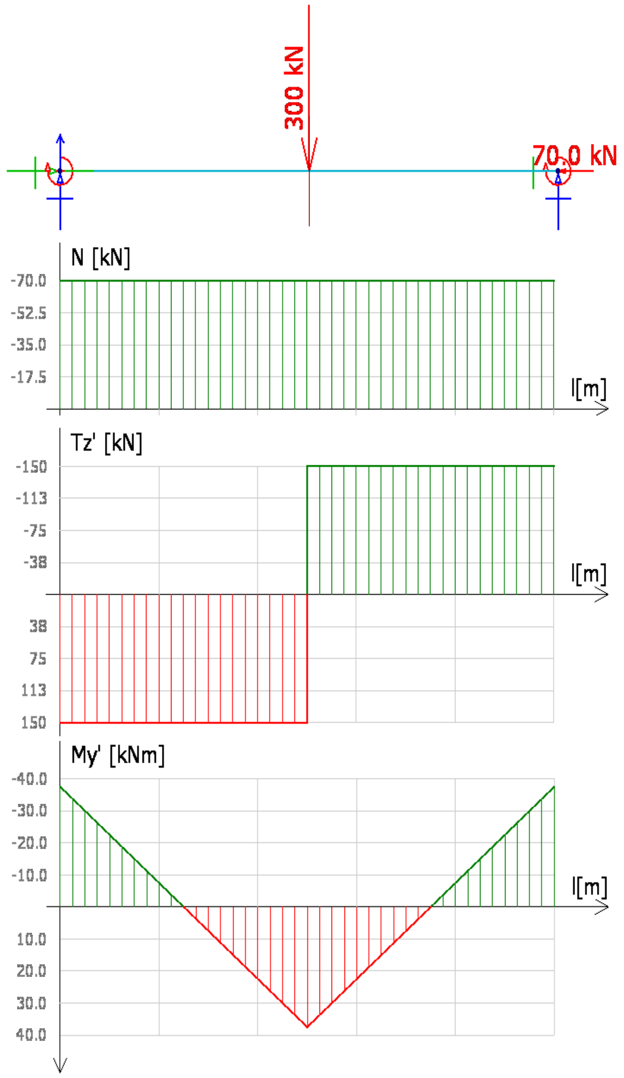
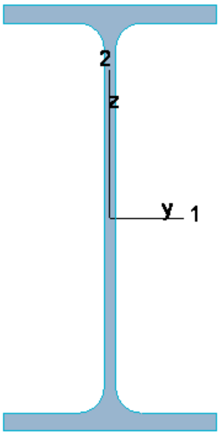


Figure 9.3.1.2 – The statical system and internal forces in FEM-Design

The detailed results with the interaction utilization based on FEM-Design are shown in Fig. 9.3.1.3.

IPE 200



P	=	768 mm	f_y	=	235 N/mm^2
A	=	2848 mm^2	ε	=	1.00
I_y	=	1.943e+07 mm^4	λ_1	=	93.90
I_z	=	1.424e+06 mm^4			
I_1	=	1.943e+07 mm^4			
I_2	=	1.424e+06 mm^4			
$W_{pl,1}$	=	2.206e+05 mm^3			
$W_{pl,2}$	=	4.464e+04 mm^3			
$W_{el,min,1}$	=	1.943e+05 mm^3			
$W_{el,min,2}$	=	2.847e+04 mm^3			
i_1	=	83 mm			
i_2	=	22 mm			
I_t	=	6.846e+04 mm^4			
I_w	=	1.275e+10 mm^6			

Shear resistance, 2-2 - Part 1-1: 6.2.6, 6.2.8
LC: '1', x = 0 mm

$Class_N = 1$, $Class_{M1} = 1$, $Class_{M2} = 1$

$$V_{2,pl,Rd} = \frac{A_{2,v} \cdot f_y}{\sqrt{3} \cdot \gamma_{M0}} = \frac{1400 \cdot 235}{\sqrt{3} \cdot 1.00} = 189.95 \text{ kN} \quad (6.18)$$

$$V_{2,pl,T,Rd} = \sqrt{1 - \frac{\tau_{t,Ed}}{1.25 \left(f_y / \sqrt{3}\right) / \gamma_{M0}}} \cdot V_{2,pl,Rd} =$$

$$= \sqrt{1 - \frac{0.00}{1.25 \left(235 / \sqrt{3}\right) / 1.00}} \cdot 189.95 = 189.95 \text{ kN} \quad (6.26)$$

$$\frac{V_{2,Ed}}{V_{2,pl,T,Rd}} = \frac{150.00}{189.95} = 0.79 \le 1.00 \quad (6.25) \text{ - OK}$$

Normal capacity - Part 1-1: 6.2
LC: '1', x = 1000 mm

$Class_N = 1$, $Class_{M1} = 1$, $Class_{M2} = 1$

$$V_{1,Ed} = 0.00 \text{ kN} \le 0.5 \cdot V_{1,pl,T,Rd} = 0.5 \cdot 247.42 = 123.71 \text{ kN} \dashrightarrow p_1 = 0.00$$

$$V_{2,Ed} = 150.00 \text{ kN} > 0.5 \cdot V_{2,pl,T,Rd} = 0.5 \cdot 189.95 = 94.97 \text{ kN} \dashrightarrow p_2 = \left(\frac{2 \cdot V_{2,Ed}}{V_{2,pl,T,Rd}} - 1\right)^2 = \left(\frac{2 \cdot 150.00}{189.95} - 1\right)^2 = 0.34$$

$$\frac{N_{Ed}}{N_{Rd}} + \frac{M_{1,Ed}}{M_{1,Rd}} + \frac{M_{2,Ed}}{M_{2,Rd}} = \frac{70.00}{669.38} + \frac{37.50}{46.29} + \frac{0.00}{10.49} = 0.91 \le 1.00 \quad (6.2) \text{ - OK}$$

Figure 9.3.1.3 – The detailed results from FEM-Design

The difference between the two calculations is 3%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.3.1 Interaction of normal force, bending moment and shear.str

9.3.2 Buckling of a doubly symmetric I section

The buckling stability analysis will be investigate in an IPE 240 simply supported beam (see Fig. 9.3.2.1).

Inputs:

Yield strength of structural steel	$f_y = 355\ N/mm^2$
Cross-sectional width	b = 120 mm
Cross-sectional height	h = 240 mm
Flange thickness	$t_f = 9.8$ mm
Web thickness	$t_w = 6.2$ mm
Web height	$h_w = 190.4$ mm
Radius of root fillet	r = 15 mm
Cross-sectional area	$A = 3912\ mm^2$
Inertia around strong axis	$I_y = 38916273\ mm^4$
Inertia around weak axis	$I_z = 2836341\ mm^4$
St. Venant torsional constant	$I_t = 127368\ mm^4$
Warping constant	$I_\omega = 36680292708\ mm^6$
Buckling length in both directions	$L_{cr} = 6.0$ m

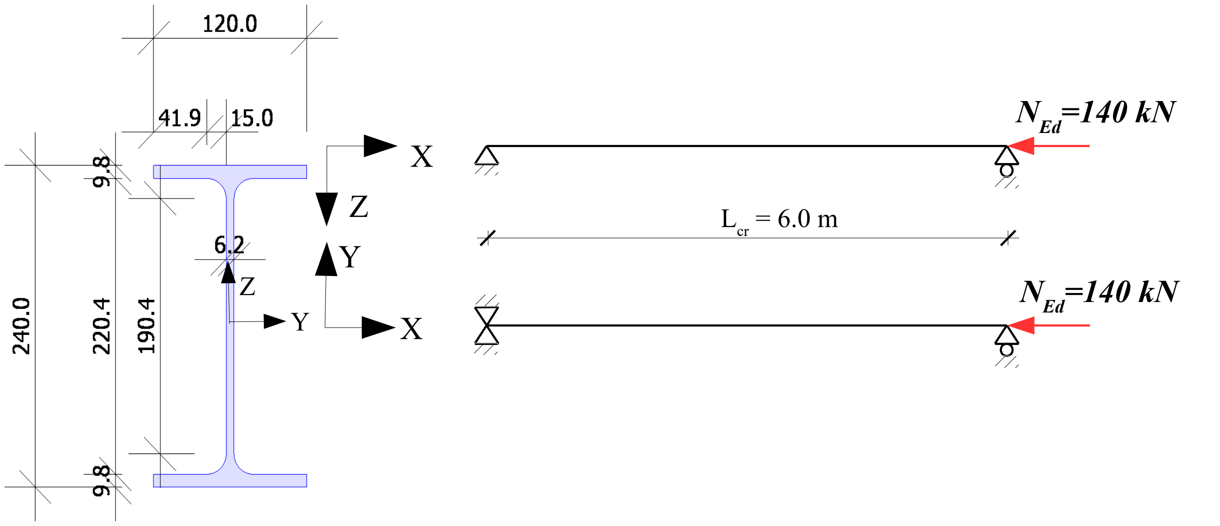


Figure 9.3.2.1 – The statical system and the cross-section

First of all we need to make the classification of the cross section for normal force:

The coefficient depending on f_y:

$$\varepsilon=\sqrt{\frac{235}{f_y}}=\sqrt{\frac{235}{355}}=0.8136$$

Classification due to normal force

Flanges:

$$c=\frac{b-t_w-2r}{2}=\frac{120-6.2-2\cdot 15}{2}=41.9\,\text{mm} \quad ; \quad \frac{c}{t}=\frac{c}{t_f}=\frac{41.9}{9.8}=4.276 \quad ; \quad \frac{c}{t}<9\,\varepsilon$$

Because $4.276<7.322$ thus the flanges are in Class 1.

Web:

$$\frac{c}{t}=\frac{h_w}{t_w}=\frac{190.4}{6.2}=30.71 \quad ; \quad \frac{c}{t}<38\,\varepsilon$$

Because $30.71<30.92$ thus the web is in Class 2.

Therefore the cross section is in Class 2 under normal force.

Flexural buckling around strong axis

The radius of gyration (y-y axis): $i_y=\sqrt{\frac{I_y}{A}}=\sqrt{\frac{38916273}{3912}}=99.74\,\text{mm}$

The non-dimensional slenderness: $\bar{\lambda}_y=\frac{L_{cr}}{i_y}\frac{1}{\lambda_1}=\frac{6000}{99.74}\frac{1}{76.41}=0.7873$, where

$$\lambda_1=\pi\sqrt{\frac{E_s}{f_y}}=\pi\sqrt{\frac{210000}{355}}=76.41$$

The imperfection α factor value based on EN 1993-1-1 Table 6.2:

$\frac{h}{b}=\frac{240}{120}=2>1.2$ rolled section (y-y axis) and $t_f=9.8\,\text{mm}<40\,\text{mm}$ therefore “a” buckling curve is relevant thus the imperfection factor is $\alpha_y=0.21$.

$$\Phi_y=0.5\left[1+\alpha_y\left(\bar{\lambda}_y-0.2\right)+\bar{\lambda}_y^{\,2}\right]=0.5\left[1+0.21\left(0.7873-0.2\right)+0.7873^2\right]=0.8716$$

Reduction factor: $\chi_y=\frac{1}{\Phi_y+\sqrt{\Phi_y^2-\bar{\lambda}_y^{\,2}}}=\frac{1}{0.8716+\sqrt{0.8716^2-0.7873^2}}=0.8029$

Flexural buckling resistance: $N_{b,y,Rd}=\frac{\chi_y A f_y}{\gamma_{M1}}=\frac{0.8029\cdot 3912\cdot 355}{1}=1115\,\text{kN}$

Flexural buckling around weak axis

The radius of gyration (z-z axis): $i_z=\sqrt{\frac{I_z}{A}}=\sqrt{\frac{2836341}{3912}}=26.93\text{ mm}$

The non-dimensional slenderness: $\bar{\lambda}_z=\frac{L_{cr}}{i_z}\frac{1}{\lambda_1}=\frac{6000}{26.93}\frac{1}{76.41}=2.916$, where

$$\lambda_1=\pi\sqrt{\frac{E_s}{f_y}}=\pi\sqrt{\frac{210000}{355}}=76.41$$

The imperfection α factor value based on EN 1993-1-1 Table 6.2:

$\frac{h}{b}=\frac{240}{120}=2>1.2$ rolled section (z-z axis) and $t_f=9.8\,\text{mm}<40\,\text{mm}$ therefore “b” buckling curve is relevant thus the imperfection factor is $\alpha_z=0.34$.

$$\Phi_z=0.5\left(1+\alpha_z\left(\bar{\lambda}_z-0.2\right)+\bar{\lambda}_z^{\,2}\right)=0.5\left(1+0.34\left(2.916-0.2\right)+2.916^2\right)=5.213$$

Reduction factor: $\chi_z=\frac{1}{\Phi_z+\sqrt{\Phi_z^2-\bar{\lambda}_z^{\,2}}}=\frac{1}{5.213+\sqrt{5.213^2-2.916^2}}=0.1049$

Flexural buckling resistance: $N_{b,z,Rd}=\frac{\chi_z A f_y}{\gamma_{M1}}=\frac{0.1049\cdot 3912\cdot 355}{1}=145.7\,\text{kN}$

Torsional buckling

The elastic torsional buckling critical force:

$$N_{cr,T}=\frac{1}{i_0^2}\left(G I_t+\frac{\pi^2 E I_\omega}{L_{cr}^2}\right)=\frac{1}{103.3^2}\left(80769\cdot 127368+\frac{\pi^2\cdot 210000\cdot 36680292708}{6000^2}\right)=1162\text{ kN}$$

where i_0 is the polar radius of gyration:

$$i_0=\sqrt{i_y^2+i_z^2+y_0^2+z_0^2}=\sqrt{99.74^2+26.93^2+0^2+0^2}=103.3\,\text{mm}$$,

where y_0 and z_0 are the distances between the center of gravity and the shear center of the cross-section respect to the principal directions.

The torsional-flexural buckling is not relevant because the cross-section is doubly symmetric.

The non-dimensional slenderness for torsional buckling: $\bar{\lambda}_T = \sqrt{\frac{A f_y}{N_{cr,T}}} = \sqrt{\frac{3912 \cdot 355}{1162000}} = 1.093$

The imperfection α factor value based on EN 1993-1-1 Table 6.2:

$\frac{h}{b} = \frac{240}{120} = 2 > 1.2$ rolled section (z-z axis) and $t_f = 9.8\,\text{mm} < 40\,\text{mm}$ therefore “b” buckling curve is relevant thus the imperfection factor is $\alpha_T = 0.34$.

$$\Phi_T = 0.5\left(1 + \alpha_T\left(\bar{\lambda}_T - 0.2\right) + \bar{\lambda}_T^{\,2}\right) = 0.5\left(1 + 0.34\left(1.093 - 0.2\right) + 1.093^2\right) = 1.249$$

Reduction factor: $\chi_T = \frac{1}{\Phi_T + \sqrt{\Phi_T^2 - \bar{\lambda}_T^{\,2}}} = \frac{1}{1.249 + \sqrt{1.249^2 - 1.093^2}} = 0.5395$

Torsional buckling resistance: $N_{b,T,Rd} = \frac{\chi_T A f_y}{\gamma_{M1}} = \frac{0.5395 \cdot 3912 \cdot 355}{1} = 749.2\,\text{kN}$

The statical system and the internal forces shown in Fig. 9.3.2.2.

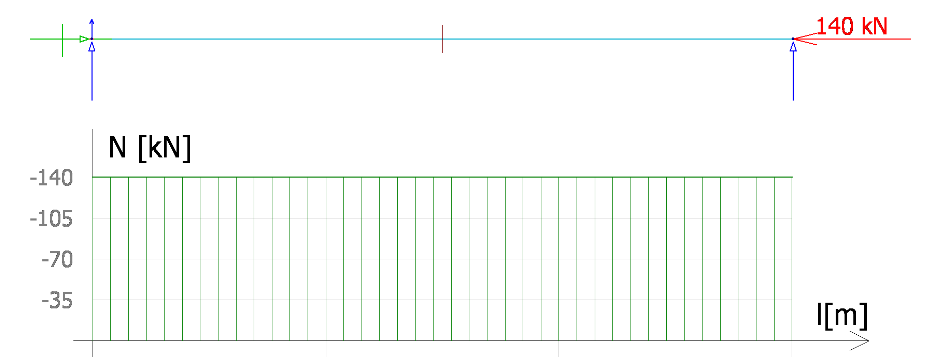


Figure 9.3.2.2 – The statical system and the internal force in FEM-Design

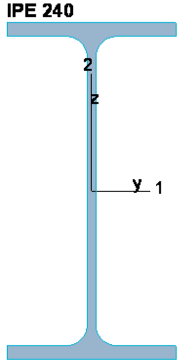
Fig. 9.3.2.3 shows the detailed results about flexural and torsional buckling in FEM-Design.

The numerical result are almost identical with the hand calculations. The difference is less than 0.5%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.3.2 Buckling of a doubly symmetric I section.str

IPE 240



P	=	922 mm
A	=	3912 mm^2
I_y	=	3.892e+07 mm^4
I_z	=	2.836e+06 mm^4
I_1	=	3.892e+07 mm^4
I_2	=	2.836e+06 mm^4
$W_{pl,1}$	=	3.666e+05 mm^3
$W_{pl,2}$	=	7.396e+04 mm^3
$W_{el,min,1}$	=	3.243e+05 mm^3
$W_{el,min,2}$	=	4.727e+04 mm^3
i_1	=	100 mm
i_2	=	27 mm
I_t	=	1.274e+05 mm^4
I_w	=	3.668e+10 mm^6

f_y = 355 N/mm^2

ε = 0.81

λ_1 = 76.40

Flexural buckling, 1-1 - Part 1-1: 6.3.1

LC: '1', x = 3500 mm

$Class_N = 2$, $Class_{M1} = 1$, $Class_{M2} = 1$

$$\bar{\lambda}_1 = \frac{L_{cr,1}}{i_1 \cdot \lambda_1} = \frac{6000}{100 \cdot 76.40} = 0.79 \quad (6.50)$$

$\alpha_1 = 0.21$ (Buckling curve: a)

$$\varphi_1 = 0.5\left[1 + \alpha_1 \cdot (\bar{\lambda}_1 - 0.2) + \bar{\lambda}_1^2\right] = 0.5\left[1 + 0.21 \cdot (0.79 - 0.2) + 0.79^2\right] = 0.87$$

$$\chi_1 = \min\left(\frac{1}{\varphi_1 + \sqrt{\varphi_1^2 - \bar{\lambda}_1^2}}, 1.0\right) = \min\left(\frac{1}{0.87 + \sqrt{0.87^2 - 0.79^2}}, 1.0\right) = 0.80 \quad (6.49)$$

$$N_{b,Rd,1} = \frac{\chi_1 \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.80 \cdot 3912 \cdot 355}{1.00} = 1114.85\ \text{kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,1}} = \frac{140.00}{1114.85} = 0.13 \le 1.00 \quad (6.46)\ \text{- OK}$$

Flexural buckling, 2-2 - Part 1-1: 6.3.1

LC: '1', x = 3500 mm

$Class_N = 2$, $Class_{M1} = 1$, $Class_{M2} = 1$

$$\bar{\lambda}_2 = \frac{L_{cr,2}}{i_2 \cdot \lambda_1} = \frac{6000}{27 \cdot 76.40} = 2.92 \quad (6.50)$$

$\alpha_2 = 0.34$ (Buckling curve: b)

$$\varphi_2 = 0.5\left[1 + \alpha_2 \cdot (\bar{\lambda}_2 - 0.2) + \bar{\lambda}_2^2\right] = 0.5\left[1 + 0.34 \cdot (2.92 - 0.2) + 2.92^2\right] = 5.21$$

$$\chi_2 = \min\left(\frac{1}{\varphi_2 + \sqrt{\varphi_2^2 - \bar{\lambda}_2^2}}, 1.0\right) = \min\left(\frac{1}{5.21 + \sqrt{5.21^2 - 2.92^2}}, 1.0\right) = 0.10 \quad (6.49)$$

$$N_{b,Rd,2} = \frac{\chi_2 \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.10 \cdot 3912 \cdot 355}{1.00} = 145.59\ \text{kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,2}} = \frac{140.00}{145.59} = 0.96 \le 1.00 \quad (6.46)\ \text{- OK}$$

Torsional-flexural buckling - Part 1-1: 6.3.1

LC: '1', x = 3500 mm

$Class_N = 2$, $Class_{M1} = 1$, $Class_{M2} = 1$

$$i_0 = \sqrt{i_1^2 + i_2^2 + y_0^2 + z_0^2} = \sqrt{100^2 + 27^2 + 0^2 + 0^2} = 103\ \text{mm}$$

$$N_{cr,T} = \frac{1}{i_0^2}\left(G \cdot I_t + \frac{\pi^2 \cdot E \cdot I_w}{l_T^2}\right) =$$

$$= \frac{1}{103^2}\left(80769 \cdot 1.274e{+}05 + \frac{\pi^2 \cdot 210000 \cdot 3.668e{+}10}{6000^2}\right) = 1161.62\ \text{kN}$$

$N_{cr,TF} = 1161.62$ kN

$N_{cr} = \min(N_{cr,T}, N_{cr,TF}) = \min(1161.62, 1161.62) = 1161.62$ kN

$$\bar{\lambda}_T = \sqrt{\frac{A \cdot f_y}{N_{cr}}} = \sqrt{\frac{3912 \cdot 355}{1161.62}} = 1.09 \quad (6.53)$$

$\alpha_T = 0.34$ (Buckling curve: b)

$$\varphi_T = 0.5\left[1 + \alpha_T \cdot (\bar{\lambda}_T - 0.2) + \bar{\lambda}_T^2\right] = 0.5\left[1 + 0.34 \cdot (1.09 - 0.2) + 1.09^2\right] = 1.25$$

$$\chi_T = \min\left(\frac{1}{\varphi_T + \sqrt{\varphi_T^2 - \bar{\lambda}_T^2}}, 1.0\right) = \min\left(\frac{1}{1.25 + \sqrt{1.25^2 - 1.09^2}}, 1.0\right) = 0.54 \quad (6.49)$$

$$N_{b,Rd,T} = \frac{\chi_T \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.54 \cdot 3912 \cdot 355}{1.00} = 748.75\ \text{kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,T}} = \frac{140.00}{748.75} = 0.19 \le 1.00\ \text{- OK}$$

Figure 9.3.2.3 – Detailed results based on FEM-Design

9.3.3 Buckling of a doubly symmetric + section

Yield strength of structural steel	f_y = 355 N/mm^2
Cross sectional width	b = 200 mm
Thickness of the parts	t = 4 mm
Cross-sectional area	A = 1584 mm^2
Inertia around strong axis	I_y = 2667712 mm^4
Inertia around weak axis	I_z = 2667712 mm^4
St. Venant torsional constant	I_t = 8529 mm^4
Warping constant	I_ω = 7097545 mm^6
Buckling length	L_{cr} = 2 m

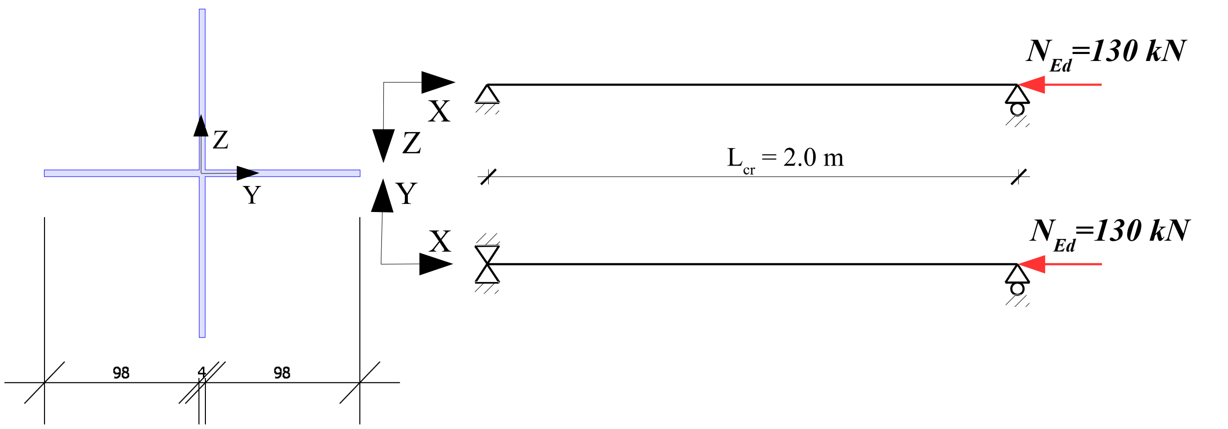


Figure 9.3.3.1 – The statical system and the cross-section

First of all we need to make the classification of the cross section for normal force:

The coefficient depending on f_y:

$$\varepsilon=\sqrt{\frac{235}{f_y}}=\sqrt{\frac{235}{355}}=0.8136$$

<u>Classification due to normal force</u>

<u>Outstand flanges:</u>

$$c=\frac{b-t}{2}=\frac{200-4}{2}=98\,\text{mm} \quad ; \quad \frac{c}{t}=\frac{98}{4}=24.45 \quad ; \quad \frac{c}{t}>14\,\varepsilon$$

Because $24.45>11.39$ thus the flanges (and the whole section) are in Class 4.

Calculation of the effective cross-section

$k_\sigma = 0.43$

$$\bar{\lambda}_p = \frac{\frac{c}{t}}{28.4\,\varepsilon\sqrt{k_\sigma}} = \frac{\frac{98}{4}}{28.4 \cdot 0.8136\sqrt{0.43}} = 1.617$$

because $\bar{\lambda}_p = 1.617 > 0.748$ thus: $\rho = \frac{\bar{\lambda}_p - 0.188}{\bar{\lambda}_p^{\,2}} = \frac{1.617 - 0.188}{1.617^2} = 0.5465$

$$b_{eff} = \rho\, c = 0.5465 \cdot 98 = 53.56\,\text{mm}$$

$$A_{eff} = t^2 + 4\,b_{eff}\,t = 4^2 + 4 \cdot 53.56 \cdot 4 = 872.9\,\text{mm}^2$$

Flexural buckling

The radius of gyration (based on the gross section): $i_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{2667712}{1584}} = 41.04\,\text{mm}$

The non-dimensional slenderness: $\bar{\lambda} = \frac{L_{cr}}{i_y}\frac{1}{\lambda_1}\sqrt{\frac{A_{eff}}{A}} = \frac{2000}{41.04}\frac{1}{76.41}\sqrt{\frac{872.9}{1584}} = 0.4733$, where

$$\lambda_1 = \pi\sqrt{\frac{E_s}{f_y}} = \pi\sqrt{\frac{210000}{355}} = 76.41$$

In EN 1993-1-1 Table 6.2 this type of section is not included thus “c” buckling curve was chosen. The imperfection factor is: $\alpha = 0.49$.

$$\Phi = 0.5\left[1 + \alpha\left(\bar{\lambda} - 0.2\right) + \bar{\lambda}^2\right] = 0.5\left[1 + 0.49\left(0.4733 - 0.2\right) + 0.4733^2\right] = 0.6790$$

Reduction factor: $\chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} = \frac{1}{0.6790 + \sqrt{0.6790^2 - 0.4733^2}} = 0.8577$

Flexural buckling resistance: $N_{b,Rd} = \frac{\chi\, A_{eff}\, f_y}{\gamma_{M1}} = \frac{0.8577 \cdot 872.9 \cdot 355}{1} = 265.8\,\text{kN}$

Torsional buckling

The elastic torsional buckling critical force:

$$N_{cr,T} = \frac{1}{i_0^2}\left(G\,I_t + \frac{\pi^2 E\, I_\omega}{L_{cr}^2}\right) = \frac{1}{58.04^2}\left(80769 \cdot 8529 + \frac{\pi^2 \cdot 210000 \cdot 7097545}{2000^2}\right) = 205.6\,\text{kN}$$

where i_0 is the polar radius of gyration:

$$i_0 = \sqrt{i_y^2 + i_z^2 + y_0^2 + z_0^2} = \sqrt{41.04^2 + 41.04^2 + 0^2 + 0^2} = 58.04\,\text{mm} \quad ,$$

where y_0 and z_0 are the distances between the center of gravity and the shear center of the cross-section respect to the principal directions.

The torsional-flexural buckling is not relevant because the cross-section is doubly symmetric.

The non-dimensional slenderness for torsional buckling: $\bar{\lambda}_T = \sqrt{\frac{A_{eff} f_y}{N_{cr,T}}} = \sqrt{\frac{872.9 \cdot 355}{205600}} = 1.228$

In EN 1993-1-1 Table 6.2 this type of section is not included thus "c" buckling curve was chosen. The imperfection factor is: $\alpha_T = 0.49$.

$$\Phi_T = 0.5\left(1 + \alpha_T\left(\bar{\lambda}_T - 0.2\right) + \bar{\lambda}_T^{\,2}\right) = 0.5\left(1 + 0.49\left(1.228 - 0.2\right) + 1.228^2\right) = 1.506$$

Reduction factor: $\chi_T = \frac{1}{\Phi_T + \sqrt{\Phi_T^2 - \bar{\lambda}_T^{\,2}}} = \frac{1}{1.506 + \sqrt{1.506^2 - 1.228^2}} = 0.4206$

Torsional buckling resistance: $N_{b,T,Rd} = \frac{\chi_T A_{eff} f_y}{\gamma_{M1}} = \frac{0.4206 \cdot 872.9 \cdot 355}{1} = 130.3\ \text{kN}$

The statical system and the normal forces shown in Fig. 9.3.3.2.

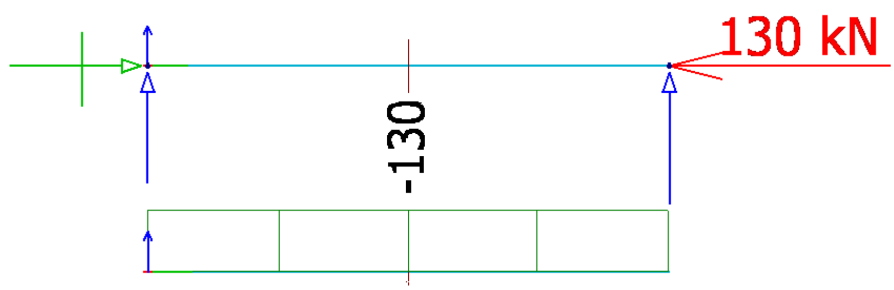
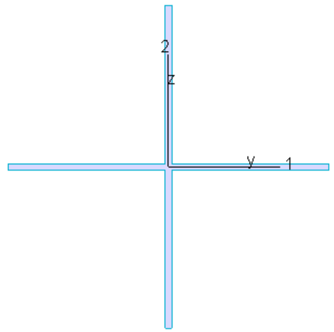


Figure 9.3.3.2 – The statical system and the normal force diagram [kN] *in FEM-Design*

Fig. 9.3.3.3 shows the detailed results about flexural and torsional buckling in FEM-Design.

The numerical result are almost identical with the hand calculations. The difference is less than 0.1%.



S 355

E	= 210000 N/mm^2	$\gamma_{M0,acc/seis}$	= 1.00		
G	= 80769 N/mm^2	$\gamma_{M1,acc/seis}$	= 1.00		
$\gamma_{M0,ult}$	= 1.00	$\gamma_{M2,acc/seis}$	= 1.00		
$\gamma_{M1,ult}$	= 1.00	P	= 800 mm		
$\gamma_{M2,ult}$	= 1.25	A	= 1584 mm^2		
		I_y	= 2.668e+06 mm^4		
f_y	= 355 N/mm^2	I_z	= 2.668e+06 mm^4		
ε	= 0.81	I_1	= 2.668e+06 mm^4		
λ_1	= 76.40	I_2	= 2.668e+06 mm^4		
		$W_{pl,1}$	= 4.078e+04 mm^3		
		$W_{pl,2}$	= 4.078e+04 mm^3		
		$W_{el,min,1}$	= 2.668e+04 mm^3		
		$W_{el,min,2}$	= 2.668e+04 mm^3		
		i_1	= 41 mm		
		i_2	= 41 mm		
		I_t	= 8.529e+03 mm^4		
		I_w	= 7.098e+06 mm^6		

Flexural buckling, 1-1 - Part 1-1: 6.3.1
LC: '1', x = 1500 mm

Class$_N$ = 4, Class$_{M1}$ = 4, Class$_{M2}$ = 4

$\bar{\lambda}_1 = \frac{L_{cr,1}}{i_1 \cdot \lambda_1} \cdot \sqrt{\frac{A_{eff}}{A}} = \frac{2000}{41 \cdot 76.40} \cdot \sqrt{\frac{873}{1584}} = 0.47$ (6.51)

$\alpha_1 = 0.49$ (Buckling curve: c)

$\varphi_1 = 0.5\left[1 + \alpha_1 \cdot (\bar{\lambda}_1 - 0.2) + \bar{\lambda}_1^2\right] = 0.5\left[1 + 0.49 \cdot (0.47 - 0.2) + 0.47^2\right] = 0.68$

$\chi_1 = \min\left(\frac{1}{\varphi_1 + \sqrt{\varphi_1^2 - \bar{\lambda}_1^2}}, 1.0\right) = \min\left(\frac{1}{0.68 + \sqrt{0.68^2 - 0.47^2}}, 1.0\right) = 0.86$ (6.49)

$N_{b,Rd,1} = \frac{\chi_1 \cdot A_{eff} \cdot f_y}{\gamma_{M1}} = \frac{0.86 \cdot 873 \cdot 355}{1.00} = 265.80$ kN (6.48)

$\frac{N_{Ed}}{N_{b,Rd,1}} = \frac{130.00}{265.80} = 0.49 \leq 1.00$ (6.46) - OK

Flexural buckling, 2-2 - Part 1-1: 6.3.1
LC: '1', x = 1500 mm

Class$_N$ = 4, Class$_{M1}$ = 4, Class$_{M2}$ = 4

$\bar{\lambda}_2 = \frac{L_{cr,2}}{i_2 \cdot \lambda_1} \cdot \sqrt{\frac{A_{eff}}{A}} = \frac{2000}{41 \cdot 76.40} \cdot \sqrt{\frac{873}{1584}} = 0.47$ (6.51)

$\alpha_2 = 0.49$ (Buckling curve: c)

$\varphi_2 = 0.5\left[1 + \alpha_2 \cdot (\bar{\lambda}_2 - 0.2) + \bar{\lambda}_2^2\right] = 0.5\left[1 + 0.49 \cdot (0.47 - 0.2) + 0.47^2\right] = 0.68$

$\chi_2 = \min\left(\frac{1}{\varphi_2 + \sqrt{\varphi_2^2 - \bar{\lambda}_2^2}}, 1.0\right) = \min\left(\frac{1}{0.68 + \sqrt{0.68^2 - 0.47^2}}, 1.0\right) = 0.86$ (6.49)

$N_{b,Rd,2} = \frac{\chi_2 \cdot A_{eff} \cdot f_y}{\gamma_{M1}} = \frac{0.86 \cdot 873 \cdot 355}{1.00} = 265.80$ kN (6.48)

$\frac{N_{Ed}}{N_{b,Rd,2}} = \frac{130.00}{265.80} = 0.49 \leq 1.00$ (6.46) - OK

Torsional-flexural buckling - Part 1-1: 6.3.1
LC: '1', x = 1500 mm

Class$_N$ = 4, Class$_{M1}$ = 4, Class$_{M2}$ = 4

$i_0 = \sqrt{i_1^2 + i_2^2 + y_0^2 + z_0^2} = \sqrt{41^2 + 41^2 + 0^2 + 0^2} = 58$ mm

$N_{cr,T} = \frac{1}{i_0^2}\left(G \cdot I_t + \frac{\pi^2 \cdot E \cdot I_w}{l_T^2}\right) =$

$= \frac{1}{58^2}\left(80769 \cdot 8.529e{+}03 + \frac{\pi^2 \cdot 210000 \cdot 7.098e{+}06}{2000^2}\right) = 205.61$ kN

$i_0^2\,(N - N_{b,Rd,1})\,(N - N_{b,Rd,2})\,(N - N_{cr,T}) - N^2 y_0^2\,(N - N_{b,Rd,2}) - N^2 z_0^2\,(N - N_{b,Rd,1}) = 0$

Smallest root of the above equation:

$N_{cr,TF} = 205.61$ kN

$N_{cr} = \min(N_{cr,T}, N_{cr,TF}) = \min(205.61, 205.61) = 205.61$ kN

$\bar{\lambda}_T = \sqrt{\frac{A_{eff} \cdot f_y}{N_{cr}}} = \sqrt{\frac{873 \cdot 355}{205.61}} = 1.23$ (6.52)

$\alpha_T = 0.49$ (Buckling curve: c)

$\varphi_T = 0.5\left[1 + \alpha_T \cdot (\bar{\lambda}_T - 0.2) + \bar{\lambda}_T^2\right] = 0.5\left[1 + 0.49 \cdot (1.23 - 0.2) + 1.23^2\right] = 1.51$

$\chi_T = \min\left(\frac{1}{\varphi_T + \sqrt{\varphi_T^2 - \bar{\lambda}_T^2}}, 1.0\right) = \min\left(\frac{1}{1.51 + \sqrt{1.51^2 - 1.23^2}}, 1.0\right) = 0.42$ (6.49)

$N_{b,Rd,T} = \frac{\chi_T \cdot A_{eff} \cdot f_y}{\gamma_{M1}} = \frac{0.42 \cdot 873 \cdot 355}{1.00} = 130.40$ kN (6.48)

$\frac{N_{Ed}}{N_{b,Rd,T}} = \frac{130.00}{130.40} = 1.00 \leq 1.00$ - OK

Figure 9.3.3.3 – Detailed results based on FEM-Design

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.3.3 Buckling of a doubly symmetric + section.str

9.3.4 Buckling of a mono-symmetric channel section

Yield strength of structural steel	$f_y = 275$ N/mm^2
Thickness of the parts	t = 20 mm
Cross-sectional area	$A = 9200$ mm^2
Inertia around strong axis	$I_y = 35158841$ mm^4
Inertia around weak axis	$I_z = 13426667$ mm^4
St. Venant torsional constant	$I_t = 1217153$ mm^4
Warping constant	$I_\omega = 54039544948$ mm^6
Buckling length	$L_{cr} = 6$ m

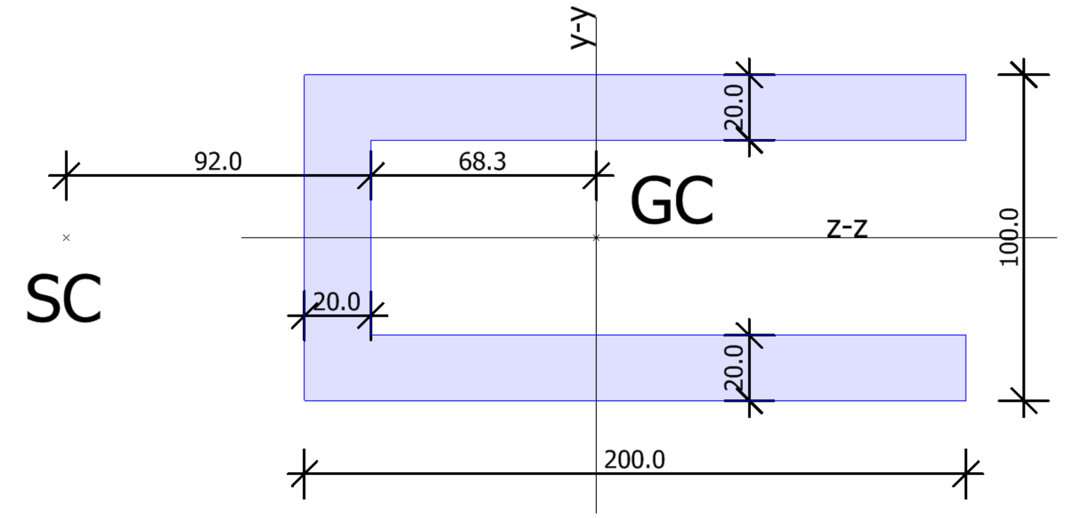


Figure 9.3.4.1 – The channel section with the dimensions [mm] *and the positions of the gravity and shear centers*

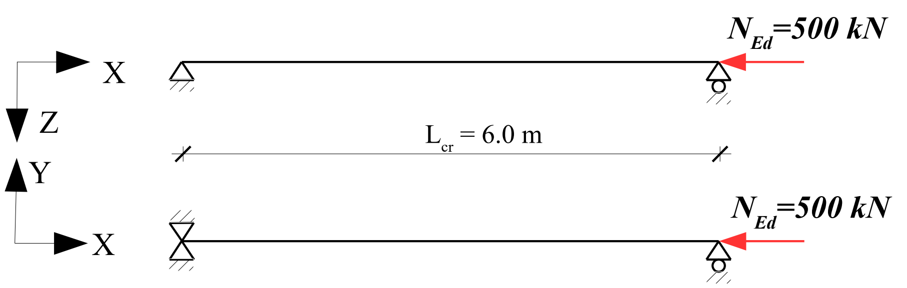


Figure 9.3.4.2 – The statical system and design forces

First of all we need to make the classification of the cross section for normal force.

The coefficient depending on f_y:

$$\varepsilon=\sqrt{\frac{235}{f_y}}=\sqrt{\frac{235}{275}}=0.9244$$

Classification due to normal force

Flanges:

$$c=200-20=180\,\text{mm} \quad ; \quad \frac{c}{t}=\frac{180}{20}=9.0 \quad ; \quad 9\varepsilon<\frac{c}{t}<10\varepsilon$$

Because $7.395<9.0<9.244$ thus the flanges are in Class 2.

Web:

$$c=100-20-20=60\,\text{mm} \quad ; \quad \frac{c}{t}=\frac{60}{20}=3.0 \quad ; \quad \frac{c}{t}<33\,\varepsilon$$

Because $3.0<30.51$ thus the web is in Class 1.

According to these calculations the section is in Class 2 due to normal force.

Flexural buckling around strong axis

The radius of gyration (y-y axis, see Fig. 9.3.4.1): $i_y=\sqrt{\frac{I_y}{A}}=\sqrt{\frac{35158841}{9200}}=61.82\,\text{mm}$

The non-dimensional slenderness: , $\bar{\lambda}_y=\frac{L_{cr}}{i_y}\frac{1}{\lambda_1}=\frac{6000}{61.82}\frac{1}{86.81}=1.118$ where

$$\lambda_1=\pi\sqrt{\frac{E_s}{f_y}}=\pi\sqrt{\frac{210000}{275}}=86.81$$

According to EN 1993-1-1 Table 6.2 “c” buckling curve was chosen. The imperfection factor is: $\alpha_y=0.49$.

$$\Phi_y=0.5\left[1+\alpha_y\left(\bar{\lambda}_y-0.2\right)+\bar{\lambda}_y^{\,2}\right]=0.5\left[1+0.49\left(1.118-0.2\right)+1.118^2\right]=1.350$$

Reduction factor: $\chi_y=\frac{1}{\Phi_y+\sqrt{\Phi_y^2-\bar{\lambda}_y^{\,2}}}=\frac{1}{1.350+\sqrt{1.350^2-1.118^2}}=0.4747$

Flexural buckling resistance: $N_{b,y,Rd}=\frac{\chi_y A f_y}{\gamma_{MI}}=\frac{0.4747\cdot 9200\cdot 275}{1}=1201\,\text{kN}$

Flexural buckling around weak axis

The radius of gyration (z-z axis, see Fig. 9.3.4.1): $i_z=\sqrt{\frac{I_z}{A}}=\sqrt{\frac{13426667}{9200}}=38.20\,\text{mm}$

The non-dimensional slenderness: $\bar{\lambda}_z=\frac{L_{cr}}{i_z}\frac{1}{\lambda_1}=\frac{6000}{38.20}\frac{1}{86.81}=1.809$, where

$$\lambda_1=\pi\sqrt{\frac{E_s}{f_y}}=\pi\sqrt{\frac{210000}{275}}=86.81$$

According to EN 1993-1-1 Table 6.2 “c” buckling curve was chosen. The imperfection factor is: $\alpha_z=0.49$.

$$\Phi_z=0.5\left(1+\alpha_z\left(\bar{\lambda}_z-0.2\right)+\bar{\lambda}_z^{\,2}\right)=0.5\left(1+0.49\left(1.809-0.2\right)+1.809^2\right)=2.530$$

Reduction factor: $\chi_z=\frac{1}{\Phi_z+\sqrt{\Phi_z^2-\bar{\lambda}_z^{\,2}}}=\frac{1}{2.530+\sqrt{2.530^2-1.809^2}}=0.2326$

Flexural buckling resistance: $N_{b,z,Rd}=\frac{\chi_z A f_y}{\gamma_{M1}}=\frac{0.2326\cdot 9200\cdot 275}{1}=588.5\,\text{kN}$

Torsional buckling

The elastic torsional buckling critical force:

$$N_{cr,T}=\frac{1}{i_0^2}\left(G I_t+\frac{\pi^2 E I_\omega}{L_{cr}^2}\right)=\frac{1}{176.0^2}\left(80769\cdot 1217153+\frac{\pi^2\cdot 210000\cdot 54039544948}{6000^2}\right)=3274\,\text{kN}$$

where i_0 is the polar radius of gyration:

$$i_0=\sqrt{i_y^2+i_z^2+y_0^2+z_0^2}=\sqrt{61.82^2+38.20^2+0^2+(68.3+92.0)^2}=176.0\,\text{mm} \quad ,$$

where y_0 and z_0 are the distances between the center of gravity and the shear center of the cross-section respect to the principal directions (see Fig. 9.3.4.1).

The non-dimensional slenderness for torsional buckling: $\bar{\lambda}_T=\sqrt{\frac{A f_y}{N_{cr,T}}}=\sqrt{\frac{9200\cdot 275}{3274000}}=0.8791$

According to EN 1993-1-1 Table 6.2 “c” buckling curve was chosen. The imperfection factor is: $\alpha_T=0.49$.

$$\Phi_T=0.5\left(1+\alpha_T\left(\bar{\lambda}_T-0.2\right)+\bar{\lambda}_T^{\,2}\right)=0.5\left(1+0.49\left(0.8791-0.2\right)+0.8791^2\right)=1.053$$

Reduction factor: $\chi_T = \frac{1}{\Phi_T + \sqrt{\Phi_T^2 - \bar{\lambda}_T^2}} = \frac{1}{1.053 + \sqrt{1.053^2 - 0.8791^2}} = 0.6125$

Torsional buckling resistance: $N_{b,T,Rd} = \frac{\chi_T A f_y}{\gamma_{M1}} = \frac{0.6125 \cdot 9200 \cdot 275}{1} = 1550\,\text{kN}$

Torsional-flexural buckling.

The torsional-flexural buckling could be relevant by a mono symmetric section.

We need to find the roots of the following equation:

$$i_0^2(N - N_{cr,y})(N - N_{cr,z})(N - N_{cr,T}) - N^2 y_0^2(N - N_{cr,z}) - N^2 z_0^2(N - N_{cr,y}) = 0$$

Where in addition the quantities already calculated:

$$N_{cr,y} = \frac{\pi^2 EI_y}{L_{cr}^2} = \frac{\pi^2 210000 \cdot 35158841}{6000^2} = 2024\,\text{kN} \quad \text{and}$$

$$N_{cr,z} = \frac{\pi^2 EI_z}{L_{cr}^2} = \frac{\pi^2 210000 \cdot 13426667}{6000^2} = 773.0\,\text{kN}$$

$y_0 = 0\,\text{mm}$; $z_0 = 68.3 + 92 = 160.3\,\text{mm}$ the distance between the gravity center and shear center respect to the principal directions (see Fig. 9.3.4.1).

$$i_0 = \sqrt{i_y^2 + i_z^2 + y_0^2 + z_0^2} = \sqrt{61.82^2 + 38.20^2 + 0^2 + 160.3^2} = 176.0\,\text{mm}$$

Thus the third degree polynomial:

$$176^2(N - 2024)(N - 773)(N - 3274) - N^2 0^2 (N - 773) - N^2 160.3^2 (N - 2024) = 0$$

The roots of the equation (see Fig. 9.3.4.3):

$$N_1 = 642.7\,\text{kN} \quad ; \quad N_2 = 2024\,\text{kN} \quad ; \quad N_3 = 23097\,\text{kN}$$

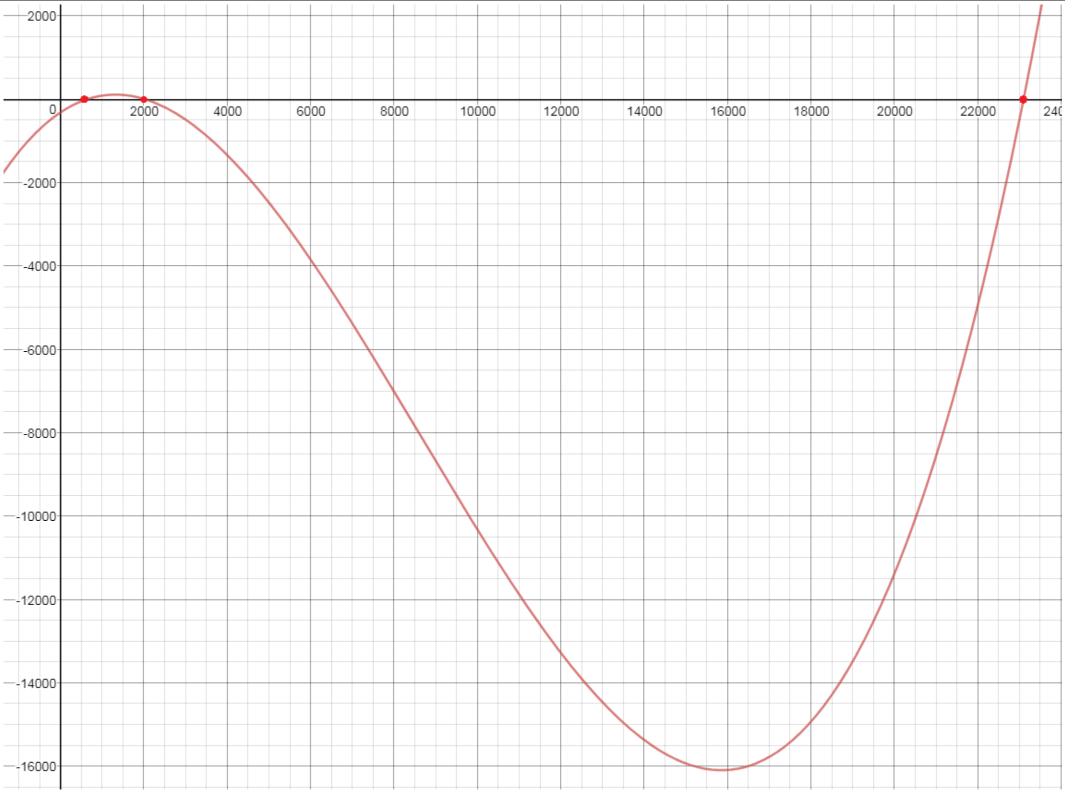


Figure 9.3.4.3 – The roots of the third degree polynomial

Because the smallest root is smaller than the smallest clear critical elastic force thus the torsional-flexural buckling is relevant in this case.

$$N_1 = 642.7\,\text{kN} < N_{cr,z} = 773.0\,\text{kN} < N_2 = N_{cr,y} = 2024\,\text{kN} < N_{cr,T} = 3274\,\text{kN} < N_3 = 23097\,\text{kN}$$

Therefore:

$$N_{cr,TF} = N_1 = 642.7\,\text{kN}$$

The non-dimensional slenderness for torsional-flexural buckling:

$$\bar{\lambda}_{TF} = \sqrt{\frac{A f_y}{N_{cr,TF}}} = \sqrt{\frac{9200 \cdot 275}{642700}} = 1.984$$

$$\Phi_{TF} = 0.5\left(1 + \alpha_{TF}\left(\bar{\lambda}_{TF} - 0.2\right) + \bar{\lambda}_{TF}^{\,2}\right) = 0.5\left(1 + 0.49\left(1.984 - 0.2\right) + 1.984^2\right) = 2.905$$

Reduction factor: $\chi_{TF} = \dfrac{1}{\Phi_{TF} + \sqrt{\Phi_{TF}^2 - \bar{\lambda}_{TF}^2}} = \dfrac{1}{2.905 + \sqrt{2.905^2 - 1.984^2}} = 0.1989$

Torsional-flexural buckling resistance: $N_{b,TF,Rd} = \dfrac{\chi_{TF} A f_y}{\gamma_{M1}} = \dfrac{0.1989 \cdot 9200 \cdot 275}{1} = 503.2\,\text{kN}$

The statical system and the normal forces shown in Fig. 9.3.4.4.

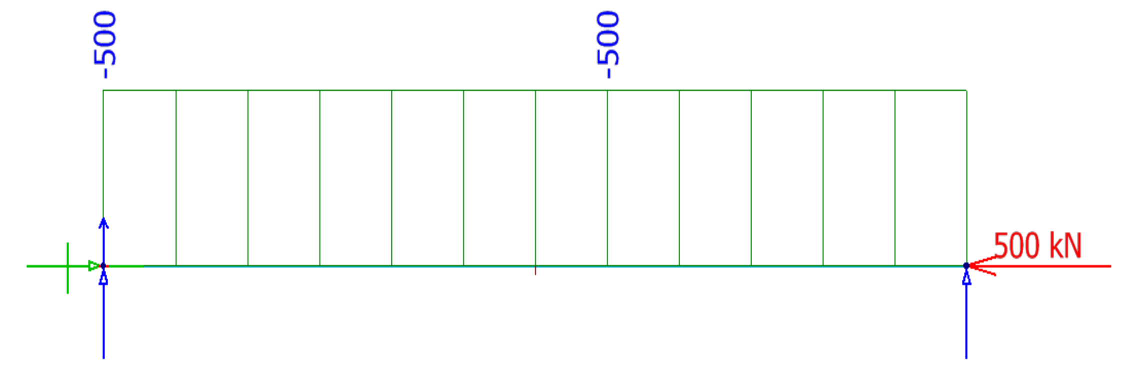


Figure 9.3.4.4 – The statical system and the normal force diagram [kN] *in FEM-Design*

Fig. 9.3.4.5 shows the detailed results about torsional-flexural and torsional buckling in FEM-Design.

The numerical result are almost identical with the hand calculations. The difference is less than 0.1%.

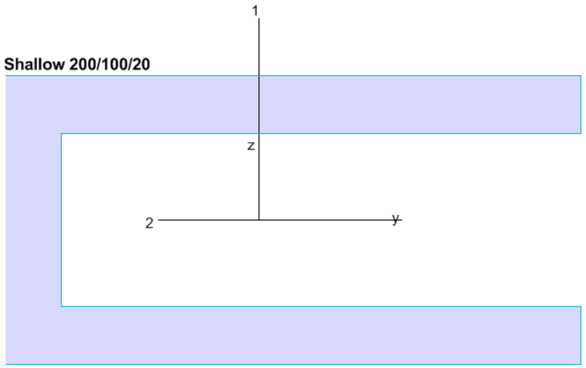
Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.3.4 Buckling of a mono symmetric channel section.str

S 275

E	=	210000 N/mm^2	$\gamma_{M0,ult}$	=	1.00	$\gamma_{M0,acc/seis}$	=	1.00
G	=	80769 N/mm^2	$\gamma_{M1,ult}$	=	1.00	$\gamma_{M1,acc/seis}$	=	1.00
			$\gamma_{M2,ult}$	=	1.25	$\gamma_{M2,acc/seis}$	=	1.00

Shallow 200/100/20



P	=	960 mm
A	=	9200 mm^2
I_y	=	1.343e+07 mm^4
I_z	=	3.516e+07 mm^4
I_1	=	3.516e+07 mm^4
I_2	=	1.343e+07 mm^4
$W_{pl,1}$	=	4.990e+05 mm^3
$W_{pl,2}$	=	3.380e+05 mm^3
$W_{el,min,1}$	=	3.147e+05 mm^3
$W_{el,min,2}$	=	2.685e+05 mm^3
i_1	=	62 mm
i_2	=	38 mm
I_t	=	1.217e+06 mm^4
I_w	=	5.404e+10 mm^6

f_y	=	275 N/mm^2
ε	=	0.92
λ_1	=	86.80

Flexural buckling, 1-1 - Part 1-1: 6.3.1
LC: '1', x = 3500 mm

$Class_N = 2$, $Class_{M1} = 2$, $Class_{M2} = 1$

$$\bar{\lambda}_1 = \frac{L_{cr,1}}{i_1 \cdot \lambda_1} = \frac{6000}{62 \cdot 86.80} = 1.12 \quad (6.50)$$

$\alpha_1 = 0.49$ (Buckling curve: c)

$$\varphi_1 = 0.5\left[1 + \alpha_1 \cdot (\bar{\lambda}_1 - 0.2) + \bar{\lambda}_1^2\right] = 0.5\left[1 + 0.49 \cdot (1.12 - 0.2) + 1.12^2\right] = 1.35$$

$$\chi_1 = \min\left(\frac{1}{\varphi_1 + \sqrt{\varphi_1^2 - \bar{\lambda}_1^2}}, 1.0\right) = \min\left(\frac{1}{1.35 + \sqrt{1.35^2 - 1.12^2}}, 1.0\right) = 0.47 \quad (6.49)$$

$$N_{b,Rd,1} = \frac{\chi_1 \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.47 \cdot 9200 \cdot 275}{1.00} = 1200.96 \text{ kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,1}} = \frac{500.00}{1200.96} = 0.42 \le 1.00 \quad (6.46) \text{ - OK}$$

Flexural buckling, 2-2 - Part 1-1: 6.3.1
LC: '1', x = 3500 mm

$Class_N = 2$, $Class_{M1} = 2$, $Class_{M2} = 1$

$$\bar{\lambda}_2 = \frac{L_{cr,2}}{i_2 \cdot \lambda_1} = \frac{6000}{38 \cdot 86.80} = 1.81 \quad (6.50)$$

$\alpha_2 = 0.49$ (Buckling curve: c)

$$\varphi_2 = 0.5\left[1 + \alpha_2 \cdot (\bar{\lambda}_2 - 0.2) + \bar{\lambda}_2^2\right] = 0.5\left[1 + 0.49 \cdot (1.81 - 0.2) + 1.81^2\right] = 2.53$$

$$\chi_2 = \min\left(\frac{1}{\varphi_2 + \sqrt{\varphi_2^2 - \bar{\lambda}_2^2}}, 1.0\right) = \min\left(\frac{1}{2.53 + \sqrt{2.53^2 - 1.81^2}}, 1.0\right) = 0.23 \quad (6.49)$$

$$N_{b,Rd,2} = \frac{\chi_2 \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.23 \cdot 9200 \cdot 275}{1.00} = 588.20 \text{ kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,2}} = \frac{500.00}{588.20} = 0.85 \le 1.00 \quad (6.46) \text{ - OK}$$

Torsional-flexural buckling - Part 1-1: 6.3.1
LC: '1', x = 3500 mm

$Class_N = 2$, $Class_{M1} = 2$, $Class_{M2} = 1$

$$i_0 = \sqrt{i_1^2 + i_2^2 + y_0^2 + z_0^2} = \sqrt{62^2 + 38^2 + 0^2 + 160^2} = 176 \text{ mm}$$

$$N_{cr,1} = \frac{\pi^2 \cdot E \cdot I_1}{L_{cr,1}^2} = \frac{\pi^2 \cdot 210000 \cdot 35158841}{6000^2} = 2024.19 \text{ kN}$$

$$N_{cr,2} = \frac{\pi^2 \cdot E \cdot I_2}{L_{cr,2}^2} = \frac{\pi^2 \cdot 210000 \cdot 13426667}{6000^2} = 773.01 \text{ kN}$$

$$N_{cr,T} = \frac{1}{i_0^2}\left(G \cdot I_t + \frac{\pi^2 \cdot E \cdot I_w}{l_T^2}\right) =$$

$$= \frac{1}{176^2}\left(80769 \cdot 1.217e{+}06 + \frac{\pi^2 \cdot 210000 \cdot 5.404e{+}10}{6000^2}\right) = 3272.88 \text{ kN}$$

$$i_0^2 (N - N_{cr,1})(N - N_{cr,2})(N - N_{cr,T}) - N^2 y_0^2 (N - N_{cr,2}) - N^2 z_0^2 (N - N_{cr,1}) = 0$$

Smallest root of the above equation:

$N_{cr,TF} = 642.72$ kN

$N_{cr} = \min(N_{cr,T}, N_{cr,TF}) = \min(3272.88, 642.72) = 642.72$ kN

$$\bar{\lambda}_T = \sqrt{\frac{A \cdot f_y}{N_{cr}}} = \sqrt{\frac{9200 \cdot 275}{642.72}} = 1.98 \quad (6.53)$$

$\alpha_T = 0.49$ (Buckling curve: c)

$$\varphi_T = 0.5\left[1 + \alpha_T \cdot (\bar{\lambda}_T - 0.2) + \bar{\lambda}_T^2\right] = 0.5\left[1 + 0.49 \cdot (1.98 - 0.2) + 1.98^2\right] = 2.91$$

$$\chi_T = \min\left(\frac{1}{\varphi_T + \sqrt{\varphi_T^2 - \bar{\lambda}_T^2}}, 1.0\right) = \min\left(\frac{1}{2.91 + \sqrt{2.91^2 - 1.98^2}}, 1.0\right) = 0.20 \quad (6.49)$$

$$N_{b,Rd,T} = \frac{\chi_T \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.20 \cdot 9200 \cdot 275}{1.00} = 503.22 \text{ kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,T}} = \frac{500.00}{503.22} = 0.99 \le 1.00 \text{ - OK}$$

Figure 9.3.4.5 – Detailed results based on FEM-Design

9.3.5 Lateral torsional buckling of a doubly symmetric I section

In this sub-chapter we will calculate the lateral torsional buckling resistance of an IPE 240 beam under concentrated load. We will calculate according to EN1993-1-1:6.3.2.2 (general case) and EN1993-1-1:6.3.2.4 (simplified assessment). The section is in Class 1 due to pure bending.

Yield strength of structural steel	f_y = 355 MPa
Poisson's ratio	ν = 0.3
Young's modulus	E = 210 GPa
Shear modulus	G = E / (2(1+ν)) = 80.77 GPa
Distance between lateral restraints	L = 6.0 m
Height of the cross-section	h = 240 mm
Inertia around weak axis	I_z = 2836341 mm^4
St. Venant torsional constant	I_t = 127368 mm^4
Warping constant	I_ω = 36680292708 mm^6
Plastic cross-sectional modulus around strong axis	$W_{pl,y}$ = 366645 mm^3

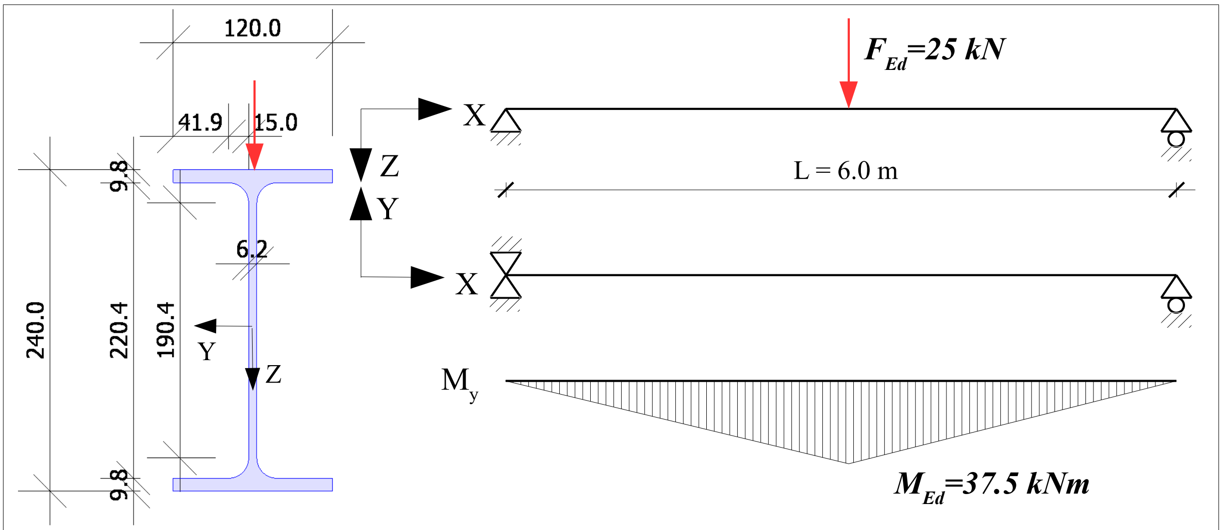


Figure 9.3.5.1 – The statical system and the cross-section

General case EN1993-1-1:6.3.2.2

$k=1.0\,;k_\omega=1.0$ free to rotate about z axis and restraint against movements and free to warp but restraint against rotation about the longitudinal axis.

The C_i coefficients are depending on the loading and end restraint conditions:

$$C_1=1.35\,;C_2=0.63\,;C_3=1.73$$

Because the cross section is doubly symmetric: $z_j=0$.

The distance between the point of load application and the shear centre (this value has a sign):

$z_g = \frac{h}{2} = \frac{240}{2} = 120\,\text{mm}$ see Fig. 9.3.5.1.

The calculation of the elastic critical moment:

$$M_{cr} = C_1 \frac{\pi^2 E I_z}{(k L)^2} \left(\sqrt{\left(\frac{k}{k_\omega}\right)^2 \frac{I_\omega}{I_z} + \frac{(k L)^2 G I_t}{\pi^2 E I_z} + (C_2 z_g - C_3 z_j)^2} - (C_2 z_g - C_3 z_j) \right)$$

$$Z = C_2 z_g - C_3 z_j = 0.63 \cdot 120 - 1.73 \cdot 0 = 75.6\,\text{mm}$$

$$M_{cr} = 1.35 \frac{\pi^2 210000 \cdot 2836341}{(1 \cdot 6000)^2} \left(\sqrt{\left(\frac{1}{1}\right)^2 \frac{3.668 \cdot 10^{10}}{2836341} + \frac{(1 \cdot 6000)^2 80769 \cdot 127368}{\pi^2 210000 \cdot 2836341} + 75.6^2} - 75.6 \right)$$

$$M_{cr} = 46.32\,\text{kNm}$$

Non-dimensional slenderness: $\bar{\lambda}_{LT} = \sqrt{\frac{W_{pl,y} f_y}{M_{cr}}} = \sqrt{\frac{366645 \cdot 355}{46320000}} = 1.676$

$$\Phi_{LT} = \frac{1 + \alpha_{LT}(\bar{\lambda}_{LT} - 0.2) + \bar{\lambda}_{LT}^2}{2} = \frac{1 + 0.21(1.676 - 0.2) + 1.676^2}{2} = 2.059$$

where the imperfection α_{LT} factor value based on EN 1993-1-1 Table 6.4:

because $\frac{h}{b} = \frac{240}{120} = 2 \leq 2$ rolled section therefore “a” buckling curve is relevant thus the imperfection factor is $\alpha_{LT} = 0.21$.

Reduction factor for lateral torsional buckling:

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \bar{\lambda}_{LT}^2}} = \frac{1}{2.059 + \sqrt{2.059^2 - 1.676^2}} = 0.3072$$

The lateral torsional buckling resistance:

$$M_{b,Rd} = \chi_{LT} W_{pl,y} \frac{f_y}{\gamma_{M1}} = 0.3072 \frac{366645 \cdot 355}{1.0} = 39.98\,\text{kNm}$$

Simplified assessment EN1993-1-1:6.3.2.4

In this method we check the buckling of an equivalent T section, where the compressed flange is the same as in the original section, the web height is third of the compressed web height.

Note: We simplify the equivalent section, we are not considering roundings.

The height of the section: 220.4/2/3+9.8 = 46.5 mm (see Fig. 9.3.5.1-2).

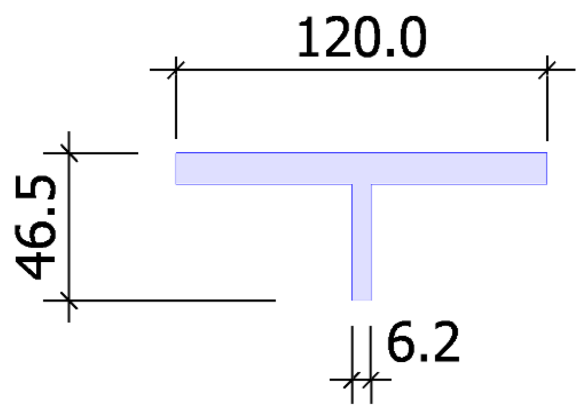


Figure 9.3.5.2 – The equivalent T section

The area and the inertia about the original weak axis of the cross section:

$$A_f = 9.8 \cdot 120 + 6.2 \cdot (46.5 - 9.8) = 1404\,\text{mm}^2$$

$$I_{f,z} = 9.8 \cdot 120^3 / 12 + (46.5 - 9.8) 6.2^3 / 12 = 1412000\,\text{mm}^4$$

The relevant radius of gyration of the equivalent T section:

$$i_{f,z} = \sqrt{\frac{I_{f,z}}{A_f}} = \sqrt{\frac{1412000}{1403}} = 31.72\,\text{mm}$$

Non-dimensional slenderness:

$$\bar{\lambda}_f = \frac{k_c L}{i_{fz} \lambda_1} = \frac{0.86 \cdot 6000}{31.72 \cdot 76.41} = 2.129$$

where $\lambda_1 = \pi \sqrt{\frac{E}{f_y}} = \pi \sqrt{\frac{210000}{355}} = 76.41$ and k_c depends on the moment distribution.

$$\Phi_f = \frac{1 + \alpha_f (\bar{\lambda}_f - 0.2) + \bar{\lambda}_f^2}{2} = \frac{1 + 0.49(2.129 - 0.2) + 2.129^2}{2} = 3.239$$

where the imperfection factor comes from buckling curve “c”: $\alpha_f = 0.49$

Reduction factor for lateral torsional buckling:

$$\chi_{LT}=\frac{1}{\Phi_f+\sqrt{\Phi_f^2-\bar{\lambda}_f^2}}=\frac{1}{3.239+\sqrt{3.239^2-2.129^2}}=0.1760$$

The lateral torsional buckling resistance:

$$M_{b,Rd}=k_{fl}\chi_{LT}W_{pl,y}\frac{f_y}{\gamma_{M1}}=1.1\cdot 0.1760\cdot 366645\frac{355}{1.0}=25.20\,\text{kNm}$$

Fig. 9.3.5.3 shows the FEM-Design statical system and the bending moment diagram.

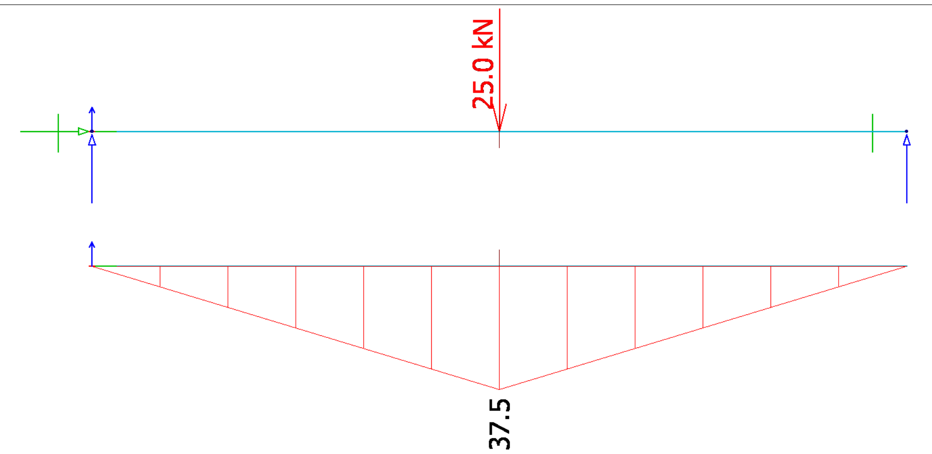


Figure 9.3.5.3 – The statical system and the bending moment diagram [kNm] *based on FEM-Design*

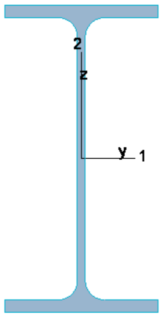
The differences between the hand and FEM-Design calculations are less than 2%.

See Fig. 9.3.5.4 about the detailed results of FEM-Design.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.3.5 Lateral torsional buckling of a doubly symmetric I section.str

IPE 240



P	=	922 mm
A	=	3912 mm^2
I_y	=	3.892e+07 mm^4
I_z	=	2.836e+06 mm^4
I_1	=	3.892e+07 mm^4
I_2	=	2.836e+06 mm^4
$W_{pl,1}$	=	3.666e+05 mm^3
$W_{pl,2}$	=	7.396e+04 mm^3
$W_{el,min,1}$	=	3.243e+05 mm^3
$W_{el,min,2}$	=	4.727e+04 mm^3
i_1	=	100 mm
i_2	=	27 mm
I_t	=	1.274e+05 mm^4
I_w	=	3.668e+10 mm^6

S 355

E = 210000 N/mm^2

G = 80769 N/mm^2

$\gamma_{M0,ult}$	= 1.00		$\gamma_{M0,acc/seis}$	=	1.00
$\gamma_{M1,ult}$	= 1.00		$\gamma_{M1,acc/seis}$	=	1.00
$\gamma_{M2,ult}$	= 1.25		$\gamma_{M2,acc/seis}$	=	1.00

f_y = 355 N/mm^2

ε = 0.81

λ_1 = 76.40

Lateral torsional buckling - Part 1-1: 6.3.2.2

LC: '1', x = 3000 mm

$Class_N = 2,\quad Class_{M1} = 1,\quad Class_{M2} = 1$

$$N_{cr,LT} = \frac{\pi^2 \cdot E \cdot I_z}{(k_z \cdot L_{cr})^2} = \frac{\pi^2 \cdot 2.100e{+}05 \cdot 2.836e{+}06}{(1.00 \cdot 6000)^2} = 163.30\ \text{kN}$$

Loaded on top edge.

$$Z = (C_2 \cdot z_g - C_3 \cdot z_j) = (0.63 \cdot 120 - 1.73 \cdot 0) = 75.60\ \text{mm}$$

$$M_{cr} = C_1 \cdot N_{cr,LT} \cdot \left\{\left[\left(\frac{k_z}{k_w}\right)^2 \cdot \frac{I_w}{I_z} + \frac{G \cdot I_t}{N_{cr,LT}} + Z^2\right]^{0.5} - Z\right\} =$$

$$= 1.35 \cdot 1.633e{+}05 \cdot \left\{\left[\left(\frac{1.00}{1.00}\right)^2 \cdot \frac{3.668e{+}10}{2.836e{+}06} + \frac{8.077e{+}04 \cdot 1.274e{+}05}{1.633e{+}05} + 75.60^2\right]^{0.5} - 75.60\right\} =$$

$$= 46.32\ \text{kNm}$$

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_y \cdot f_y}{M_{cr}}} = \sqrt{\frac{366645 \cdot 355}{4.632e{+}07}} = 1.68$$

$\alpha_{LT} = 0.21$ (Buckling curve: a)

$$\varphi_{LT} = 0.5\left[1 + \alpha_{LT} \cdot (\bar{\lambda}_{LT} - 0.2) + \bar{\lambda}_{LT}^{\,2}\right] =$$

$$= 0.5\left[1 + 0.21 \cdot (1.68 - 0.2) + 1.68^2\right] = 2.06$$

$$\chi_{LT} = \min\left(\frac{1}{\varphi_{LT} + \sqrt{\varphi_{LT}^{\,2} - \bar{\lambda}_{LT}^{\,2}}},\ 1.0\right) = \min\left(\frac{1}{2.06 + \sqrt{2.06^2 - 1.68^2}},\ 1.0\right) = 0.31 \qquad (6.56)$$

$$M_{y,b,Rd} = \frac{\chi_{LT} \cdot W_y \cdot f_y}{\gamma_{M1}} = \frac{0.31 \cdot 366645 \cdot 355}{1.00} = 39.96\ \text{kNm} \qquad (6.55)$$

$$\frac{M_{1,Ed}}{M_{y,b,Rd}} = \frac{37.50}{39.96} = 0.94 \le 1.00 \qquad (6.54)\ \text{- OK}$$

Lateral torsional buckling - Part 1-1: 6.3.2.4

LC: '1', x = 3000 mm

$Class_N = 2,\quad Class_{M1} = 1,\quad Class_{M2} = 1$

$$\bar{\lambda}_{f,y} = \frac{k_c \cdot L_{cr}}{i_{f,z} \cdot \lambda_1} = \frac{0.86 \cdot 6000}{32 \cdot 76.40} = 2.11 \qquad (6.59)$$

$\alpha_y = 0.49$ (Buckling curve: c)

$$\varphi_y = 0.5\left[1 + \alpha_y \cdot (\bar{\lambda}_{f,y} - 0.2) + \bar{\lambda}_{f,y}^{\,2}\right] =$$

$$= 0.5\left[1 + 0.49 \cdot (2.11 - 0.2) + 2.11^2\right] = 3.19$$

$$\chi_y = \min\left(\frac{1}{\varphi_y + \sqrt{\varphi_y^{\,2} - \bar{\lambda}_{f,y}^{\,2}}},\ 1.0\right) = \min\left(\frac{1}{3.19 + \sqrt{3.19^2 - 2.11^2}},\ 1.0\right) = 0.18 \qquad (6.49)$$

$$M_{y,c,Rd} = W_y \frac{f_y}{\gamma_{M1}} = 366645 \frac{355}{1.00} = 130.16\ \text{kNm}$$

$$M_{y,b,Rd} = \min(k_{fl} \cdot \chi_y \cdot M_{y,c,Rd},\ M_{y,c,Rd}) = \min(1.10 \cdot 0.18 \cdot 130.16,\ 130.16) = 25.68\ \text{kNm} \qquad (6.60)$$

$$\frac{M_{1,Ed}}{M_{y,b,Rd}} = \frac{37.50}{25.68} = 1.46 > 1.00 \qquad (6.54)\ \text{- Not OK}$$

Figure 9.3.5.4 – The detailed results about the lateral torsional buckling based on FEM-Design

9.3.6 Interaction of biaxial bending and axial compression in an RHS section

In this sub-chapter the stability interaction of an RHS section (KKR 200x100x10, cold formed hollow section) will be investigated. The statical system and the design load values are indicated in Fig. 9.3.6.1, furthermore the other general input data are in the table below.

Yield strength of structural steel	f_y = 355 MPa
Poisson's ratio	ν = 0.3
Young's modulus	E = 210 GPa
Shear modulus	G = E / (2(1+ν)) = 80.77 GPa
Span length	L = 6.0 m
Cross-sectional height	h = 200 mm
Cross-sectional width	b = 100 mm
Cross-sectional thickness	t = 10 mm
Cross-sectional area	A = 5257 mm^2
Inertia around strong axis	$I_{y'}$ = 24443956 mm^4
Inertia around weak axis	$I_{z'}$ = 8177434 mm^4
St. Venant torsional constant	I_t = 21571134 mm^4
Warping constant	I_ω = 4313360830 mm^6
Elastic cross-sectional modulus around strong axis	$W_{el,y'}$ = 244440 mm^3
Elastic cross-sectional modulus around weak axis	$W_{el,z'}$ = 163549 mm^3
Plastic cross-sectional modulus around strong axis	$W_{pl,y'}$ = 318082 mm^3
Plastic cross-sectional modulus around weak axis	$W_{pl,z'}$ = 195250 mm^3

According to the EN1993-1-1 the section is in Class 1 due to pure bending in both directions and also under normal force. The calculation will be performed with EN1993-1-1 Annex A – Method 1 and with Annex B – Method 2 regarding to get k_{ij} interaction factors.

The characteristic normal force resistance:

$$N_{Rk}=A\,f_y=5257\cdot 355=1866\,\text{kN}$$

The characteristic bending moment resistance around strong axis:

$$M_{y',Rk}=W_{pl,y'}\,f_y=318082\cdot 355=112.9\,\text{kNm}$$

The characteristic bending moment resistance around weak axis:

$$M_{z',Rk}=W_{pl,z'}\,f_y=195250\cdot 355=69.31\,\text{kNm}$$

Flexural buckling around strong axis

The radius of gyration (strong axis): $i_y = \sqrt{\frac{I_{y'}}{A}} = \sqrt{\frac{24443956}{5257}} = 68.19\,\text{mm}$

The non-dimensional slenderness: $\bar{\lambda}_y = \frac{L_{cr}}{i_y}\frac{1}{\lambda_1} = \frac{6000}{68.19}\frac{1}{76.41} = 1.152$, where

$$\lambda_1 = \pi\sqrt{\frac{E_s}{f_y}} = \pi\sqrt{\frac{210000}{355}} = 76.41 \quad .$$

The imperfection α factor value based on EN 1993-1-1 Table 6.2:

cold formed hollow section therefore "c" buckling curve is relevant thus the imperfection factor is $\alpha_y = 0.49$.

$$\Phi_y = 0.5\left[1+\alpha_y(\bar{\lambda}_y - 0.2)+\bar{\lambda}_y^2\right] = 0.5\left[1+0.49(1.152-0.2)+1.152^2\right] = 1.386$$

Reduction factor: $\chi_y = \frac{1}{\Phi_y + \sqrt{\Phi_y^2 - \bar{\lambda}_y^2}} = \frac{1}{1.386+\sqrt{1.386^2 - 1.152^2}} = 0.4637$

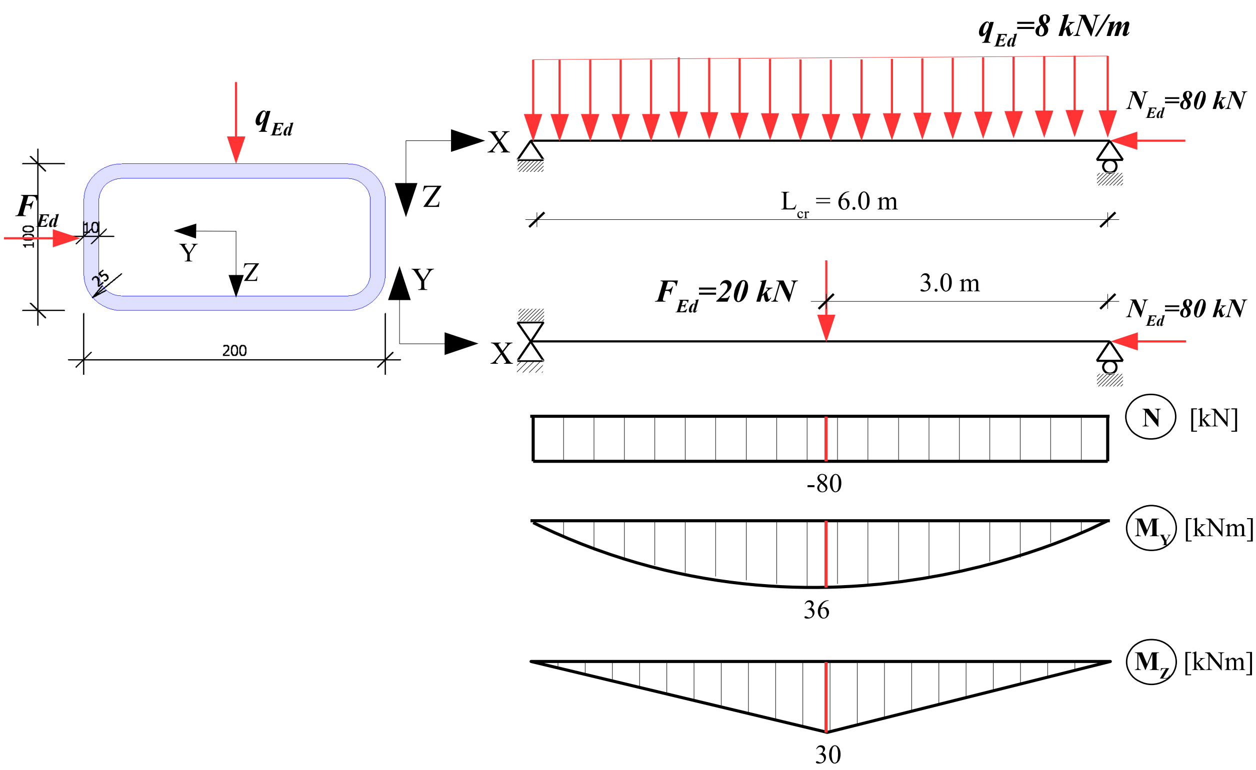


Figure 9.3.6.1 – The statical system, the cross-section and the design loads and internal forces in the global system

Flexural buckling around weak axis

The radius of gyration (weak axis): $i_z = \sqrt{\frac{I_{z'}}{A}} = \sqrt{\frac{8177434}{5257}} = 39.44\,\text{mm}$

The non-dimensional slenderness: $\bar{\lambda}_z = \frac{L_{cr}}{i_z}\frac{1}{\lambda_1} = \frac{6000}{39.44}\frac{1}{76.41} = 1.991$, where

$\lambda_1 = \pi\sqrt{\frac{E_s}{f_y}} = \pi\sqrt{\frac{210000}{355}} = 76.41$.

The imperfection α factor value based on EN 1993-1-1 Table 6.2:

cold formed hollow section therefore "c" buckling curve is relevant thus the imperfection factor is $\alpha_z = 0.49$.

$$\Phi_z = 0.5\left(1+\alpha_z\left(\bar{\lambda}_z - 0.2\right)+\bar{\lambda}_z^{\,2}\right) = 0.5\left(1+0.49\left(1.991-0.2\right)+1.991^2\right) = 2.921$$

Reduction factor: $\chi_z = \frac{1}{\Phi_z + \sqrt{\Phi_z^2 - \bar{\lambda}_z^{\,2}}} = \frac{1}{2.921+\sqrt{2.921^2 - 1.991^2}} = 0.1977$

Lateral torisonal buckling general case according to EN1993-1-1:6.3.2.2

The major axis bending (around strong axis) is relevant according to EN1993-1-1:6.3.2.1

$k = 1.0\,;\,k_\omega = 1.0$ free to rotate about weak axis and restraint against movements and free to warp but restraint against rotation about the longitudinal axis.

The C_i coefficients are depending on the loading and end restraint conditions:

$C_1 = 1.35\,;\,C_2 = 0.63\,;\,C_3 = 1.73$

Because the cross section is doubly symmetric: $z_j = 0$.

The distance between the point of load application and the shear centre in the relevant direction (this value has a sign): $z_g = \frac{h}{2} = \frac{200}{2} = 100\,\text{mm}$ see Fig. 9.3.6.1.

The calculation of the elastic critical moment:

$$M_{cr} = C_1\frac{\pi^2 E I_z}{(k L)^2}\left(\sqrt{\left(\frac{k}{k_\omega}\right)^2\frac{I_\omega}{I_z}+\frac{(k L)^2 G I_t}{\pi^2 E I_z}+(C_2 z_g - C_3 z_j)^2} - (C_2 z_g - C_3 z_j)\right)$$

$$Z = C_2 z_g - C_3 z_j = 0.63\cdot 100 - 1.73\cdot 0 = 63.0\,\text{mm}$$

$$M_{cr} = 1.35\frac{\pi^2 210000\cdot 8177434}{(1\cdot 6000)^2}\left(\sqrt{\left(\frac{1}{1}\right)^2\frac{4.313\cdot 10^9}{8177434}+\frac{(1\cdot 6000)^2 80769\cdot 21571134}{\pi^2 210000\cdot 8177434}+63.0^2} - 63.0\right)$$

$$M_{cr}=1183\,\mathrm{kNm}$$

Non-dimensional slenderness: $\bar{\lambda}_{LT}=\sqrt{\frac{W_{pl,y'}f_y}{M_{cr}}}=\sqrt{\frac{318082\cdot 355}{1183000000}}=0.3090$

$$\Phi_{LT}=\frac{1+\alpha_{LT}(\bar{\lambda}_{LT}-0.2)+\bar{\lambda}_{LT}^2}{2}=\frac{1+0.76(0.3090-0.2)+0.3090^2}{2}=0.5892$$

where the imperfection α_{LT} factor value based on EN 1993-1-1 Table 6.4 thus “d” buckling curve is relevant. The imperfection factor is $\alpha_{LT}=0.76$.

Reduction factor for lateral torsional buckling:

$$\chi_{LT}=\frac{1}{\Phi_{LT}+\sqrt{\Phi_{LT}^2-\bar{\lambda}_{LT}^2}}=\frac{1}{0.5892+\sqrt{0.5892^2-0.3090^2}}=0.9167$$

<u>Interaction factors according to EN1993-1-1 Annex A – Method 1</u>

Auxiliary terms:

$$\bar{\lambda}_{max}=\max\begin{bmatrix}\bar{\lambda}_y\\ \bar{\lambda}_z\end{bmatrix}=\max\begin{bmatrix}1.152\\ 1.991\end{bmatrix}=1.991$$

Calculation of the non-dimensional slenderness due to uniform bending moment:

The calculation of the elastic critical moment due to uniform bending moment:

$$M_{cr}=\frac{\pi^2 E I_z}{L^2}\sqrt{\frac{I_\omega}{I_z}+\frac{L^2 G I_t}{\pi^2 E I_z}}=\frac{\pi^2 210000\cdot 8177434}{6000^2}\sqrt{\frac{4.313\cdot 10^9}{8177434}+\frac{6000^2\cdot 80769\cdot 21571134}{\pi^2 210000\cdot 8177434}}$$

$$M_{cr}=905.7\,\mathrm{kNm}$$

Non-dimensional slenderness: $\bar{\lambda}_0=\sqrt{\frac{W_{pl,y'}f_y}{M_{cr}}}=\sqrt{\frac{318082\cdot 355}{905700000}}=0.3531$

The elastic flexural buckling forces:

$$N_{cr,y}=\frac{\pi^2 EI_{y'}}{L^2}=\frac{\pi^2 210000\cdot 24443956}{6000^2}=1407\,\mathrm{kN}$$

$$N_{cr,z}=\frac{\pi^2 EI_{z'}}{L^2}=\frac{\pi^2 210000\cdot 8177434}{6000^2}=470.8\,\mathrm{kN}$$

Because the section is doubly symmetric the elastic torsional-flexural buckling force:

$$N_{cr,TF}=N_{cr,T}$$

The elastic critical torsional force:

$$N_{cr,T}=\frac{1}{i_y^2+i_z^2}\left(GI_t+\frac{\pi^2 EI_\omega}{L_{cr}^2}\right)=\frac{1}{68.19^2+39.44^2}\left(80769\cdot 21571134+\frac{\pi^2\cdot 210000\cdot 4.313\cdot 10^9}{6000^2}\right)$$

$$N_{cr,TF}=N_{cr,T}=280800\ \mathrm{kN}$$

Because:

$$\bar{\lambda}_0=0.3531>0.2\sqrt{C_1}\sqrt[4]{\left(1-\frac{N_{Ed}}{N_{cr,z}}\right)\left(1-\frac{N_{Ed}}{N_{cr,TF}}\right)}=0.2\sqrt{1.35}\sqrt[4]{\left(1-\frac{80}{470.8}\right)\left(1-\frac{80}{280800}\right)}=0.2218$$

Thus:

$$C_{my,0}=1-0.18\frac{N_{Ed}}{N_{cr,y}}=1-0.18\frac{80}{1407}=0.9898$$

$$\mathrm{a}_{\mathrm{LT}}=1-\frac{I_t}{I_y}=1-\frac{21571134}{24443956}=0.1175$$

$$\varepsilon_y=\frac{M_{y,Ed}}{N_{Ed}}\frac{A}{W_{el,y}}=\frac{30\cdot 10^6}{80\cdot 10^3}\frac{5257}{244440}=8.065$$

$$C_{my}=C_{my,0}+(1-C_{my,0})\frac{\sqrt{\varepsilon_y}\mathrm{a}_{\mathrm{LT}}}{1+\sqrt{\varepsilon_y}\mathrm{a}_{\mathrm{LT}}}=0.9898+(1-0.9898)\frac{\sqrt{8.065}\cdot 0.1175}{1+\sqrt{8.065}\cdot 0.1175}=0.9924$$

$$C_{mz}=C_{mz,0}=1+0.03\frac{N_{Ed}}{N_{cr,z}}=1+0.03\frac{80}{470.8}=1.005$$

$$C_{mLT}=\max\begin{bmatrix}C_{my}^2\frac{\mathrm{a}_{\mathrm{LT}}}{\sqrt{\left(1-\frac{N_{Ed}}{N_{cr,z}}\right)\left(1-\frac{N_{Ed}}{N_{cr,T}}\right)}}\\1\end{bmatrix}=\max\begin{bmatrix}0.9924^2\frac{0.1175}{\sqrt{\left(1-\frac{80}{470.8}\right)\left(1-\frac{80}{280800}\right)}}\\1\end{bmatrix}$$

$$C_{mLT}=\max\begin{bmatrix}0.1270\\1\end{bmatrix}=1$$

$$\mathrm{b}_{\mathrm{LT}}=0.5\,\mathrm{a}_{\mathrm{LT}}\bar{\lambda}_0^2\frac{M_{y,Ed}}{\chi_{LT}M_{pl,y,Rd}}\frac{M_{z,Ed}}{M_{pl,z,Rd}}=0.5\cdot 0.1175\cdot 0.3531^2\frac{30}{0.9167\cdot 112.9}\frac{36}{69.31}=0.001103$$

$$\mathrm{c}_{\mathrm{LT}}=10\,\mathrm{a}_{\mathrm{LT}}\frac{\bar{\lambda}_0^2}{5+\bar{\lambda}_z^4}\frac{M_{y,Ed}}{C_{my}\chi_{LT}M_{pl,y,Rd}}=10\cdot 0.1175\frac{0.3531^2}{5+1.991^4}\frac{30}{0.9924\cdot 0.9167\cdot 112.9}=0.002066$$

$$\mathrm{d}_{\mathrm{LT}}=2\,\mathrm{a}_{\mathrm{LT}}\frac{\bar{\lambda}_0}{0.1+\bar{\lambda}_z^4}\frac{M_{y,Ed}}{C_{my}\chi_{LT}M_{pl,y,Rd}}\frac{M_{z,Ed}}{C_{mz}M_{pl,z,Rd}}$$

$$d_{LT}=2\cdot 0.1175\frac{0.3531}{0.1+1.991^4}\frac{30}{0.9924\cdot 0.9167\cdot 112.9}\frac{36}{1.005\cdot 69.31}=0.0007921$$

$$e_{LT}=1.7a_{LT}\frac{\bar{\lambda}_0}{0.1+\bar{\lambda}_z^{\,4}}\frac{M_{y,Ed}}{C_{my}\chi_{LT}M_{pl,y,Rd}}=\frac{1.7\cdot 0.1175\cdot 0.3531}{0.1+1.991^4}\frac{30}{0.9924\cdot 0.9167\cdot 112.9}=0.001303$$

$$w_y=\min\begin{bmatrix}\frac{W_{pl,y'}}{W_{el,y'}}\\1.5\end{bmatrix}=\min\begin{bmatrix}\frac{318082}{244440}\\1.5\end{bmatrix}=\min\begin{bmatrix}1.301\\1.5\end{bmatrix}=1.301$$

$$w_z=\min\begin{bmatrix}\frac{W_{pl,z'}}{W_{el,z'}}\\1.5\end{bmatrix}=\min\begin{bmatrix}\frac{195250}{163549}\\1.5\end{bmatrix}=\min\begin{bmatrix}1.194\\1.5\end{bmatrix}=1.194$$

$$n_{pl}=\frac{N_{Ed}}{N_{Rk}/\gamma_{M1}}=\frac{80}{1866/1.0}=0.04287$$

$$C_{yy}=\max\begin{bmatrix}1+(w_y-1)\left[\left(2-\frac{1.6}{w_y}C_{my}^2\bar{\lambda}_{max}-\frac{1.6}{w_y}C_{my}^2\bar{\lambda}_{max}^{\,2}\right)n_{pl}-b_{LT}\right]\\\frac{W_{el,y'}}{W_{pl,y'}}\end{bmatrix}$$

$$C_{yy}=\max\begin{bmatrix}1+(1.301-1)\left[\left(2-\frac{1.6}{1.301}0.9924^2 1.991-\frac{1.6}{1.301}0.9924^2 1.991^2\right)0.04287-0.001103\right]\\\frac{244440}{318082}\end{bmatrix}$$

$$C_{yy}=\max\begin{bmatrix}0.9324\\0.7685\end{bmatrix}=0.9324$$

$$C_{yz}=\max\begin{bmatrix}1+(w_z-1)\left[\left(2-14\frac{C_{mz}^{\,2}\bar{\lambda}_{max}^{\,2}}{w_z^5}\right)n_{pl}-c_{LT}\right]\\0.6\sqrt{\frac{w_z}{w_y}}\frac{W_{el,z'}}{W_{pl,z'}}\end{bmatrix}$$

$$C_{yz}=\max\begin{bmatrix}1+(1.194-1)\left[\left(2-14\frac{1.005^2\cdot 1.991^2}{1.194^5}\right)0.04287-0.002066\right]\\0.6\sqrt{\frac{1.194}{1.301}}\frac{163549}{195250}\end{bmatrix}=\max\begin{bmatrix}0.8241\\0.4815\end{bmatrix}=0.8241$$

$$C_{zy}=\max\begin{bmatrix}1+(w_y-1)\left[\left(2-14\frac{C_{my}^{2}\bar{\lambda}_{max}^{2}}{w_y^5}\right)n_{pl}-d_{LT}\right]\\0.6\sqrt{\frac{w_y}{w_z}}\frac{W_{el,y'}}{W_{pl,y'}}\end{bmatrix}$$

$$C_{zy}=\max\begin{bmatrix}1+(1.301-1)\left[\left(2-14\frac{0.9924^2\cdot 1.991^2}{1.301^5}\right)0.04287-0.0007921\right]\\0.6\sqrt{\frac{1.301}{1.194}}\frac{244440}{318082}\end{bmatrix}$$

$$C_{zy}=\max\begin{bmatrix}0.8363\\0.4813\end{bmatrix}=0.8363$$

$$C_{zz}=\max\begin{bmatrix}1+(w_z-1)\left[\left(2-\frac{1.6}{w_z}C_{mz}^2\bar{\lambda}_{max}-\frac{1.6}{w_z}C_{mz}^2\bar{\lambda}_{max}^{2}\right)n_{pl}-e_{LT}\right]\\\frac{W_{el,z}}{W_{pl,z}}\end{bmatrix}$$

$$C_{zz}=\max\begin{bmatrix}1+(1.194-1)\left[\left(2-\frac{1.6}{1.194}1.005^2 1.991-\frac{1.6}{1.194}1.005^2 1.991^2\right)0.04287-0.001303\right]\\\frac{163549}{195250}\end{bmatrix}$$

$$C_{zz}=\max\begin{bmatrix}0.9493\\0.8376\end{bmatrix}=0.9493$$

$$\mu_y=\frac{1-\frac{N_{Ed}}{N_{cr,y}}}{1-\chi_y\frac{N_{Ed}}{N_{cr,y}}}=\frac{1-\frac{80}{1407}}{1-0.4637\frac{80}{1407}}=0.9687$$

$$\mu_z=\frac{1-\frac{N_{Ed}}{N_{cr,z}}}{1-\chi_z\frac{N_{Ed}}{N_{cr,z}}}=\frac{1-\frac{80}{470.8}}{1-0.1977\frac{80}{470.8}}=0.8589$$

And finally the interaction factors based on the auxiliary terms:

$$k_{yy}=C_{my}C_{mLT}\frac{\mu_y}{1-\frac{N_{Ed}}{N_{cr,y}}}\frac{1}{C_{yy}}=0.9924\cdot 1\frac{0.9687}{1-\frac{80}{1407}}\frac{1}{0.9324}=1.093$$

$$k_{yz}=C_{mz}\frac{\mu_y}{1-\frac{N_{Ed}}{N_{cr,z}}}\frac{1}{C_{yz}}0.6\sqrt{\frac{w_z}{w_y}}=1.005\frac{0.9687}{1-\frac{80}{470.8}}\frac{1}{0.8241}0.6\sqrt{\frac{1.194}{1.301}}=0.8180$$

$$k_{zy}=C_{my}C_{mLT}\frac{\mu_z}{1-\frac{N_{Ed}}{N_{cr,y}}}\frac{1}{C_{zy}}0.6\sqrt{\frac{w_y}{w_z}}=0.9924\cdot 1\frac{0.8589}{1-\frac{80}{1407}}\frac{1}{0.8363}0.6\sqrt{\frac{1.301}{1.194}}=0.6768$$

$$k_{zz}=C_{mz}\frac{\mu_z}{1-\frac{N_{Ed}}{N_{cr,z}}}\frac{1}{C_{zz}}=1.005\frac{0.8589}{1-\frac{80}{470.8}}\frac{1}{0.9493}=1.095$$

The interaction formulas:

$$\frac{N_{Ed}}{\frac{\chi_y N_{Rk}}{\gamma_{M1}}}+k_{yy}\frac{M_{y,Ed}}{\frac{\chi_{LT}M_{y,Rk}}{\gamma_{M1}}}+k_{yz}\frac{M_{z,Ed}}{\frac{M_{z,Rk}}{\gamma_{M1}}}=\frac{80}{\frac{0.4637\cdot 1866}{1.0}}+\frac{1.093\cdot 30}{\frac{0.9167\cdot 112.9}{1.0}}+\frac{0.8180\cdot 36}{\frac{69.31}{1.0}}=0.8342$$

$$\frac{N_{Ed}}{\frac{\chi_z N_{Rk}}{\gamma_{M1}}}+k_{zy}\frac{M_{y,Ed}}{\frac{\chi_{LT}M_{y,Rk}}{\gamma_{M1}}}+k_{zz}\frac{M_{z,Ed}}{\frac{M_{z,Rk}}{\gamma_{M1}}}=\frac{80}{\frac{0.1977\cdot 1866}{1.0}}+\frac{0.6768\cdot 30}{\frac{0.9167\cdot 112.9}{1.0}}+\frac{1.095\cdot 36}{\frac{69.31}{1.0}}=0.9818$$

The hand calculation and FEM-Design calculation are almost identical to each other considering Annex A (Method 1). The difference is less than 0.5%. See Fig. 9.3.6.2 about the detailed FEM-Design results.

Interaction factors according to EN1993-1-1 Annex B – Method 2

Due to the concentrated load in the weak direction (bending around major axis) EN 1993-1-1 Table B.3:

$$C_{my}=0.9$$

Due to the uniform load in the strong direction (bending around minor axis) EN 1993-1-1 Table B.3:

$$C_{mz}=0.95$$

The interaction factors according to EN 1993-1-1 Table B.1:

$$k_{yy}=\min\begin{bmatrix} C_{my}\left(1+(\bar{\lambda}_y-0.2)\dfrac{N_{Ed}}{\chi_y\dfrac{N_{Rk}}{\gamma_{M1}}}\right) \\ C_{my}\left(1+0.8\dfrac{N_{Ed}}{\chi_y\dfrac{N_{Rk}}{\gamma_{M1}}}\right)\end{bmatrix}=\min\begin{bmatrix} 0.9\left(1+(1.152-0.2)\dfrac{80}{0.4637\dfrac{1866}{1.0}}\right) \\ 0.9\left(1+0.8\dfrac{80}{0.4637\dfrac{1866}{1.0}}\right)\end{bmatrix}$$

$$k_{yy}=\min\begin{bmatrix}0.9792\\0.9666\end{bmatrix}=0.9666$$

$$k_{zz}=\min\begin{bmatrix} C_{mz}\left(1+(\bar{\lambda}_z-0.2)\dfrac{N_{Ed}}{\chi_z\dfrac{N_{Rk}}{\gamma_{M1}}}\right) \\ C_{mz}\left(1+0.8\dfrac{N_{Ed}}{\chi_z\dfrac{N_{Rk}}{\gamma_{M1}}}\right)\end{bmatrix}=\min\begin{bmatrix} 0.95\left(1+(1.991-0.2)\dfrac{80}{0.1977\dfrac{1866}{1.0}}\right) \\ 0.95\left(1+0.8\dfrac{80}{0.1977\dfrac{1866}{1.0}}\right)\end{bmatrix}$$

$$k_{zz}=\min\begin{bmatrix}1.319\\1.115\end{bmatrix}=1.115$$

$$k_{yz}=0.6\,k_{zz}=0.6\cdot 1.115=0.669 \quad ; \quad k_{zy}=0.6\,k_{yy}=0.6\cdot 0.9666=0.580$$

The interaction formulas:

$$\frac{N_{Ed}}{\dfrac{\chi_y N_{Rk}}{\gamma_{M1}}}+k_{yy}\frac{M_{y,Ed}}{\dfrac{\chi_{LT} M_{y,Rk}}{\gamma_{M1}}}+k_{yz}\frac{M_{z,Ed}}{\dfrac{M_{z,Rk}}{\gamma_{M1}}}=\frac{80}{\dfrac{0.4637\cdot 1866}{1.0}}+\frac{0.9666\cdot 30}{\dfrac{0.9167\cdot 112.9}{1.0}}+\frac{0.669\cdot 36}{\dfrac{69.31}{1.0}}=0.7201$$

$$\frac{N_{Ed}}{\dfrac{\chi_z N_{Rk}}{\gamma_{M1}}}+k_{zy}\frac{M_{y,Ed}}{\dfrac{\chi_{LT} M_{y,Rk}}{\gamma_{M1}}}+k_{zz}\frac{M_{z,Ed}}{\dfrac{M_{z,Rk}}{\gamma_{M1}}}=\frac{80}{\dfrac{0.1977\cdot 1866}{1.0}}+\frac{0.580\cdot 30}{\dfrac{0.9167\cdot 112.9}{1.0}}+\frac{1.115\cdot 36}{\dfrac{69.31}{1.0}}=0.9641$$

The hand calculation and FEM-Design calculation are identical to each other considering Annex B (Method 2). See Fig. 9.3.6.3 about the detailed FEM-Design results.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.3.6 Interaction of biaxial bending and axial compression in an RHS section.str

S 355

$E = 210000\ \mathrm{N/mm^2}$ $\quad \gamma_{M0,acc/seis} = 1.00$

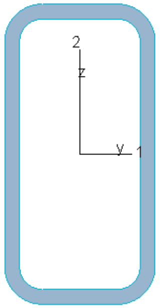
$G = 80769\ \mathrm{N/mm^2}$ $\quad \gamma_{M1,acc/seis} = 1.00$

$\gamma_{M0,ult} = 1.00$ $\quad \gamma_{M2,acc/seis} = 1.00$

$\gamma_{M1,ult} = 1.00$

$\gamma_{M2,ult} = 1.25$

KKR 200x100x10



P	=	557 mm
A	=	5257 mm^2
I_y	=	2.444e+07 mm^4
I_z	=	8.177e+06 mm^4
I_1	=	2.444e+07 mm^4
I_2	=	8.177e+06 mm^4
$W_{pl,1}$	=	3.181e+05 mm^3
$W_{pl,2}$	=	1.952e+05 mm^3
$W_{el,min,1}$	=	2.444e+05 mm^3
$W_{el,min,2}$	=	1.635e+05 mm^3
i_1	=	68 mm
i_2	=	39 mm
I_t	=	2.157e+07 mm^4
I_w	=	4.313e+09 mm^6

$f_y = 355\ \mathrm{N/mm^2}$

$\varepsilon = 0.81$

$\lambda_1 = 76.40$

Normal capacity - Part 1-1: 6.2

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$V_{1,Ed} = 0.00\ \mathrm{kN} \le 0.5 \cdot V_{1,pl,T,Rd} = 0.5 \cdot 359.13 = 179.57\ \mathrm{kN} \dashrightarrow \rho_1 = 0.00$

$V_{2,Ed} = 10.00\ \mathrm{kN} \le 0.5 \cdot V_{2,pl,T,Rd} = 0.5 \cdot 718.26 = 359.13\ \mathrm{kN} \dashrightarrow \rho_1 = 0.00$

$$\frac{N_{Ed}}{N_{Rd}} + \frac{M_{1,Ed}}{M_{1,Rd}} + \frac{M_{2,Ed}}{M_{2,Rd}} = \frac{80.00}{1866.11} + \frac{30.00}{112.92} + \frac{36.00}{69.31} = 0.83 \le 1.00 \quad (6.2)\text{ - OK}$$

Flexural buckling, 1-1 - Part 1-1: 6.3.1

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$$\bar{\lambda}_1 = \frac{L_{cr,1}}{i_1 \cdot \lambda_1} = \frac{6000}{68 \cdot 76.40} = 1.15 \quad (6.50)$$

$\alpha_1 = 0.49$ (Buckling curve: c)

$$\varphi_1 = 0.5\left[1 + \alpha_1 \cdot (\bar{\lambda}_1 - 0.2) + \bar{\lambda}_1^2\right] = 0.5\left[1 + 0.49 \cdot (1.15 - 0.2) + 1.15^2\right] = 1.40$$

$$\chi_1 = \min\left(\frac{1}{\varphi_1 + \sqrt{\varphi_1^2 - \bar{\lambda}_1^2}}, 1.0\right) = \min\left(\frac{1}{1.40 + \sqrt{1.40^2 - 1.15^2}}, 1.0\right) = 0.46$$

$$N_{b,Rd,1} = \frac{\chi_1 \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.46 \cdot 5257 \cdot 355}{1.00} = 853.69\ \mathrm{kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,1}} = \frac{80.00}{853.69} = 0.09 \le 1.00 \quad (6.46)\text{ - OK}$$

Flexural buckling, 2-2 - Part 1-1: 6.3.1

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$$\bar{\lambda}_2 = \frac{L_{cr,2}}{i_2 \cdot \lambda_1} = \frac{6000}{39 \cdot 76.40} = 1.99 \quad (6.50)$$

$\alpha_2 = 0.49$ (Buckling curve: c)

$$\varphi_2 = 0.5\left[1 + \alpha_2 \cdot (\bar{\lambda}_2 - 0.2) + \bar{\lambda}_2^2\right] = 0.5\left[1 + 0.49 \cdot (1.99 - 0.2) + 1.99^2\right] = 2.92$$

$$\chi_2 = \min\left(\frac{1}{\varphi_2 + \sqrt{\varphi_2^2 - \bar{\lambda}_2^2}}, 1.0\right) = \min\left(\frac{1}{2.92 + \sqrt{2.92^2 - 1.99^2}}, 1.0\right) = 0.20 \quad (6.49)$$

$$N_{b,Rd,2} = \frac{\chi_2 \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.20 \cdot 5257 \cdot 355}{1.00} = 368.89\ \mathrm{kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,2}} = \frac{80.00}{368.89} = 0.22 \le 1.00 \quad (6.46)\text{ - OK}$$

Torsional-flexural buckling - Part 1-1: 6.3.1

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$$i_0 = \sqrt{i_1^2 + i_2^2 + y_0^2 + z_0^2} = \sqrt{68^2 + 39^2 + 0^2 + 0^2} = 79\ \mathrm{mm}$$

$$N_{cr,T} = \frac{1}{i_0^2}\left(G \cdot I_t + \frac{\pi^2 \cdot E \cdot I_w}{l_T^2}\right) =$$

$$= \frac{1}{79^2}\left(80769 \cdot 2.157e{+}07 + \frac{\pi^2 \cdot 210000 \cdot 4.313e{+}09}{6000^2}\right) = 280793.07\ \mathrm{kN}$$

$N_{cr,TF} = 280793.07\ \mathrm{kN}$

$N_{cr} = \min(N_{cr,T}, N_{cr,TF}) = \min(280793.07, 280793.07) = 280793.07\ \mathrm{kN}$

$$\bar{\lambda}_T = \sqrt{\frac{A \cdot f_y}{N_{cr}}} = \sqrt{\frac{5257 \cdot 355}{280793.07}} = 0.08 \quad (6.53)$$

$\alpha_T = 0.49$ (Buckling curve: c)

$$\varphi_T = 0.5\left[1 + \alpha_T \cdot (\bar{\lambda}_T - 0.2) + \bar{\lambda}_T^2\right] = 0.5\left[1 + 0.49 \cdot (0.08 - 0.2) + 0.08^2\right] = 0.47$$

$$\chi_T = \min\left(\frac{1}{\varphi_T + \sqrt{\varphi_T^2 - \bar{\lambda}_T^2}}, 1.0\right) = \min\left(\frac{1}{0.47 + \sqrt{0.47^2 - 0.08^2}}, 1.0\right) = 1.00 \quad (6.49)$$

$$N_{b,Rd,T} = \frac{\chi_T \cdot A \cdot f_y}{\gamma_{M1}} = \frac{1.00 \cdot 5257 \cdot 355}{1.00} = 1866.11\ \mathrm{kN} \quad (6.47)$$

$$\frac{N_{Ed}}{N_{b,Rd,T}} = \frac{80.00}{1866.11} = 0.04 \le 1.00\text{ - OK}$$

Lateral torsional buckling - Part 1-1: 6.3.2.2

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$$N_{cr,LT} = \frac{\pi^2 \cdot E \cdot I_z}{(k_z \cdot L_{cr})^2} = \frac{\pi^2 \cdot 2.100e{+}05 \cdot 8.177e{+}06}{(1.00 \cdot 6000)^2} = 470.80\ \mathrm{kN}$$

Loaded on top edge.

$$M_{cr} = C_1 \cdot N_{cr,LT} \cdot \left\{\left[\left(\frac{k_z}{k_w}\right)^2 \cdot \frac{I_w}{I_z} + \frac{G \cdot I_t}{N_{cr,LT}} + Z^2\right]^{0.5} - Z\right\} =$$

$$= 1.35 \cdot 4.708e{+}05 \cdot \left\{\left[\left(\frac{1.00}{1.00}\right)^2 \cdot \frac{4.313e{+}09}{8.177e{+}06} + \frac{8.077e{+}04 \cdot 2.157e{+}07}{4.708e{+}05} + 63.00^2\right]^{0.5} - 63.00\right\} =$$

$= 1183.37\ \mathrm{kNm}$

$Z = (C_2 \cdot z_g - C_3 \cdot z_j) = (0.63 \cdot 100 - 1.73 \cdot 0) = 63.00\ \mathrm{mm}$

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_y \cdot f_y}{M_{cr}}} = \sqrt{\frac{318082 \cdot 355}{1.183e{+}09}} = 0.31$$

$\alpha_{LT} = 0.76$ (Buckling curve: d)

$$\varphi_{LT} = 0.5\left[1 + \alpha_{LT} \cdot (\bar{\lambda}_{LT} - 0.2) + \bar{\lambda}_{LT}^2\right] =$$

$$= 0.5\left[1 + 0.76 \cdot (0.31 - 0.2) + 0.31^2\right] = 0.59$$

$$\chi_{LT} = \min\left(\frac{1}{\varphi_{LT} + \sqrt{\varphi_{LT}^2 - \bar{\lambda}_{LT}^2}}, 1.0\right) = \min\left(\frac{1}{0.59 + \sqrt{0.59^2 - 0.31^2}}, 1.0\right) = 0.92 \quad (6.56)$$

$$M_{y,b,Rd} = \frac{\chi_{LT} \cdot W_y \cdot f_y}{\gamma_{M1}} = \frac{0.92 \cdot 318082 \cdot 355}{1.00} = 103.53\ \mathrm{kNm} \quad (6.55)$$

$$\frac{M_{1,Ed}}{M_{y,b,Rd}} = \frac{30.00}{103.53} = 0.29 \le 1.00 \quad (6.54)\text{ - OK}$$

Interaction between normal force and bending 1. - Part 1-1: 6.3.3

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

k_{ij} factors are calculated according to Method 1

$C_{my} = 0.99 \quad C_{mLT} = 1.00 \qquad C_{yy} = 0.93 \quad C_{zy} = 0.84$

$C_{mz} = 1.01 \quad M_{2,Rk} = f_y \cdot W_{pl,2} = 355 \cdot 195249 = 69.31\ \mathrm{kNm} \qquad C_{yz} = 0.82 \quad C_{zz} = 0.95$

$$\frac{N_{Ed}^{comp}}{N_{b,Rd,1}} + k_{11} \cdot \frac{M_{1,Ed}}{M_{y,b,Rd}} + k_{12} \cdot \frac{M_{2,Ed}}{\frac{M_{2,Rk}}{\gamma_{M1}}} =$$

$$= \frac{80.00}{853.69} + 1.09 \cdot \frac{30.00}{103.53} + 0.82 \cdot \frac{36.00}{\frac{69.31}{1.00}} = 0.84 \le 1.00 \quad (6.61)\text{ - OK}$$

Interaction between normal force and bending 2. - Part 1-1: 6.3.3

$$\frac{N_{Ed}^{comp}}{N_{b,Rd,2}} + k_{21} \cdot \frac{M_{1,Ed}}{M_{y,b,Rd}} + k_{22} \cdot \frac{M_{2,Ed}}{\frac{M_{2,Rk}}{\gamma_{M1}}} =$$

$$= \frac{80.00}{368.89} + 0.68 \cdot \frac{30.00}{103.53} + 1.10 \cdot \frac{36.00}{\frac{69.31}{1.00}} = 0.98 \le 1.00 \quad (6.62)\text{ - OK}$$

Fig. 9.3.6.2 – The FEM-Design results about the interaction according to Annex A

S 355

$E = 210000$ N/mm^2 $\gamma_{M0,acc/seis} = 1.00$

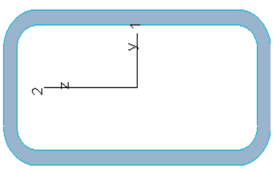
$G = 80769$ N/mm^2 $\gamma_{M1,acc/seis} = 1.00$

$\gamma_{M0,ult} = 1.00$ $\gamma_{M2,acc/seis} = 1.00$

$\gamma_{M1,ult} = 1.00$

$\gamma_{M2,ult} = 1.25$

KKR 200x100x10



P	=	557 mm	f_y	=	355 N/mm^2
A	=	5257 mm^2	ε	=	0.81
I_y	=	2.444e+07 mm^4	λ_1	=	76.40
I_z	=	8.177e+06 mm^4			
I_1	=	2.444e+07 mm^4			
I_2	=	8.177e+06 mm^4			
$W_{pl,1}$	=	3.181e+05 mm^3			
$W_{pl,2}$	=	1.952e+05 mm^3			
$W_{el,min,1}$	=	2.444e+05 mm^3			
$W_{el,min,2}$	=	1.635e+05 mm^3			
i_1	=	68 mm			
i_2	=	39 mm			
I_t	=	2.157e+07 mm^4			
I_w	=	4.313e+09 mm^6			

Normal capacity - Part 1-1: 6.2

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$V_{1,Ed} = 0.00\ \text{kN} \le 0.5 \cdot V_{1,pl,T,Rd} = 0.5 \cdot 359.13 = 179.57\ \text{kN} \dashrightarrow \rho_1 = 0.00$

$V_{2,Ed} = 10.00\ \text{kN} \le 0.5 \cdot V_{2,pl,T,Rd} = 0.5 \cdot 718.26 = 359.13\ \text{kN} \dashrightarrow \rho_1 = 0.00$

$\frac{N_{Ed}}{N_{Rd}} + \frac{M_{1,Ed}}{M_{1,Rd}} + \frac{M_{2,Ed}}{M_{2,Rd}} = \frac{80.00}{1866.11} + \frac{30.00}{112.92} + \frac{36.00}{69.31} = 0.83 \le 1.00$ (6.2) - OK

Flexural buckling, 1-1 - Part 1-1: 6.3.1

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$\bar{\lambda}_1 = \frac{L_{cr,1}}{i_1 \cdot \lambda_1} = \frac{6000}{68 \cdot 76.40} = 1.15$ (6.50)

$\alpha_1 = 0.49$ (Buckling curve: c)

$\varphi_1 = 0.5\left[1 + \alpha_1 \cdot (\bar{\lambda}_1 - 0.2) + \bar{\lambda}_1^2\right] = 0.5\left[1 + 0.49 \cdot (1.15 - 0.2) + 1.15^2\right] = 1.40$

$\chi_1 = \min\left\{\frac{1}{\varphi_1 + \sqrt{\varphi_1^2 - \bar{\lambda}_1^2}},\ 1.0\right\} = \min\left\{\frac{1}{1.40 + \sqrt{1.40^2 - 1.15^2}},\ 1.0\right\} = 0.46$ (6.49)

$N_{b,Rd,1} = \frac{\chi_1 \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.46 \cdot 5257 \cdot 355}{1.00} = 853.69\ \text{kN}$ (6.47)

$\frac{N_{Ed}}{N_{b,Rd,1}} = \frac{80.00}{853.69} = 0.09 \le 1.00$ (6.46) - OK

Flexural buckling, 2-2 - Part 1-1: 6.3.1

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$\bar{\lambda}_2 = \frac{L_{cr,2}}{i_2 \cdot \lambda_1} = \frac{6000}{39 \cdot 76.40} = 1.99$ (6.50)

$\alpha_2 = 0.49$ (Buckling curve: c)

$\varphi_2 = 0.5\left[1 + \alpha_2 \cdot (\bar{\lambda}_2 - 0.2) + \bar{\lambda}_2^2\right] = 0.5\left[1 + 0.49 \cdot (1.99 - 0.2) + 1.99^2\right] = 2.92$

$\chi_2 = \min\left\{\frac{1}{\varphi_2 + \sqrt{\varphi_2^2 - \bar{\lambda}_2^2}},\ 1.0\right\} = \min\left\{\frac{1}{2.92 + \sqrt{2.92^2 - 1.99^2}},\ 1.0\right\} = 0.20$ (6.49)

$N_{b,Rd,2} = \frac{\chi_2 \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.20 \cdot 5257 \cdot 355}{1.00} = 368.89\ \text{kN}$ (6.47)

$\frac{N_{Ed}}{N_{b,Rd,2}} = \frac{80.00}{368.89} = 0.22 \le 1.00$ (6.46) - OK

Lateral torsional buckling - Part 1-1: 6.3.2.2

LC: '1', x = 3000 mm

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

$N_{cr,LT} = \frac{\pi^2 \cdot E \cdot I_z}{(k_z \cdot L_{cr})^2} = \frac{\pi^2 \cdot 2.100e{+}05 \cdot 8.177e{+}06}{(1.00 \cdot 6000)^2} = 470.80\ \text{kN}$

Loaded on top edge.

$Z = (C_2 \cdot z_g - C_3 \cdot z_j) = (0.63 \cdot 100 - 1.73 \cdot 0) = 63.00$ mm

$M_{cr} = C_1 \cdot N_{cr,LT} \cdot \left\{\left[\left(\frac{k_z}{k_w}\right)^2 \cdot \frac{I_w}{I_z} + \frac{G \cdot I_t}{N_{cr,LT}} + Z^2\right]^{0.5} - Z\right\} =$

$= 1.35 \cdot 4.708e{+}05 \cdot \left\{\left[\left(\frac{1.00}{1.00}\right)^2 \cdot \frac{4.313e{+}09}{8.177e{+}06} + \frac{8.077e{+}04 \cdot 2.157e{+}07}{4.708e{+}05} + 63.00^2\right]^{0.5} - 63.00\right\} =$

$= 1183.37$ kNm

$\bar{\lambda}_{LT} = \sqrt{\frac{W_y \cdot f_y}{M_{cr}}} = \sqrt{\frac{318082 \cdot 355}{1.183e{+}09}} = 0.31$

$\alpha_{LT} = 0.76$ (Buckling curve: d)

$\varphi_{LT} = 0.5\left[1 + \alpha_{LT} \cdot (\bar{\lambda}_{LT} - 0.2) + \bar{\lambda}_{LT}^2\right] =$

$= 0.5\left[1 + 0.76 \cdot (0.31 - 0.2) + 0.31^2\right] = 0.59$

$\chi_{LT} = \min\left\{\frac{1}{\varphi_{LT} + \sqrt{\varphi_{LT}^2 - \bar{\lambda}_{LT}^2}},\ 1.0\right\} = \min\left\{\frac{1}{0.59 + \sqrt{0.59^2 - 0.31^2}},\ 1.0\right\} = 0.92$ (6.56)

$M_{y,b,Rd} = \frac{\chi_{LT} \cdot W_y \cdot f_y}{\gamma_{M1}} = \frac{0.92 \cdot 318082 \cdot 355}{1.00} = 103.53\ \text{kNm}$ (6.55)

$\frac{M_{1,Ed}}{M_{y,b,Rd}} = \frac{30.00}{103.53} = 0.29 \le 1.00$ (6.54) - OK

Interaction between normal force and bending 1.

LC: '1', x = 3000 mm **Part 1-1: 6.3.3**

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

k_{ij} factors are calculated according to Method 2

α_{my}	=	0.00	ψ_{my}	=	1.00	C_{my}	=	0.90
α_{mz}	=	0.00	ψ_{mz}	=	1.00	C_{mz}	=	0.95
α_{mLT}	=	0.00	ψ_{mLT}	=	1.00	C_{mLT}	=	0.90

$M_{2,Rk} = f_y \cdot W_{pl,2} = 355 \cdot 195249 = 69.31$ kNm

$\frac{N_{Ed}^{comp}}{N_{b,Rd,1}} + k_{11} \cdot \frac{M_{1,Ed}}{M_{y,b,Rd}} + k_{12} \cdot \frac{M_{2,Ed}}{\frac{M_{2,Rk}}{\gamma_{M1}}} =$

$= \frac{80.00}{853.69} + 0.97 \cdot \frac{30.00}{103.53} + 0.67 \cdot \frac{36.00}{\frac{69.31}{1.00}} = 0.72 \le 1.00$ (6.61) - OK

Interaction between normal force and bending 2.

LC: '1', x = 3000 mm **Part 1-1: 6.3.3**

$Class_N = 1,\ Class_{M1} = 1,\ Class_{M2} = 1$

k_{ij} factors are calculated according to Method 2

α_{my}	=	0.00	ψ_{my}	=	1.00	C_{my}	=	0.90
α_{mz}	=	0.00	ψ_{mz}	=	1.00	C_{mz}	=	0.95
α_{mLT}	=	0.00	ψ_{mLT}	=	1.00	C_{mLT}	=	0.90

$M_{2,Rk} = f_y \cdot W_{pl,2} = 355 \cdot 195249 = 69.31$ kNm

$\frac{N_{Ed}^{comp}}{N_{b,Rd,2}} + k_{21} \cdot \frac{M_{1,Ed}}{M_{y,b,Rd}} + k_{22} \cdot \frac{M_{2,Ed}}{\frac{M_{2,Rk}}{\gamma_{M1}}} =$

$= \frac{80.00}{368.89} + 0.58 \cdot \frac{30.00}{103.53} + 1.11 \cdot \frac{36.00}{\frac{69.31}{1.00}} = 0.96 \le 1.00$ (6.62) - OK

Fig. 9.3.6.3 – The FEM-Design results about the interaction according to Annex B

9.3.7 Interaction calculation with a Class 4 section

This chapter is unfinished.

9.4 Timber design

This chapter is unfinished.

9.4.1 Design of timber bars

9.4.2 CLT panel design with laminated composite shell theory

9.4.2.1 Calculation of the homogenized shell material stiffness matrix

In the next sub-chapters of this example we will calculate the homogenized stiffness values with different settings using the layer composition what you can see in Fig. 18.

No	Material	Thickness [mm]	Theta [°]	Ex [N/mm2]	Ey [N/mm2]	nuxy [-]	Gxy [N/mm2]	Gxz [N/mm2]	Gyz [N/mm2]	Rho [kg/m3]
1	C24	15.00	0.00	11600	450	0.40	690	690	100	420
2	C16	40.00	90	8000	270	0.40	500	500	50	370
3	C24	35.00	0.00	11600	450	0.40	690	690	100	420

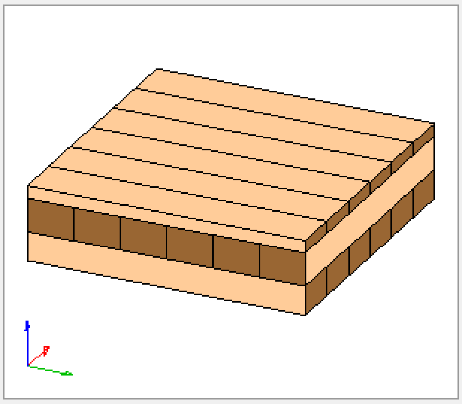


Figure 18 – The adjusted layer composition

In the next calculations (and also by FEM-Design “Display stiffness” button) we will calculate the stiffnesses relate to the mid-surface (center) of the shell layout composition.

The co-ordinates of the edge of the different layers related to the mid-surface:

$$z=\begin{bmatrix} 0.045 \\ 0.030 \\ -0.010 \\ -0.045 \end{bmatrix} \mathrm{m}$$

If someone adjusts the k_{def} deformation factors by the panel properties in the different limit states then the homogenization method will be the same but before the calculation the Young's moduli and shear moduli will be divided with $1+k_{def}$ and these reduced moduli will be the input values of the homogenization method as it was stated in Chapter 5.5.

9.4.2.1.1 Calculation with shear coupling

In this sub-chapter we will show the stiffness calculation in case of shear coupling between the layers when they are working together. In the first Chapter 6.1.1.1 we will show the calculation with the glue at narrow side option, then in Chapter 6.1.1.2 with no glue at narrow side option.

9.4.2.1.1.1 Glue at narrow side

First of all we should calculate the material stiffness matrices of the layers interpreted in the grain directions.

Material of layer No. 1 and 3 are the same therefore these matrices are the same:

In-plane part:

$$\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} = \begin{bmatrix} \dfrac{E_x}{1-\nu_{xy}^2 \dfrac{E_y}{E_x}} & \nu_{xy}\dfrac{E_y}{1-\nu_{xy}^2 \dfrac{E_y}{E_x}} & 0 \\ \nu_{xy}\dfrac{E_y}{1-\nu_{xy}^2 \dfrac{E_y}{E_x}} & \dfrac{E_y}{1-\nu_{xy}^2 \dfrac{E_y}{E_x}} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{11600}{1-0.4^2 \dfrac{450}{11600}} & 0.4\dfrac{450}{1-0.4^2 \dfrac{450}{11600}} & 0 \\ 0.4\dfrac{450}{1-0.4^2 \dfrac{450}{11600}} & \dfrac{450}{1-0.4^2 \dfrac{450}{11600}} & 0 \\ 0 & 0 & 690 \end{bmatrix} = \begin{bmatrix} 11672 & 181.1 & 0 \\ 181.1 & 452.8 & 0 \\ 0 & 0 & 690 \end{bmatrix} \text{MPa}$$

Transverse part:

$$\begin{bmatrix} Q_{55} & 0 \\ 0 & Q_{44} \end{bmatrix} = \begin{bmatrix} G_{xz} & 0 \\ 0 & G_{yz} \end{bmatrix} = \begin{bmatrix} 690 & 0 \\ 0 & 100 \end{bmatrix} \text{MPa}$$

Layer No. 2:

In-plane part:

$$\begin{bmatrix} \dfrac{8000}{1-0.4^2 \dfrac{270}{8000}} & 0.4\dfrac{270}{1-0.4^2 \dfrac{270}{8000}} & 0 \\ 0.4\dfrac{270}{1-0.4^2 \dfrac{270}{8000}} & \dfrac{270}{1-0.4^2 \dfrac{270}{8000}} & 0 \\ 0 & 0 & 500 \end{bmatrix} = \begin{bmatrix} 8043 & 108.6 & 0 \\ 108.6 & 271.5 & 0 \\ 0 & 0 & 500 \end{bmatrix} \text{MPa}$$

Transverse part:

$$\begin{bmatrix} Q_{55} & 0 \\ 0 & Q_{44} \end{bmatrix} = \begin{bmatrix} 500 & 0 \\ 0 & 50 \end{bmatrix} \text{MPa}$$

According to Fig. 18 the grain direction of layer No. 1 and 3 are in the local x' direction of the shell (0° orientation). It means that the previously calculated material stiffness will be the one what we need to use during the homogenization:

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_{1\&3} = \begin{bmatrix} 11672 & 181.1 & 0 \\ 181.1 & 452.8 & 0 \\ 0 & 0 & 690 \end{bmatrix} \mathrm{MPa}$$

$$\begin{bmatrix} \overline{\overline{Q}}_{55} & \overline{\overline{Q}}_{45} \\ \overline{\overline{Q}}_{45} & \overline{\overline{Q}}_{44} \end{bmatrix}_{1\&3} = \begin{bmatrix} 690 & 0 \\ 0 & 100 \end{bmatrix} \mathrm{MPa}$$

But by layer No. 2 we can see that the grain direction is perpendicular to the x' local system direction (90° orientation). Thus we should transform the material stiffness into the local x' direction. With the transformation matrix in the theoretical desciption we will get:

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_{2} = \begin{bmatrix} 271.5 & 108.6 & 0 \\ 108.6 & 8043 & 0 \\ 0 & 0 & 500 \end{bmatrix} \mathrm{MPa}$$

$$\begin{bmatrix} \overline{\overline{Q}}_{55} & \overline{\overline{Q}}_{45} \\ \overline{\overline{Q}}_{45} & \overline{\overline{Q}}_{44} \end{bmatrix}_{2} = \begin{bmatrix} 50 & 0 \\ 0 & 500 \end{bmatrix} \mathrm{MPa}$$

After these steps we can calculate the ABD and shear part of the stiffness matrix.

<u>The bending part</u>:

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^3 - z_{k+1}^3) \qquad (i,j = 1,2,6)$$

$$D_{11} = \frac{1}{3}\left[11672(0.045^3-0.03^3)+271.5(0.03^3-(-0.01^3))+11672((-0.01)^3-(-0.045)^3)\right] =$$

$$= 0.6027\frac{\mathrm{MNm}^2}{\mathrm{m}} = 602.7\,\mathrm{kNm}$$

$$D_{12} = \frac{1}{3}\left[181.1(0.045^3-0.03^3)+108.6(0.03^3-(-0.01^3))+181.1((-0.01)^3-(-0.045)^3)\right] =$$

$$= 0.01033\frac{\mathrm{MNm}^2}{\mathrm{m}} = 10.33\,\mathrm{kNm}$$

$$D_{22} = \frac{1}{3}\left[452.8(0.045^3-0.03^3)+8043(0.03^3-(-0.01^3))+452.8((-0.01)^3-(-0.045)^3)\right] =$$

$$= 0.09835\frac{\mathrm{MNm}^2}{\mathrm{m}} = 98.35\,\mathrm{kNm}$$

$$D_{66} = \frac{1}{3}\left[690(0.045^3-0.03^3)+500(0.03^3-(-0.01^3))+690((-0.01)^3-(-0.045)^3)\right] =$$

$$= 0.04014\frac{\mathrm{MNm}^2}{\mathrm{m}} = 40.14\,\mathrm{kNm}$$

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:

$$D_{16} = D_{26} = 0 \quad .$$

The membrane part:

$$A_{ij} = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k - z_{k+1}) \qquad (i, j = 1,2,6)$$

$$A_{11} = 11672(0.045-0.03) + 271.5(0.03-(-0.01)) + 11672((-0.01)-(-0.045)) =$$
$$= 594.46 \frac{\mathrm{MN}}{\mathrm{m}} = 594460 \frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{12} = 181.1(0.045-0.03) + 108.6(0.03-(-0.01)) + 181.1((-0.01)-(-0.045)) =$$
$$= 13.40 \frac{\mathrm{MN}}{\mathrm{m}} = 13400 \frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{22} = 452.8(0.045-0.03) + 8043(0.03-(-0.01)) + 452.8((-0.01)-(-0.045)) =$$
$$= 344.4 \frac{\mathrm{MN}}{\mathrm{m}} = 344400 \frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{66} = 690(0.045-0.03) + 500(0.03-(-0.01)) + 690((-0.01)-(-0.045)) =$$
$$= 54.5 \frac{\mathrm{MN}}{\mathrm{m}} = 54500 \frac{\mathrm{kN}}{\mathrm{m}}$$

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:

$$A_{16} = A_{26} = 0 \quad .$$

Eccentricity part:

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^2 - z_{k+1}^2) \qquad (i, j = 1,2,6)$$

$$B_{11} = \frac{1}{2}\left[11672(0.045^2-0.03^2) + 271.5(0.03^2-(-0.01^2)) + 11672((-0.01)^2-(-0.045)^2)\right] =$$
$$= -4.560 \frac{\mathrm{MNm}}{\mathrm{m}} = -4560\,\mathrm{kN}$$

$$B_{12} = \frac{1}{2}\left[181.1(0.045^2-0.03^2) + 108.6(0.03^2-(-0.01^2)) + 181.1((-0.01^2)-(-0.045^2))\right] =$$
$$= -0.029 \frac{\mathrm{MNm}}{\mathrm{m}} = -29\,\mathrm{kN}$$

$$B_{22} = \frac{1}{2}\left[452.8(0.045^2-0.03^2) + 8043(0.03^2-(-0.01^2)) + 452.8((-0.01)^2-(-0.045)^2)\right] =$$
$$= 3.036 \frac{\mathrm{MNm}}{\mathrm{m}} = 3036\,\mathrm{kN}$$

$$B_{66}=\frac{1}{2}\left[690\left(0.045^2-0.03^2\right)+500\left(0.03^2-\left(-0.01^2\right)\right)+690\left(\left(-0.01\right)^2-\left(-0.045\right)^2\right)\right]=$$

$$=-0.076\frac{\mathrm{MNm}}{\mathrm{m}}=-76\,\mathrm{kN}$$

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:

$$B_{16}=B_{26}=0\quad .$$

<u>Shear part</u>:

In this case the calculation of transverse shear stiffnesses is much easier than by a general case (see the theory description also about this topic).

Here: $\begin{bmatrix} S_{55} & S_{45} \\ S_{45} & S_{44} \end{bmatrix}=\begin{bmatrix} \bar{\bar{S}}_{55} & 0 \\ 0 & \bar{\bar{S}}_{44} \end{bmatrix}$

With the aim of the following equations first of all we should calculate the shear correction factors in the main stiffness direction (and perpendicular to it) which now coincide with the local system of the shell (see Fig. 18).

The shear correction factors:

$$\rho_{13}=\frac{R_1^2}{d_1\int\limits_{-\frac{t}{2}}^{\frac{t}{2}}\frac{g_1^2(z)}{\bar{\bar{Q}}_{55}(z)}\mathrm{d}\,z}=0.1638\quad ;\quad \rho_{23}=\frac{R_2^2}{d_2\int\limits_{-\frac{t}{2}}^{\frac{t}{2}}\frac{g_2^2(z)}{\bar{\bar{Q}}_{44}(z)}\mathrm{d}\,z}=0.8528$$

Based on these the shear stiffnesses:

$$\bar{\bar{S}}_{55}=\rho_{13}\sum_{k=1}^{N}\left(\bar{\bar{Q}}_{55}\right)_k\left(z_k-z_{k+1}\right)=$$

$$=0.1638\left[690\left(0.045-0.03\right)+50\left(0.03-\left(-0.010\right)\right)+690\left(\left(-0.01\right)-\left(-0.045\right)\right)\right]=5979\frac{\mathrm{kN}}{\mathrm{m}}$$

$$\bar{\bar{S}}_{44}=\rho_{23}\sum_{k=1}^{N}\left(\bar{\bar{Q}}_{44}\right)_k\left(z_k-z_{k+1}\right)=$$

$$=0.8528\left[100\left(0.045-0.03\right)+500\left(0.03-\left(-0.010\right)\right)+100\left(\left(-0.01\right)-\left(-0.045\right)\right)\right]=21320\frac{\mathrm{kN}}{\mathrm{m}}$$

Thus the final homogenized laminated shell stiffnesses (in kNm, kN and kN/m):

$$\begin{bmatrix} 602.7 & 10.33 & 0 & -4560 & -29 & 0 & 0 & 0 \\ 10.33 & 98.35 & 0 & -29 & 3036 & 0 & 0 & 0 \\ 0 & 0 & 40.14 & 0 & 0 & -76 & 0 & 0 \\ -4560 & -29 & 0 & 594460 & 13400 & 0 & 0 & 0 \\ -29 & 3036 & 0 & 13400 & 344400 & 0 & 0 & 0 \\ 0 & 0 & -76 & 0 & 0 & 54500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5979 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21320 \end{bmatrix}$$

Fig. 19 shows the values based on FEM-Design. The values are the same.

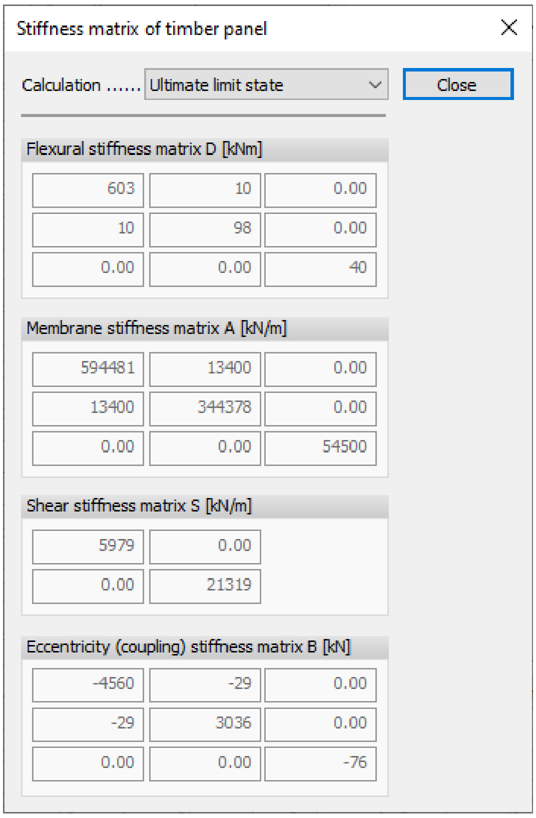


Figure 19 – The relevant stiffness values based on FEM-Design

9.4.2.1.1.2 No glue at narrow side

This calculation only differs from Chapter 6.1.1.1 that in the whole homogenization method the Young's moduli of the layers in perpendicular direction to the grains assumed to zero independently from the input data, see Fig. 18.

Material of layer No. 1 and 3 are the same therefore these matrices are the same:

In-plane part:

$$\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} = \begin{bmatrix} \dfrac{E_x}{1-\nu_{xy}^2 \dfrac{E_y}{E_x}} & \nu_{xy} \dfrac{E_y}{1-\nu_{xy}^2 \dfrac{E_y}{E_x}} & 0 \\ \nu_{xy} \dfrac{E_y}{1-\nu_{xy}^2 \dfrac{E_y}{E_x}} & \dfrac{E_y}{1-\nu_{xy}^2 \dfrac{E_y}{E_x}} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{11600}{1-0.4^2 \dfrac{0}{11600}} & 0.4 \dfrac{0}{1-0.4^2 \dfrac{0}{11600}} & 0 \\ 0.4 \dfrac{0}{1-0.4^2 \dfrac{0}{11600}} & \dfrac{0}{1-0.4^2 \dfrac{0}{11600}} & 0 \\ 0 & 0 & 690 \end{bmatrix} = \begin{bmatrix} 11600 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 690 \end{bmatrix} \mathrm{MPa}$$

Transverse part:

$$\begin{bmatrix} Q_{55} & 0 \\ 0 & Q_{44} \end{bmatrix} = \begin{bmatrix} G_{xz} & 0 \\ 0 & G_{yz} \end{bmatrix} = \begin{bmatrix} 690 & 0 \\ 0 & 100 \end{bmatrix} \mathrm{MPa}$$

Layer No. 2:

In-plane part:

$$\begin{bmatrix} \dfrac{8000}{1-0.4^2 \dfrac{0}{8000}} & 0.4 \dfrac{0}{1-0.4^2 \dfrac{0}{8000}} & 0 \\ 0.4 \dfrac{0}{1-0.4^2 \dfrac{0}{8000}} & \dfrac{0}{1-0.4^2 \dfrac{0}{8000}} & 0 \\ 0 & 0 & 500 \end{bmatrix} = \begin{bmatrix} 8000 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 500 \end{bmatrix} \mathrm{MPa}$$

Transverse part:

$$\begin{bmatrix} Q_{55} & 0 \\ 0 & Q_{44} \end{bmatrix} = \begin{bmatrix} 500 & 0 \\ 0 & 50 \end{bmatrix} \mathrm{MPa}$$

According to Fig. 18 the grain direction of layer No. 1 and 3 are in the local x' direction of the shell (0° orientation). It means that the previously calculated material stiffness will be the one what we need to use during the homogenization.

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_{1\&3} = \begin{bmatrix} 11600 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 690 \end{bmatrix} \mathrm{MPa}$$

$$\begin{bmatrix} \overline{\overline{Q}}_{55} & \overline{\overline{Q}}_{45} \\ \overline{\overline{Q}}_{45} & \overline{\overline{Q}}_{44} \end{bmatrix}_{1\&3} = \begin{bmatrix} 690 & 0 \\ 0 & 100 \end{bmatrix} \mathrm{MPa}$$

But by layer No. 2 we can see that the grain direction is perpendicular to the x' local system direction (90° orientation). Thus we should transform the former material stiffness into the local x' direction. With the transformation matrix in the theoretical description we will get:

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_{2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8000 & 0 \\ 0 & 0 & 500 \end{bmatrix} \mathrm{MPa}$$

$$\begin{bmatrix} \overline{\overline{Q}}_{55} & \overline{\overline{Q}}_{45} \\ \overline{\overline{Q}}_{45} & \overline{\overline{Q}}_{44} \end{bmatrix}_{2} = \begin{bmatrix} 50 & 0 \\ 0 & 500 \end{bmatrix} \mathrm{MPa}$$

After these steps we can calculate the ABD and shear part of the stiffness matrix.

<u>The bending part</u>:

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^3 - z_{k+1}^3) \qquad (i,j = 1,2,6)$$

$$D_{11} = \frac{1}{3}\left[11600(0.045^3 - 0.03^3) + 0(0.03^3 - (-0.01^3)) + 11600((-0.01)^3 - (-0.045)^3)\right] =$$

$$= 0.5964 \frac{\mathrm{MNm}^2}{\mathrm{m}} = 596.4\,\mathrm{kNm}$$

$$D_{22} = \frac{1}{3}\left[0(0.045^3 - 0.03^3) + 8000(0.03^3 - (-0.01^3)) + 0((-0.01)^3 - (-0.045)^3)\right] =$$

$$= 0.07467 \frac{\mathrm{MNm}^2}{\mathrm{m}} = 74.67\,\mathrm{kNm}$$

$$D_{66} = \frac{1}{3}\left[690(0.045^3 - 0.03^3) + 500(0.03^3 - (-0.01^3)) + 690((-0.01)^3 - (-0.045)^3)\right] =$$

$$= 0.04014 \frac{\mathrm{MNm}^2}{\mathrm{m}} = 40.14\,\mathrm{kNm}$$

$$D_{12} = D_{16} = D_{26} = 0 \quad .$$

The membrane part:

$$A_{ij}=\sum_{k=1}^{N}(\overline{Q}_{ij})_k(z_k-z_{k+1}) \qquad (i,j=1,2,6)$$

$$A_{11}=11600(0.045-0.03)+0(0.03-(-0.01))+11600((-0.01)-(-0.045))=$$
$$=580\frac{\mathrm{MN}}{\mathrm{m}}=580000\frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{22}=0(0.045-0.03)+8000(0.03-(-0.01))+0((-0.01)-(-0.045))=$$
$$=320\frac{\mathrm{MN}}{\mathrm{m}}=320000\frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{66}=690(0.045-0.03)+500(0.03-(-0.01))+690((-0.01)-(-0.045))=$$
$$=54.5\frac{\mathrm{MN}}{\mathrm{m}}=54500\frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{12}=A_{16}=A_{26}=0 \quad .$$

Eccentricity part:

$$B_{ij}=\frac{1}{2}\sum_{k=1}^{N}(\overline{Q}_{ij})_k(z_k^2-z_{k+1}^2) \qquad (i,j=1,2,6)$$

$$B_{11}=\frac{1}{2}\left[11600(0.045^2-0.03^2)+0(0.03^2-(-0.01^2))+11600((-0.01)^2-(-0.045)^2)\right]=$$
$$=-4.64\frac{\mathrm{MNm}}{\mathrm{m}}=-4640\,\mathrm{kN}$$

$$B_{22}=\frac{1}{2}\left[0(0.045^2-0.03^2)+8000(0.03^2-(-0.01^2))+0((-0.01)^2-(-0.045)^2)\right]=$$
$$=3.2\frac{\mathrm{MNm}}{\mathrm{m}}=3200\,\mathrm{kN}$$

$$B_{66}=\frac{1}{2}\left[690(0.045^2-0.03^2)+500(0.03^2-(-0.01^2))+690((-0.01)^2-(-0.045)^2)\right]=$$
$$=-0.076\frac{\mathrm{MNm}}{\mathrm{m}}=-76\,\mathrm{kN}$$

$$B_{12}=B_{16}=B_{26}=0 \quad .$$

Shear part:

In this case the calculation of the shear correction is much easier than by a general case (see the theory desciption also about this topic).

Here: $\begin{bmatrix} S_{55} & S_{45} \\ S_{45} & S_{44} \end{bmatrix}=\begin{bmatrix} \overline{\overline{S}}_{55} & 0 \\ 0 & \overline{\overline{S}}_{44} \end{bmatrix}$

With the aim of the following equations first of all we should calculate the shear correction factors in the main stiffness direction (and perpendicular to it) which now coincide with the local system of the shell (see Fig. 18).

The shear correction factors:

$$\rho_{13}=\frac{R_1^2}{d_1\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{g_1^2(z)}{\overline{\overline{Q}}_{55}(z)}\mathrm{d}z}=0.1640 \quad ; \quad \rho_{23}=\frac{R_2^2}{d_2\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{g_2^2(z)}{\overline{\overline{Q}}_{44}(z)}\mathrm{d}z}=0.6667$$

Based on these the shear stiffnesses:

$$\overline{\overline{S}}_{55}=\rho_{13}\sum_{k=1}^{N}(\overline{\overline{Q}}_{55})_k(z_k-z_{k+1})=$$

$$=0.1640[690(0.045-0.03)+50(0.03-(-0.010))+690((-0.01)-(-0.045))]=5986\frac{\mathrm{kN}}{\mathrm{m}}$$

$$\overline{\overline{S}}_{44}=\rho_{23}\sum_{k=1}^{N}(\overline{\overline{Q}}_{44})_k(z_k-z_{k+1})=$$

$$=0.6667[100(0.045-0.03)+500(0.03-(-0.010))+100((-0.01)-(-0.045))]=16668\frac{\mathrm{kN}}{\mathrm{m}}$$

Thus the final homogenized laminated shell stiffnesses (in kNm, kN and kN/m):

$$\begin{bmatrix} 596.4 & 0 & 0 & -4640 & 0 & 0 & 0 & 0 \\ 0 & 74.67 & 0 & 0 & 3200 & 0 & 0 & 0 \\ 0 & 0 & 40.14 & 0 & 0 & -76 & 0 & 0 \\ -4640 & 0 & 0 & 580000 & 0 & 0 & 0 & 0 \\ 0 & 3200 & 0 & 0 & 320000 & 0 & 0 & 0 \\ 0 & 0 & -76 & 0 & 0 & 54500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5986 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16668 \end{bmatrix}$$

Fig. 20 shows the values based on FEM-Design. The values are the same.

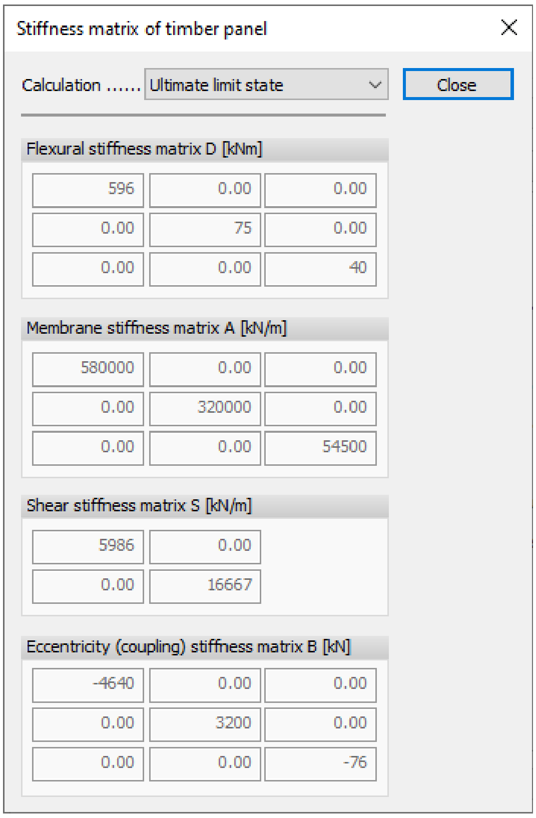


Figure 20 – The relevant stiffness values based on FEM-Design

9.4.2.1.2 Calculation without shear coupling

In this sub-chapter we will show the stiffness calculation in case of without shear coupling between the layers when they are NOT working together. In Chapter 6.1.2.1 we will show the calculation with the glue at narrow side option, then in Chapter 6.1.2.2 with no glue at narrow side option.

9.4.2.1.2.1 Glue at narrow side

The material stiffness properties are the same layer-by-layer in the shell local system just like in Chapter 6.1.1.1.

Layer No. 1 and 3:

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_{1\,\&\,3} = \begin{bmatrix} 11672 & 181.1 & 0 \\ 181.1 & 452.8 & 0 \\ 0 & 0 & 690 \end{bmatrix} \mathrm{MPa}$$

$$\begin{bmatrix} \overline{\overline{Q}}_{55} & \overline{\overline{Q}}_{45} \\ \overline{\overline{Q}}_{45} & \overline{\overline{Q}}_{44} \end{bmatrix}_{1\,\&\,3} = \begin{bmatrix} 690 & 0 \\ 0 & 100 \end{bmatrix} \mathrm{MPa}$$

Layer No. 2:

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_{2} = \begin{bmatrix} 271.5 & 108.6 & 0 \\ 108.6 & 8043 & 0 \\ 0 & 0 & 500 \end{bmatrix} \mathrm{MPa}$$

$$\begin{bmatrix} \overline{\overline{Q}}_{55} & \overline{\overline{Q}}_{45} \\ \overline{\overline{Q}}_{45} & \overline{\overline{Q}}_{44} \end{bmatrix}_{2} = \begin{bmatrix} 50 & 0 \\ 0 & 500 \end{bmatrix} \mathrm{MPa}$$

Based on these information the ABD and shear part of the stiffness matrix can be calculated.

<u>The bending part</u>:

$$D_{ij} = \sum_{k=1}^{N} (\overline{Q}_{ij})_k \frac{(z_k - z_{k+1})^3}{12}$$

$$D_{11} = \frac{1}{12}\left[11672(0.045-0.03)^3 + 271.5(0.03-(-0.01))^3 + 11672((-0.01)-(-0.045))^3\right] =$$

$$= 0.04643\,\frac{\mathrm{MNm}^2}{\mathrm{m}} = 46.43\,\mathrm{kNm}$$

$$D_{12} = \frac{1}{12}\left[181.1(0.045-0.03)^3 + 108.6(0.03-(-0.01))^3 + 181.1((-0.01)-(-0.045))^3\right] =$$

$$= 0.001277\,\frac{\mathrm{MNm}^2}{\mathrm{m}} = 1.277\,\mathrm{kNm}$$

$$D_{22}=\frac{1}{12}\left[452.8(0.045-0.03)^3+8043(0.03-(-0.01))^3+452.8((-0.01)-(-0.045))^3\right]=$$

$$=0.04464\frac{\mathrm{MNm}^2}{\mathrm{m}}=44.64\,\mathrm{kNm}$$

$$D_{66}=\frac{1}{12}\left[690(0.045-0.03)^3+500(0.03-(-0.01))^3+690((-0.01)-(-0.045))^3\right]=$$

$$=0.005325\frac{\mathrm{MNm}^2}{\mathrm{m}}=5.325\,\mathrm{kNm}$$

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:

$$D_{16}=D_{26}=0\quad.$$

<u>The membrane part</u>:

$$A_{ij}=\sum_{k=1}^{N}(\overline{Q}_{ij})_k(z_k-z_{k+1})\qquad (i,j=1,2,6)$$

$$A_{11}=11672(0.045-0.03)+271.5(0.03-(-0.01))+11672((-0.01)-(-0.045))=$$

$$=594.46\frac{\mathrm{MN}}{\mathrm{m}}=594460\frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{12}=181.1(0.045-0.03)+108.6(0.03-(-0.01))+181.1((-0.01)-(-0.045))=$$

$$=13.40\frac{\mathrm{MN}}{\mathrm{m}}=13400\frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{22}=452.8(0.045-0.03)+8043(0.03-(-0.01))+452.8((-0.01)-(-0.045))=$$

$$=344.4\frac{\mathrm{MN}}{\mathrm{m}}=344400\frac{\mathrm{kN}}{\mathrm{m}}$$

$$A_{66}=690(0.045-0.03)+500(0.03-(-0.01))+690((-0.01)-(-0.045))=$$

$$=54.5\frac{\mathrm{MN}}{\mathrm{m}}=54500\frac{\mathrm{kN}}{\mathrm{m}}$$

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:

$$A_{16}=A_{26}=0\quad.$$

<u>Eccentricity part</u>:

$$B_{ij}=0\qquad (i,j=1,2,6)$$

<u>Shear part</u>:

In this case the calculation of the shear stiffnesses are similar than the calculation of the shear stiffnesses by a homogeneous isotropic slab:

$$S_{ij}=\frac{5}{6}\sum_{k=1}^{N}(\overline{Q}_{ij})_k(z_k-z_{k+1}) \qquad (i,j=4,5)$$

$$S_{55}=\frac{5}{6}[690(0.045-0.03)+50(0.03-(-0.010))+690((-0.01)-(-0.045))]=30417\frac{\text{kN}}{\text{m}}$$

$$S_{44}=\frac{5}{6}[100(0.045-0.03)+500(0.03-(-0.010))+100((-0.01)-(-0.045))]=20833\frac{\text{kN}}{\text{m}}$$

Thus the final homogenized laminated shell stiffnesses (in kNm and kN/m):

$$\begin{bmatrix} 46.43 & 1.277 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.277 & 44.64 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.325 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 594460 & 13400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13400 & 344400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 54500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 30417 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20833 \end{bmatrix}$$

Fig. 21 shows the values based on FEM-Design. The values are the same.

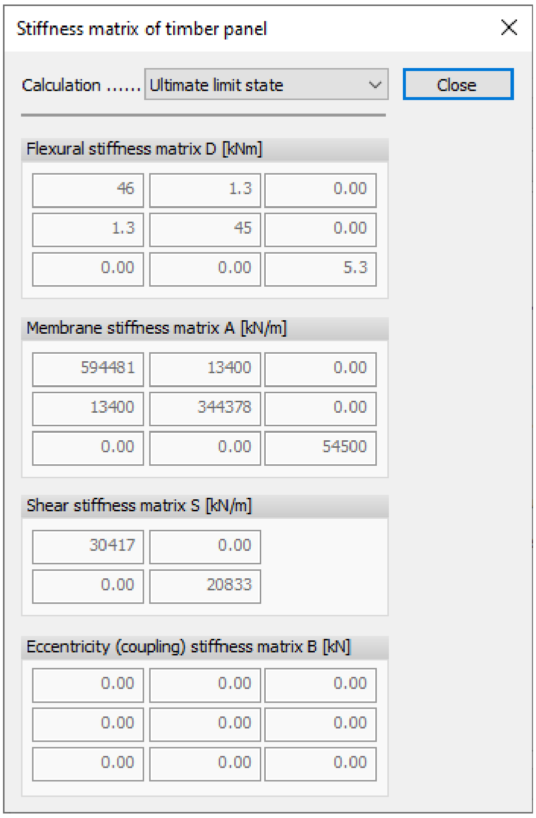


Figure 21 – The relevant stiffness values based on FEM-Design

9.4.2.1.2.2 No glue at narrow side

The material stiffness properties are the same layer-by-layer in the shell local system just like in Chapter 6.1.1.2.

Layer No. 1 and 3:

$$\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_{1\,\&\,3} = \begin{bmatrix} 11600 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 690 \end{bmatrix} \mathrm{MPa}$$

$$\begin{bmatrix} \bar{\bar{Q}}_{55} & \bar{\bar{Q}}_{45} \\ \bar{\bar{Q}}_{45} & \bar{\bar{Q}}_{44} \end{bmatrix}_{1\,\&\,3} = \begin{bmatrix} 690 & 0 \\ 0 & 100 \end{bmatrix} \mathrm{MPa}$$

Layer No. 2:

$$\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_{2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8000 & 0 \\ 0 & 0 & 500 \end{bmatrix} \mathrm{MPa}$$

$$\begin{bmatrix} \bar{\bar{Q}}_{55} & \bar{\bar{Q}}_{45} \\ \bar{\bar{Q}}_{45} & \bar{\bar{Q}}_{44} \end{bmatrix}_{2} = \begin{bmatrix} 50 & 0 \\ 0 & 500 \end{bmatrix} \mathrm{MPa}$$

Based on these information the ABD and shear part of the stiffness matrix can be calculated.

<u>The bending part</u>:

$$D_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k \frac{(z_k - z_{k+1})^3}{12}$$

$$D_{11} = \frac{1}{12}\left[11600(0.045-0.03)^3 + 0(0.03-(-0.01))^3 + 11600((-0.01)-(-0.045))^3\right] =$$

$$= 0.04471 \frac{\mathrm{MNm}^2}{\mathrm{m}} = 44.71\,\mathrm{kNm}$$

$$D_{22} = \frac{1}{12}\left[0(0.045-0.03)^3 + 8000(0.03-(-0.01))^3 + 0((-0.01)-(-0.045))^3\right] =$$

$$= 0.04267 \frac{\mathrm{MNm}^2}{\mathrm{m}} = 42.67\,\mathrm{kNm}$$

$$D_{66} = \frac{1}{12}\left[690(0.045-0.03)^3 + 500(0.03-(-0.01))^3 + 690((-0.01)-(-0.045))^3\right] =$$

$$= 0.005325 \frac{\mathrm{MNm}^2}{\mathrm{m}} = 5.325\,\mathrm{kNm}$$

$$D_{12} = D_{16} = D_{26} = 0 \quad .$$

The membrane part:

$$A_{ij}=\sum_{k=1}^{N}(\overline{Q}_{ij})_k(z_k-z_{k+1}) \qquad (i,j=1,2,6)$$

$$A_{11}=11600(0.045-0.03)+0(0.03-(-0.01))+11600((-0.01)-(-0.045))=$$
$$=580\frac{\text{MN}}{\text{m}}=580000\frac{\text{kN}}{\text{m}}$$

$$A_{22}=0(0.045-0.03)+8000(0.03-(-0.01))+0((-0.01)-(-0.045))=$$
$$=320\frac{\text{MN}}{\text{m}}=320000\frac{\text{kN}}{\text{m}}$$

$$A_{66}=690(0.045-0.03)+500(0.03-(-0.01))+690((-0.01)-(-0.045))=$$
$$=54.5\frac{\text{MN}}{\text{m}}=54500\frac{\text{kN}}{\text{m}}$$

$$A_{12}=A_{16}=A_{26}=0 \quad .$$

Eccentricity part:

$$B_{ij}=0 \qquad (i,j=1,2,6)$$

Shear part:

In this case the calculation of the shear stiffnesses are similar than the calculation of the shear stiffnesses by a homogeneous isotropic slab:

$$S_{ij}=\frac{5}{6}\sum_{k=1}^{N}(\overline{Q}_{ij})_k(z_k-z_{k+1}) \qquad (i,j=4,5)$$

$$S_{55}=\frac{5}{6}[690(0.045-0.03)+50(0.03-(-0.010))+690((-0.01)-(-0.045))]=30417\frac{\text{kN}}{\text{m}}$$

$$S_{44}=\frac{5}{6}[100(0.045-0.03)+500(0.03-(-0.010))+100((-0.01)-(-0.045))]=20833\frac{\text{kN}}{\text{m}}$$

Thus the final homogenized laminated shell stiffnesses (in kNm and kN/m):

$$\begin{bmatrix} 44.71 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 42.67 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.325 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 580000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 320000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 54500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 30417 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20833 \end{bmatrix}$$

Fig. 22 shows the values based on FEM-Design. The values are the same.

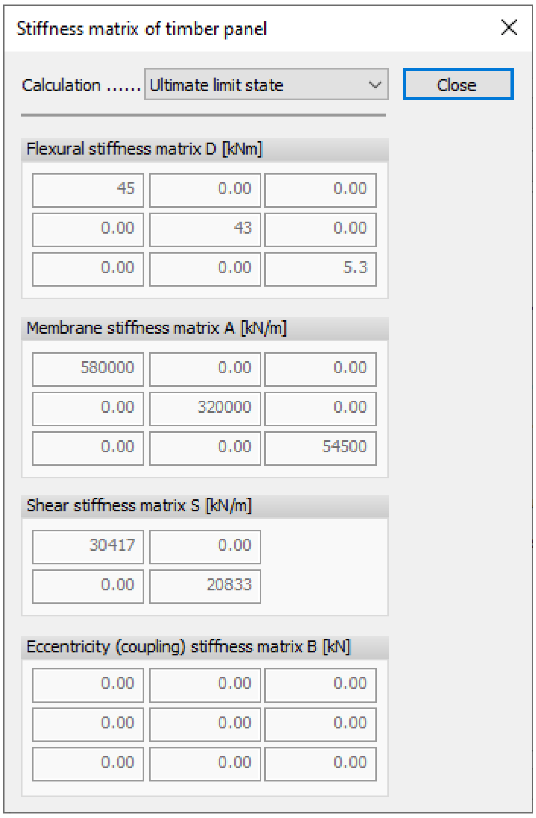


Figure 22 – The relevant stiffness values based on FEM-Design

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.1 Calculation of the homogenized shell material stiffness matrix.str

9.4.2.2 Deflection and stresses of a CLT panel supported on two opposite edges

In this example we analyze a one-way CLT panel with beam-a-like behaviour and material properties to verify the calculation results compared with hand calculation. The geomtry (10 m x 2.45 m) and the total distributed load can be seen with the supports in Fig. 23.

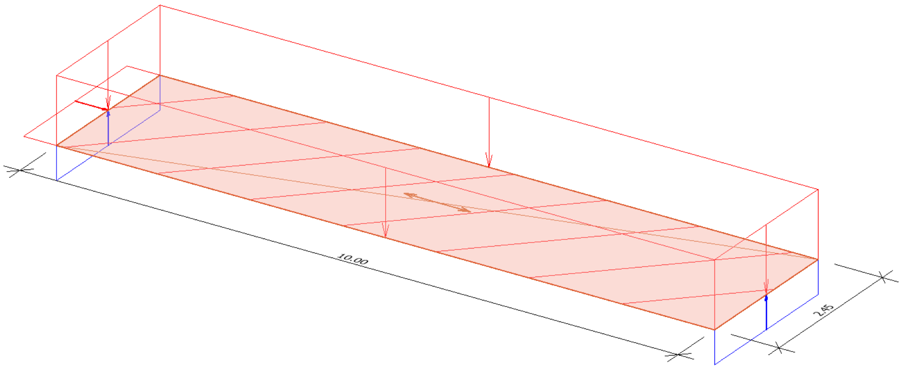


Figure 23 – The one-way CLT slab supported on two opposite edges, the longer direction is x'

The material properties are reduced to consider beam-a-like behaviour (see Fig. 24) and to better comparison between the FEM-Design calculation and hand calculation, but in reality the Poisson's ratio and the E_y modulus by a CLT panel is not equal to zero. The grain direction of the top layer is in the span direction.

Layers (top down order)

No	Material	Thickness [mm]	Beta [°]	Ex [N/mm2]	Ey [N/mm2]	nuxy [-]	Gxy [N/mm2]	Gxz [N/mm2]	Gyz [N/mm2]	Rho [kg/m3]
1	C24	30	0.00	11000	0.10	0.00	690	690	69	420
2	C24	40	90	11000	0.10	0.00	690	690	69	420
3	C24	30	0.00	11000	0.10	0.00	690	690	69	420
4	C24	40	90	11000	0.10	0.00	690	690	69	420
5	C24	30	0.00	11000	0.10	0.00	690	690	69	420
6	C24	40	90	11000	0.10	0.00	690	690	69	420
7	C24	30	0.00	11000	0.10	0.00	690	690	69	420

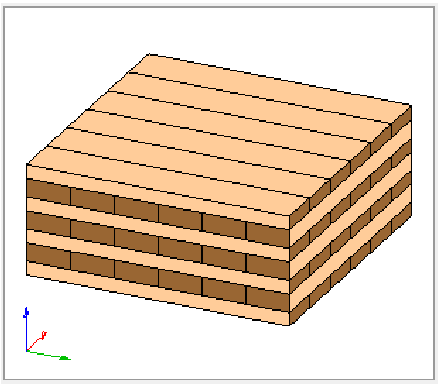


Figure 24 – The layer compositions and the reduced material properties

The specific surface load in ULS:

$$q_U = 1.35 \cdot 0.9888 + 1.5 \cdot 2 = 4.335 \, \mathrm{kN/m^2}$$

The maximum specific shear force at the support (calculated as a simply supported beam):

$$V_{max}=q_U\frac{L}{2}=\frac{4.335\cdot 10}{2}=21.68\,\mathrm{kN/m}$$

The FEM-Design result according to Fig. 20:

$$V_{max}^{FEM-Design}=21.67\,\mathrm{kN/m}$$

The maximum specific bending moment at mid-span:

$$M_{max}=q_U\frac{L^2}{8}=4.335\frac{10^2}{8}=54.19\,\mathrm{kNm/m}$$

The FEM-Design result according to Fig. 20:

$$M_{max}^{FEM-Design}=54.25\,\mathrm{kNm/m}$$

The relevant internal forces from FEM-Design can be seen in Fig. 25.

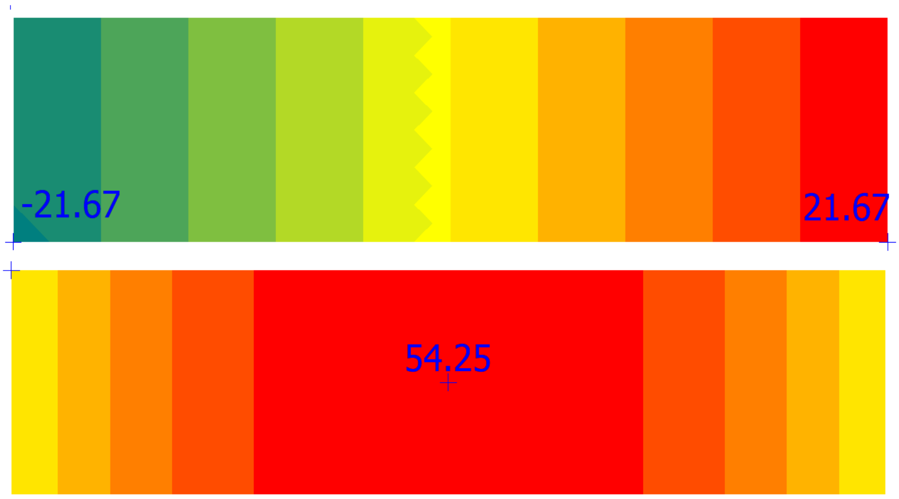


Figure 25 – The specific shear force [$q_{x'z'}$, kN/m] and the specific bending moment [$m_{x'}$, kNm/m]

First we would like to calculate the relevant normal stress in the mid-span at the extreme fibers:

The relevant specific inertia (considering the 1st, 3rd, 5th and 7th layer), neglecting the lateral layers (2nd , 4th and 6th):

$$I=2\left(\frac{2\cdot 0.03^3}{12}+0.03\cdot 0.105^2+0.03\cdot 0.035^2\right)=0.000744\,\mathrm{m^4/m}$$

The maximum normal sress in grain direction at the extreme fibers:

$$\sigma_{c/t,0,d}^{max}=\frac{M_{max}}{I}\frac{t}{2}=\frac{54.19}{0.000744}\frac{0.24}{2}=8740\,\mathrm{kPa}=8.74\,\mathrm{MPa}$$

The FEM-Design result according to Fig. 26:

$$\sigma_{max}^{FEM-Design}=8.75\,\mathrm{MPa}$$

Second we would like to calculate the relevant transverse shear stress at the mid-plane (center of the 4th layer) next to the support:

The specific maximum value of statical moment:

$$S_{max}=0.03\cdot0.105+0.03\cdot0.035=0.0042\,\mathrm{m}^3/\mathrm{m}$$

The maximum rolling shear at the mid-plane next to the support:

$$\tau_{yz,d}^{center}=\tau_{Rolling,d}^{max}=\frac{V_{max}S_{max}}{I\,b}=\frac{21.68\cdot0.0042}{0.000744\cdot1}=122.4\,\mathrm{kPa}=0.1224\,\mathrm{MPa}$$

The FEM-Design result according to Fig. 26:

$$\tau_{max}^{FEM-Design}=0.1224\,\mathrm{MPa}$$

Figure 26 – The normal stresses in the span direction at the mid-span and the relevant transverse shear stresses next to the support based on FEM-Design calculation

Third we would like to calculate the deflection in SLS considering creep.

The specific surface load in SLS quasi-permanent combination:

$$q_{Sq}=0.9888+0.3\cdot2=1.589\,\mathrm{kN/m^2}$$

The final deflection at mid-span considering creep and neglecting shear deformation:

$$w_{fin}=\frac{5}{384}\frac{q_{sq}L^4}{EI}\left(1+k_{defSq}\right)=\frac{5}{384}\frac{1.589\cdot10^4}{11000000\cdot0.000744}\left(1+0.8\right)=0.0455\,\mathrm{mm}=45.5\,\mathrm{mm}$$

The FEM-Design result according to Fig. 27:

$$w_{fin}^{FEM-Design}=47.00\,\mathrm{mm}$$

The results based on the hand calculation and FEM-Design are identical except the deflection because in FEM-Design the shear deformations are also considered, therefore the maximum deflection at the mid-span is a bit larger compared to the hand calculation.

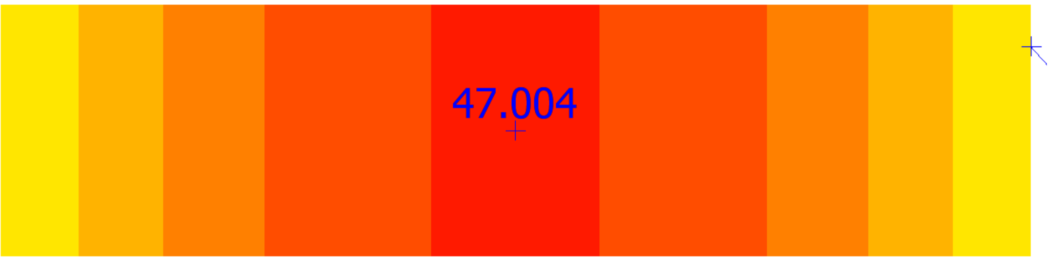


Figure 27 – The deflections in the SLS combination [mm]

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.2 Deflection and stresses of a CLT panel supported on two opposite edges.str

9.4.2.3 Deflection and stresses of a CLT simply supported two-way slab

By this verification example we calculated a simply supported two-way CLT slab (7m x 5m) with uniformly distributed load (see Fig. 28).

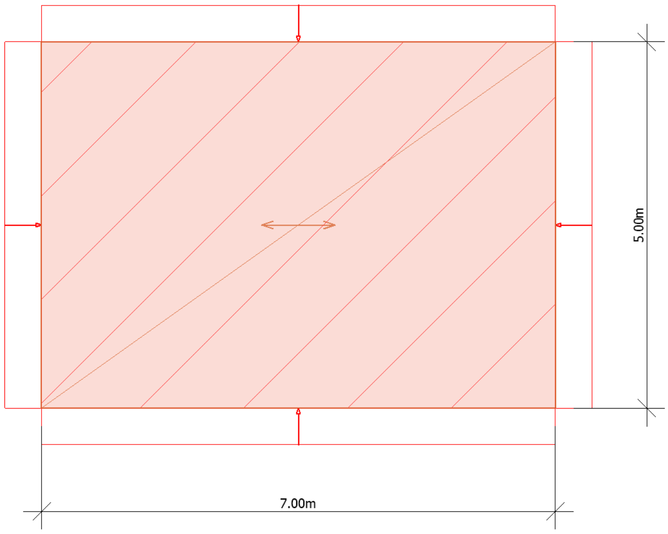


Figure 28 – The geometry of the considered plate with the uniformly distributed load

The layer composition with the mechanical properties can be seen in Fig. 29. The grain direction of the top layer is in the longer direction of the slab (see also Fig. 28). The total thickness is t = 240 mm. During the calculation we considered shear coupling between layers and glue at narrow sides.

Layers (top down order)

No	Material	Thickness [mm]	Beta [°]	Ex [N/mm2]	Ey [N/mm2]	nuxy [-]	Gxy [N/mm2]	Gxz [N/mm2]	Gyz [N/mm2]	Rho [kg/m3]
1	C24	30	0.00	11000	370	0.20	690	690	69	420
2	C24	40	90	11000	370	0.20	690	690	69	420
3	C24	30	0.00	11000	370	0.20	690	690	69	420
4	C24	40	90	11000	370	0.20	690	690	69	420
5	C24	30	0.00	11000	370	0.20	690	690	69	420
6	C24	40	90	11000	370	0.20	690	690	69	420
7	C24	30	0.00	11000	370	0.20	690	690	69	420

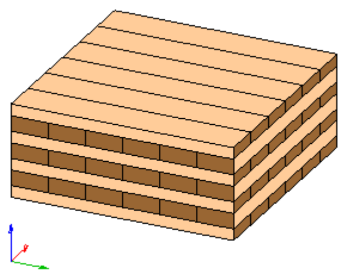


Figure 29 – The layer composition with the thicknesses and material parameters

The considered combination is a ULS combination, thus k_{defU} = 0. The design value of the considered value of the uniformly distributed surface load contains the self-weigth of the CLT panel and a variable load:

$$q_U = 1.35\left(\frac{420}{1000}9.81 \cdot 0.24\right) + 1.5 \cdot 2 = 4.335\,\frac{\text{kN}}{\text{m}^2}$$

We compared different FEM-Design results with ANSYS shell281 multilayer element results.

Fig. 30 shows the applied finite element mesh size, the average element size was 0.36m.

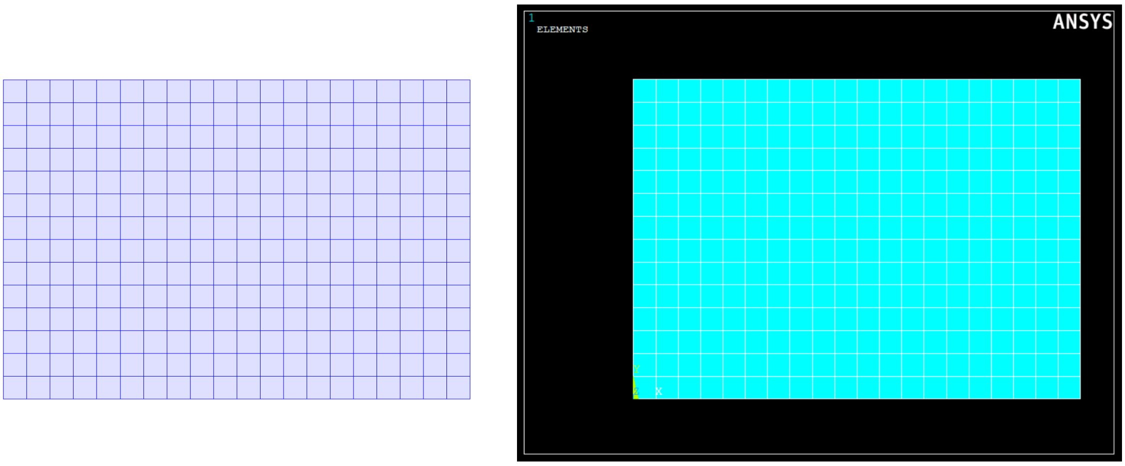


Figure 30 – The finite element mesh of the plate in FEM-Design (left) and in ANSYS (right)

Fig 31. shows the deflection colour palette results. The maximum deflection from the calculations:

$w_{max}^{FEM-Design} = 5.785\,\text{mm}$; $w_{max}^{ANSYS} = 5.786\,\text{mm}$

Practically the results are the same.

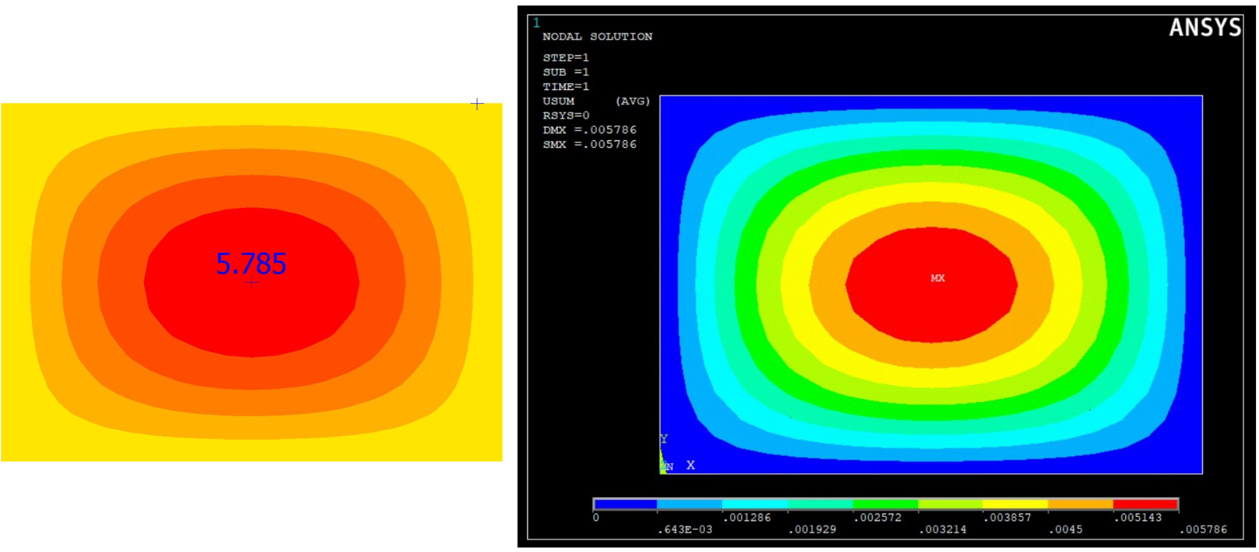


Figure 31 – The deflections of the plate in FEM-Design (left, in [mm]*) and in ANSYS (right, in* [m]*)*

The normal stress at the bottom of the lowermost layer in the x' direction in shell local system:

$\sigma_{x'bottom}^{FEM-Design} = 1.302\,\text{MPa}$; $\sigma_{x'bottom}^{ANSYS} = 1.303\,\text{MPa}$

The results are the same, see Fig. 32.

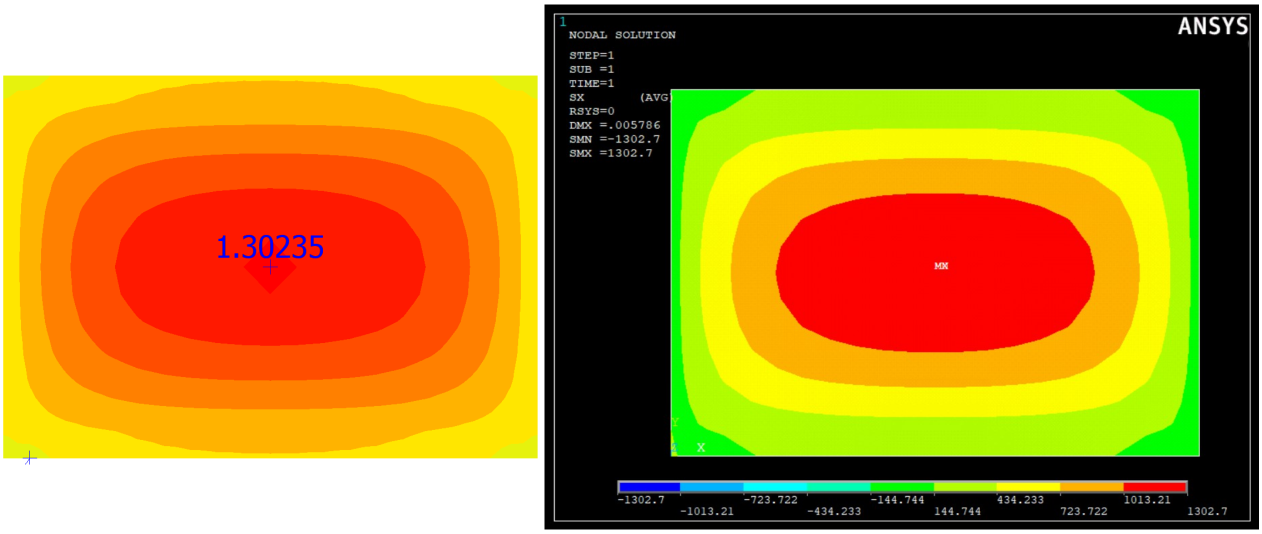


Figure 32 – The $\sigma_{x'}$ stresses at the bottom of the lowermost layer in FEM-Design (left, in [MPa]*) and in ANSYS (right, in* [kPa]*)*

The normal stress at the bottom of the lowermost layer in the y' direction in shell local system:

$\sigma_{y'bottom}^{FEM-Design} = 0.09774\,\text{MPa}$; $\sigma_{y'bottom}^{ANSYS} = 0.09771\,\text{MPa}$

The results are the same, see Fig 33.

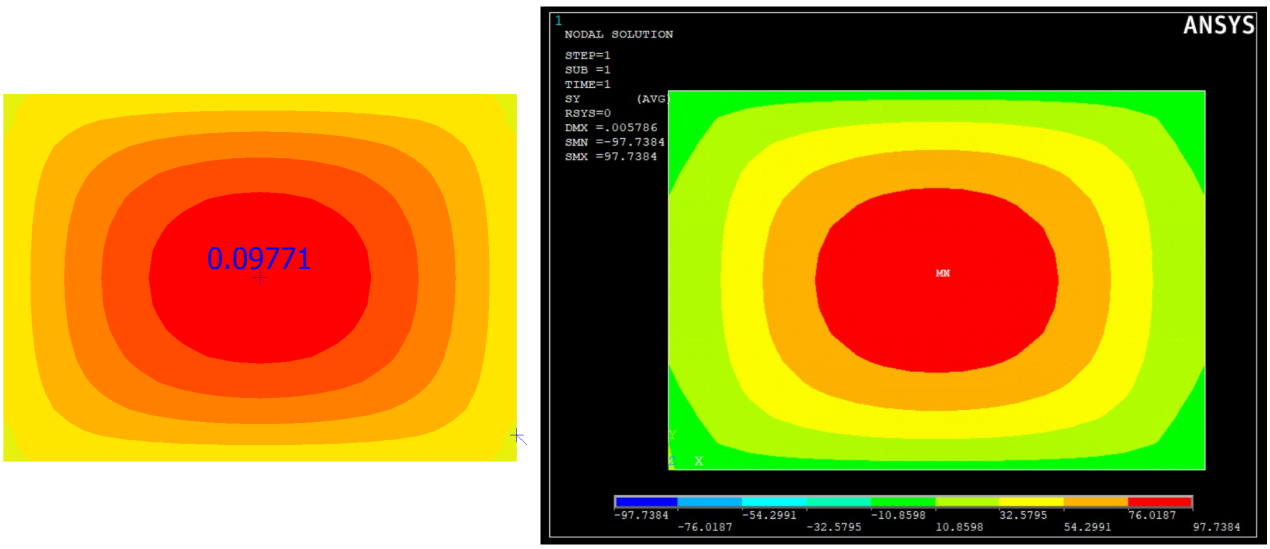


Figure 33 – The $\sigma_{y'}$ stresses at the bottom of the lowermost layer in FEM-Design (left, in [MPa]*) and in ANSYS (right, in* [kPa]*)*

The in-plane shear stress at the bottom of the lowermost layer in shell local system:

$\tau_{x'y'bottom}^{FEM-Design} = 0.2380\,\text{MPa}$; $\tau_{x'y'bottom}^{ANSYS} = 0.2442\,\text{MPa}$

The difference between the results is about 2.5%, see Fig. 34.

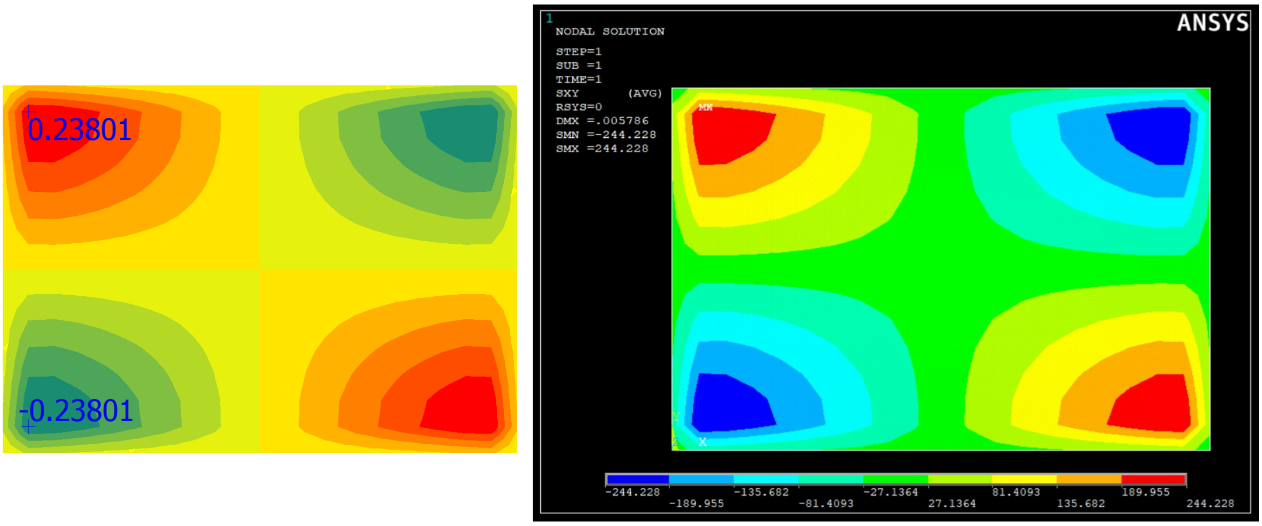


Figure 34 – The $\tau_{x'y'}$ stresses at the bottom of the lowermost layer in FEM-Design (left, in [MPa]) *and in ANSYS (right, in* [kPa])

The normal stress at the top of the second layer from above in the y' direction in shell local system:

$\sigma_{y'top}^{FEM-Design} = -1.992\,\text{MPa}$; $\sigma_{y'top}^{ANSYS} = -1.993\,\text{MPa}$

The results are the same, see Fig. 35.

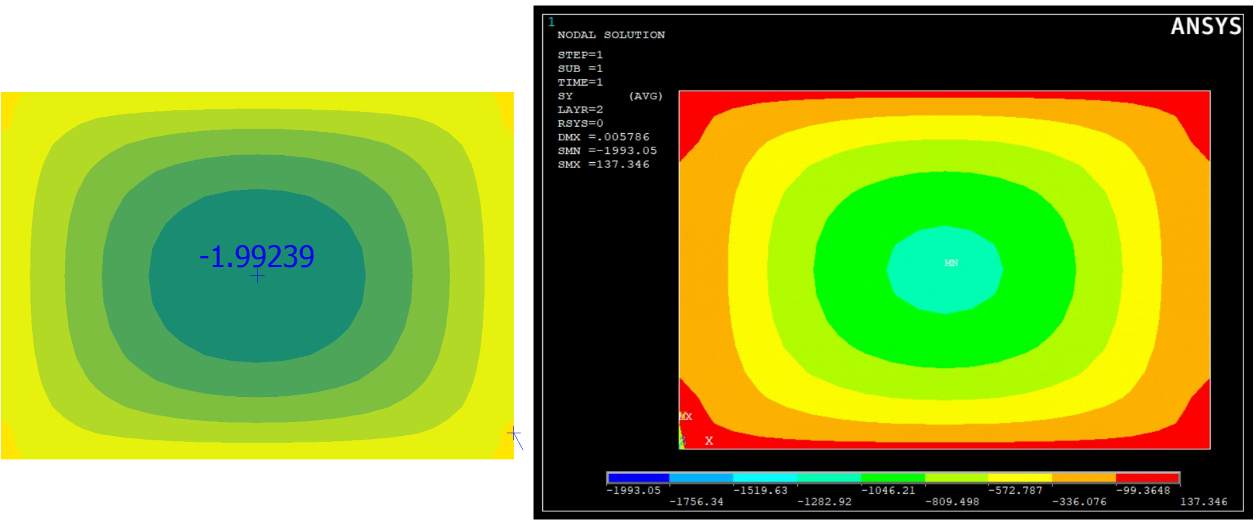


Figure 35 – The $\sigma_{y'}$ stresses at the top of the second layer counted from the top in FEM-Design (left, in [MPa]) *and in ANSYS (right, in* [kPa])

The transverse shear stresses at the middle ot the slab (center of the 4th layer) in $x'z'$ direction in shell local system:

$\tau_{x'z'middle}^{ANSYS} = 0.05865\,\text{MPa}$; $\tau_{x'z'middle}^{ANSYS} = 0.05864\,\text{MPa}$

The results are the same, see Fig. 36.

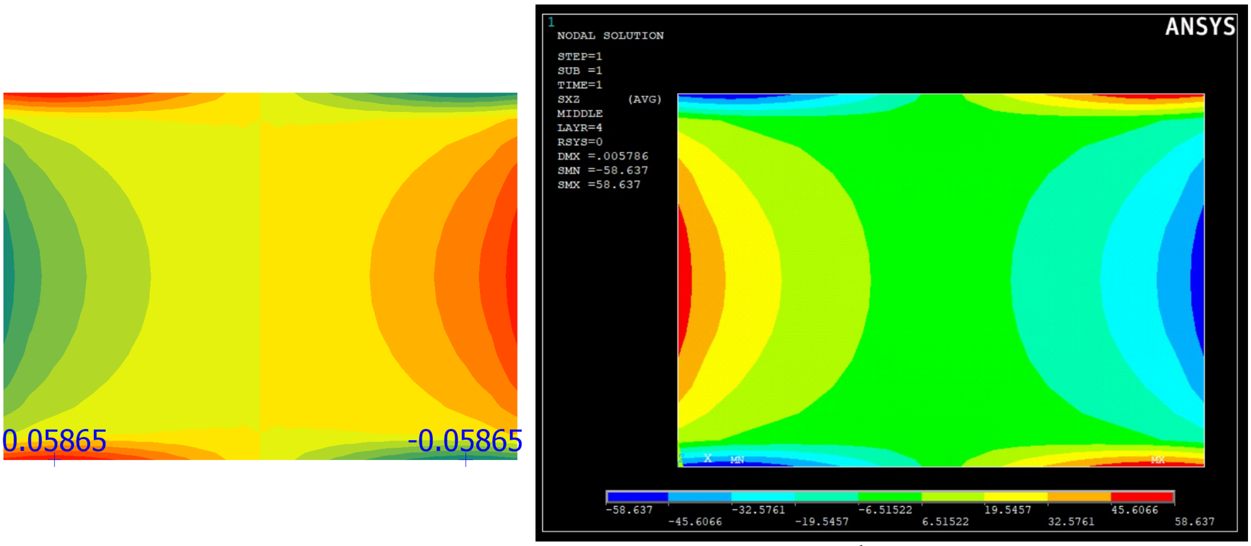


Figure 36 – The $\tau_{x'z'}$ stresses at the middle of the slab (at the center of the 4th layer) in FEM-Design (left, in [MPa]) *and in ANSYS (right, in* [kPa])

The transverse shear stresses at the middle ot the slab (center of the 4th layer) in $y'z'$ direction in shell local system:

$\tau_{y'z'middle}^{ANSYS} = 0.07181\,\text{MPa}$; $\tau_{y'z'middle}^{ANSYS} = 0.07199\,\text{MPa}$

The difference between the results is about 2.5%, see Fig. 37.

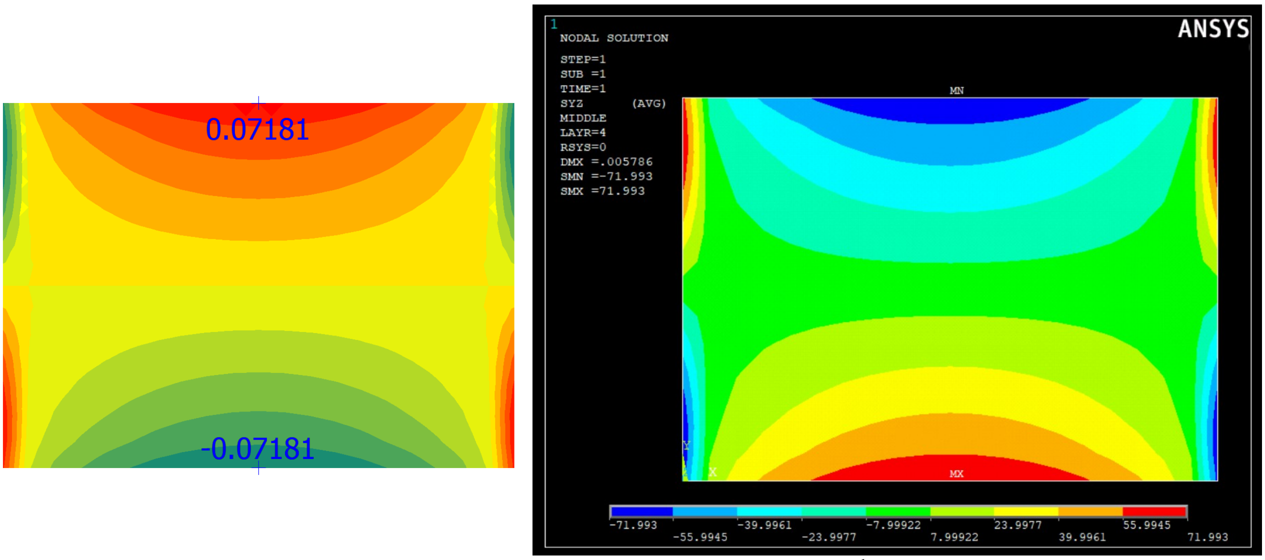


Figure 37 – The $\tau_{y'z'}$ stresses at the middle of the slab (at the center of the 4th layer) in FEM-Design (left, in [MPa]) *and in ANSYS (right, in* [kPa])

Based on the comparisons of the mentioned results we can say that FEM-Design results considering the laminated composite shell mechanical model by a two-way slab gives adequate results.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.3 Deflection and stresses of a CLT simply supported two-way slab.str

9.4.2.4 In-plane loaded CLT design check to shear failure at glued contact surface

In this example we will verify the FEM-Design design formulas in case of an in-plane loaded CLT panel according to Chapter 5.8.4 and compare the FEM-Design results with analytical/experimental data, see Ref. [40-41].

The calculated static layout can be seen in Fig. 38. The applied force is F=237 kN (thus F/2=118.5 kN, see Fig. 38. and Ref. [40]). The CLT here is an in-plane loaded panel with the following layer composition and material parameters, see Fig. 33-34. You can see in Fig. 38 that the applied plank (board) width is a = 150 mm and the number of the layers N = 3. In FEM-Design calulation we use the *No glue at narrow side* option to reach this shear failure formula by the detailed results.

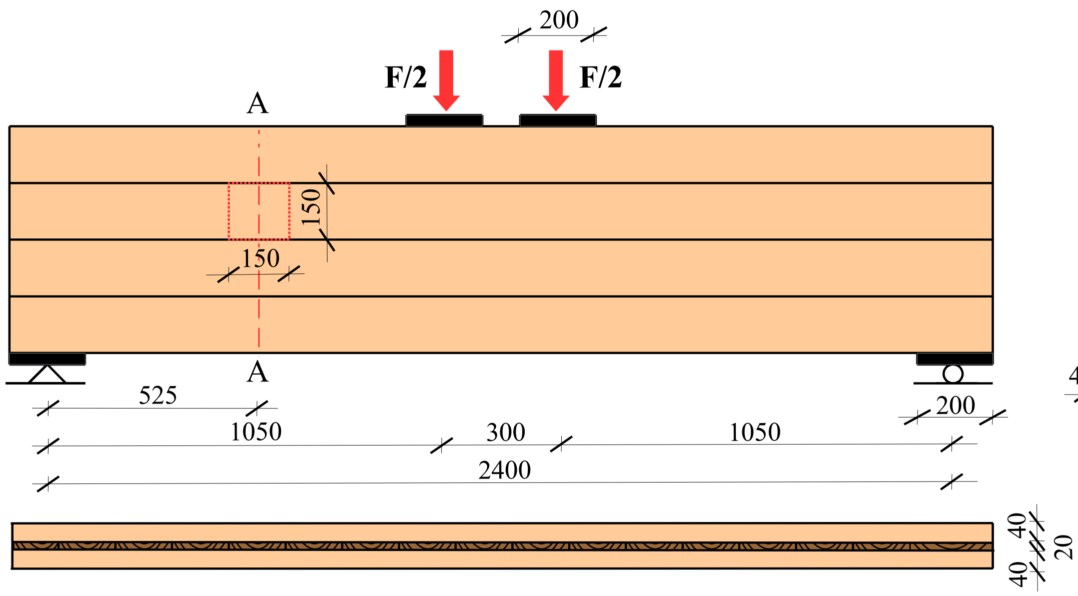


Figure 38 – The geometry, the loads and the analyzed RVE in case of in-plane loaded panel

Layers (top down order)

No	Material	Thickness [mm]	Beta [°]	Ex [N/mm2]	Ey [N/mm2]	nuxy [-]	Gxy [N/mm2]	Gxz [N/mm2]	Gyz [N/mm2]	Rho [kg/m3]
1	C24	40	0.00	12000	400	0.50	750	750	75	420
2	C24	20	90	12000	400	0.50	750	750	75	420
3	C24	40	0.00	12000	400	0.50	750	750	75	420

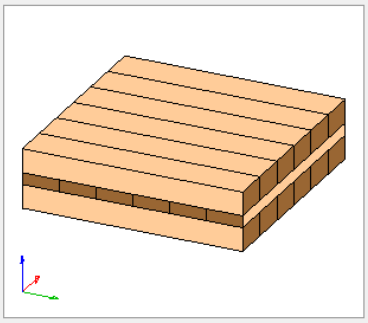


Figure 39 – The layer composition with the thicknesses and the applied material parameters

The applied material and strength parameters are according to Ref. [40] to get more comparable results in the end. Fig. 40 shows a complete model about the problem, but according to the point supports and forces in this model there is going to be singularities which prevent the comparison between the FEM-Design calculation result with the benchmark reference values, therefore only a small section including section A-A (indicated with red box in Fig. 40) was modeled and loaded with the relevant internal force system assumed as an equilibrium force system with statically determinant supports (see the downloadable sample file below).

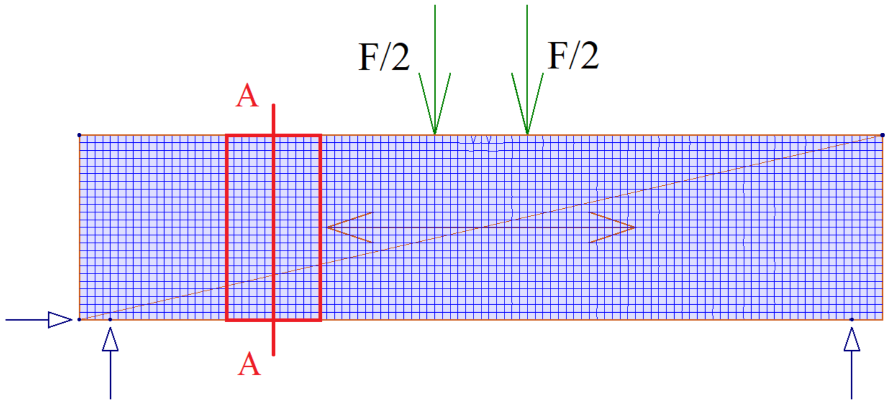


Figure 40 – The applied boundary conditions and the FE mesh

Thus the verification of failure of the glued contact surface will be performed in Section A-A, see Fig. 40.

First of all in FEM-Design the additional torsional stress at the glued surface in the mentioned RVE:

$$\tau_{tor,d}=\frac{3\,n_{xy}}{a(N-1)}=\frac{3\cdot 264.65}{0.15(3\;-1)}=2646.5\,\text{kPa}=2.647\,\text{MPa}\quad,$$

where the n_{xy} value is the in-plane specific shear force in the model, in the middle of the analyzed RVE, see Fig. 41 also.

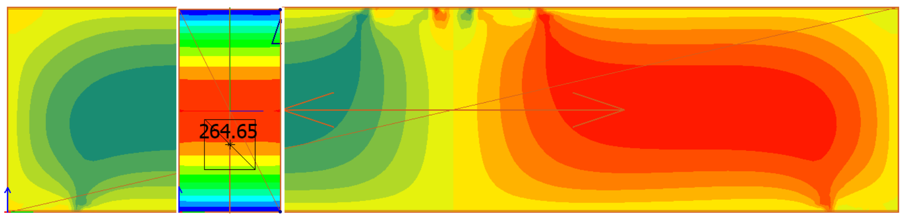
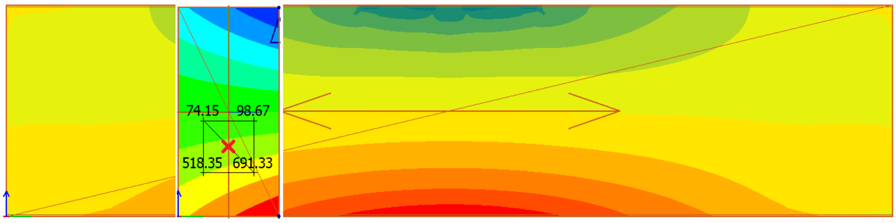


Figure 41 – The distribution of $n_{x'y'}$ specific internal forces in the local system of the shell in FEM-Design [kN/m] *the shell local x' direction is the horizontal and the shell local y' direction is the vertical one*

According to Chapter 5.8.4 the relevant failure in this example will be at the surface of the vertical middle planks. At the RVE element the $n_{x'}$ specific normal force in the shell local system (see. Fig. 42) will be the n_y value of the middle vertical board specific normal force perpendicular to the grain (see Fig 38-39).

The additional rolling shear at the glued surface in the middle vertical layer will be the following (see Fig. 42):

$$\tau_{yz,d}^{inplane}=\frac{\Delta N_y}{a^2}\approx\frac{\frac{\partial n_y}{\partial y}}{N-1}=\frac{\frac{\left|\frac{(518.3)+(74.15)}{2}-\frac{(691.3)+(98.67)}{2}\right|}{0.15}}{3-1}=329.2\,\text{kPa}=0.3292\,\text{MPa}$$



Torsion

Panel: 'TP.1.1', Layer: '2', Side: 'Top', LC: '1', k_{mod} = 0.60, PT6

$$\tau_{tor,d}=\frac{3\cdot n_{xy}}{a\cdot(N-1)}=\frac{3\cdot 264.65}{150.00\cdot(3-1)}=2.65\ \text{N/mm}^2$$

$$\tau_{yz,d,inplane}=\frac{\frac{\partial n_y}{\partial y}}{N-1}=\frac{\frac{-98.75}{150.00}}{3-1}=-0.33\ \text{N/mm}^2$$

$$\frac{|\tau_{tor,d}|}{f_{tor,d}}+\frac{|\tau_{yz,d,inplane}|+|\tau_{yz,d}|}{f_{R,d}}=\frac{|2.65|}{3.50}+\frac{|-0.33|+|0.00|}{1.50}=0.98\le 1.00\text{ - OK}$$

Figure 42 – Top: The distribution of n_x specific internal forces in the local system of the shell [kN/m] *this specific normal force if perpendicular to the middle layer grain direction*
Bottom: The relevant in-plane loaded torsion interaction formula in FEM-Design

According to the mentioned design formula in Chapter 5.8.4 the following interaction should be checked, Fig. 42 also shows this interaction formula based on FEM-Design detailed result:

$$\frac{|\tau_{tor,d}|}{f_{tor,d}}+\frac{|\tau_{yz,d}^{inplane}|}{f_{R,d}}=\frac{2.647}{3.5}+\frac{0.3292}{1.5}=0.9758\le 1.0$$

Based on Ref [40] results and the analytical theory in Ref. [41-42] the reference value of this utilization:

$$\frac{|\tau_{tor,d}|}{f_{tor,d}}+\frac{|\tau_{yz,d}^{inplane}|}{f_{R,d}}=\frac{2.686}{3.5}+\frac{0.25}{1.5}=0.9341\le 1.0$$

The literature data and the FEM-Design results are very close to each other, the difference in the interaction formula is smaller than 5 %.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.4 In-plane loaded CLT design check to shear failure at glued contact surface.str

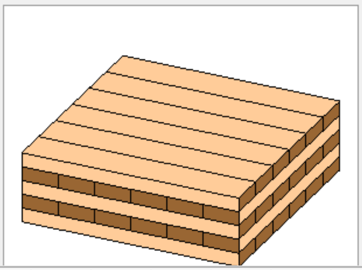
9.4.2.5 Buckling check of a CLT wall

Let's consider the following layer composition in Fig. 43, with the given elastic and strength parameters.

The load-duration class is permanent thus the applied k_{mod}=0.6 and γ_M=1.25.

Layers (top down order)

No	Material	Thickness [mm]	Beta [°]	Ex [N/mm2]	Ey [N/mm2]	nuxy [-]	Gxy [N/mm2]	Gxz [N/mm2]	Gyz [N/mm2]	Rho [kg/m3]
1	C24	20	0.00	11000	370	0.40	690	690	69	420
2	C24	20	90	11000	370	0.40	690	690	69	420
3	C24	20	0.00	11000	370	0.40	690	690	69	420
4	C24	20	90	11000	370	0.40	690	690	69	420
5	C24	20	0.00	11000	370	0.40	690	690	69	420



No	Material	fm0k [N/mm2]	fm90k [N/mm2]	ft0k [N/mm2]	ft90k [N/mm2]	fc0k [N/mm2]	fc90k [N/mm2]	fxyk [N/mm2]	fvk [N/mm2]	fRk [N/mm2]	fTork [N/mm2]
1	C24	24	24	14	0.50	21	2.5	4.0	4.0	2.0	2.5
2	C24	24	24	14	0.50	21	2.5	4.0	4.0	2.0	2.5
3	C24	24	24	14	0.50	21	2.5	4.0	4.0	2.0	2.5
4	C24	24	24	14	0.50	21	2.5	4.0	4.0	2.0	2.5
5	C24	24	24	14	0.50	21	2.5	4.0	4.0	2.0	2.5

Figure 43 – The layer composition with the thicknesses and the applied material parameters

With this CLT panel we built a wall with L height and 4 m width, applied a constant distributed load on the top edge of the wall and a surface distributed load perpendicular to the panel as you can see in Fig. 44-45. The boundary condition was simply supported. The planks are vertical at the outer layers.

In the first part (case a)) of this verification example we analyzed different height walls (L = 1,2,3,5 and 7.5 m) with only theoretically centric load (F = 543.73 kN/m; 417 kN/m; 274.84 kN/m; 123 kN/m; and 60 kN/m respectively; q = 0) on the top edge. Here the FEM-Design calculation method will be shown firstly with the mentioned buckling calculation method (Chapter 5.8.5, effective length method with reduction factor) in details. After the buckling utilizations what we get from FEM-Design calculation we show a comparison with GMNIA (geometric and material non-linear analysis with imperfections) with the aim of a structural multphysics program. Nowadays the GMNIA is the most precise engineering method to get the load bearing capacity of a structure with numerical simulation. By the verification we used in this other program solid finite elements to model more realistic structural behaviour with GMNIA.

In the second part (case b)) we show the behaviour in case of L=3m height with different vertical and horizontal loads (F = 274.84 kN/m; 252 kN/m; 234.99 kN/m; 208.33 kN/m; 171.53 kN/m; and 130.5 kN/m ; q = 0; 0.84 kN/m^2; 1.567 kN/m^2; 2.778 kN/m^2; 4.574 kN/m^2; 6.96 kN/m^2 respectively). First the FEM-Design calculation method will be shown in details. After the buckling utilizations what we get from FEM-Design calculation we show a comparison with GMNIA with the same way what was introduced in case a).

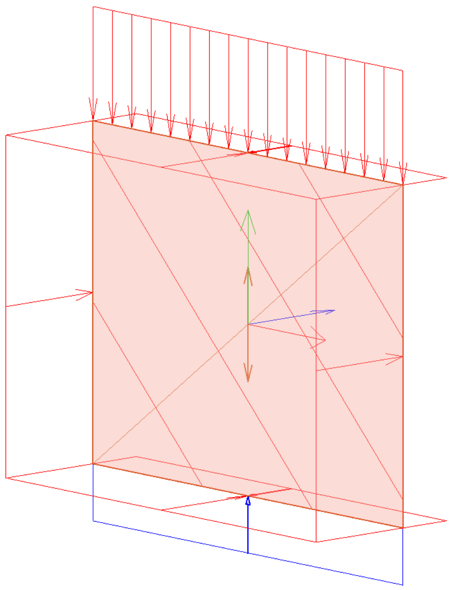


Figure 44 – The CLT wall model in FEM-Design

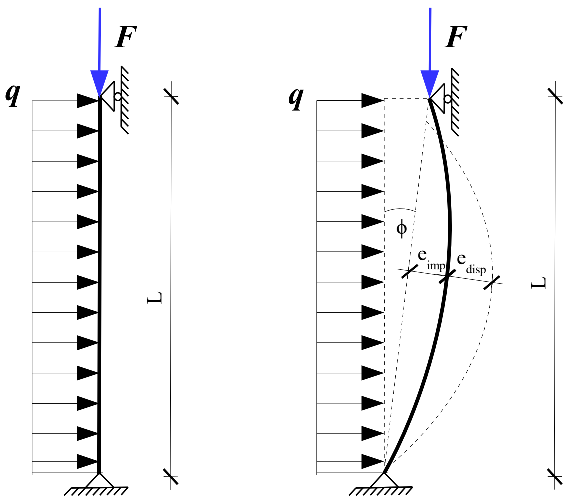


Figure 45 – The simplified static system to the effective length reduction factor method (left) and to the GMNIA (right)

Case a)

In case of centric loading (see Fig. 44-45, q=0) the analyzed height and centric loads of the walls:

L [m]	F [kN/m]	q [kN/m^2]
1	543.7	0.0
2	417.0	0.0
3	274.8	0.0
5	123.0	0.0
7.5	60.00	0.0

Based on the layers material elastic properties in Fig. 43 and with the buckling length we would like to calculate the relative slenderness and the reduction factor. The r reduction factor what was mentioned in Chapter 5.8.5 will help us to provide the characteristic values of the elastic properties to get the relative slenderness with the aim of the composite shell homogenization. The adjusted reduction factor on the elastic properties here was: r = 0.8333 (which is usually accepted by glue laminated or CLT products).

$$E_{x,05} = r\,E_{x,mean} = 0.8333 \cdot 11000 = 9170\,\mathrm{MPa} \quad ,$$

$$E_{y,05} = r\,E_{y,mean} = 0.8333 \cdot 370 = 308\,\mathrm{MPa} \quad ,$$

$$G_{xy,05} = r\,G_{xy,mean} = 0.8333 \cdot 690 = 575\,\mathrm{MPa} \quad ,$$

$$G_{xz,05} = r\,G_{xz,mean} = 0.8333 \cdot 690 = 575\,\mathrm{MPa} \quad ,$$

$$G_{yz,05} = r\,G_{yz,mean} = 0.8333 \cdot 69 = 57.5\,\mathrm{MPa} \quad .$$

From the FEM-Design laminated composite homogenization method in the background with the characteristic elastic modulus values based on Chapter 3.2 we get the following stiffness value to get the elastic critical force of the CLT panel:

$D'_{11} = 614 \dfrac{\mathrm{kNm}^2}{\mathrm{m}}$ as the relevant specific bending stiffness and

$S'_{55} = 8943\,\mathrm{kN/m}$ as the relevant specific shear stiffness.

Based on these stiffnesses we can get the elastic critical forces which depends on the buckling length of the panel. In the first part of this example we will analyze theoretically centrically loaded panels with different buckling lengths. The β factors are 1.0 because the wall was simply supported but the height of the walls are different. Thus the elastic critical forces are as follows:

$$n_{cr}^{1m} = \frac{1}{\frac{1}{\left(\frac{D'_{11}\pi^2}{(\beta L)^2}\right)} + \frac{1}{S'_{55}}} = \frac{1}{\frac{1}{\left(\frac{614\pi^2}{(1.0\cdot 1)^2}\right)} + \frac{1}{8943}} = 3612 \frac{\text{kN}}{\text{m}}$$

$$n_{cr}^{2m} = \frac{1}{\frac{1}{\left(\frac{D'_{11}\pi^2}{(\beta L)^2}\right)} + \frac{1}{S'_{55}}} = \frac{1}{\frac{1}{\left(\frac{614\pi^2}{(1.0\cdot 2)^2}\right)} + \frac{1}{8943}} = 1296 \frac{\text{kN}}{\text{m}}$$

$$n_{cr}^{3m} = \frac{1}{\frac{1}{\left(\frac{D'_{11}\pi^2}{(\beta L)^2}\right)} + \frac{1}{S'_{55}}} = \frac{1}{\frac{1}{\left(\frac{614\pi^2}{(1.0\cdot 3)^2}\right)} + \frac{1}{8943}} = 626.2 \frac{\text{kN}}{\text{m}}$$

$$n_{cr}^{5m} = \frac{1}{\frac{1}{\left(\frac{D'_{11}\pi^2}{(\beta L)^2}\right)} + \frac{1}{S'_{55}}} = \frac{1}{\frac{1}{\left(\frac{614\pi^2}{(1.0\cdot 5)^2}\right)} + \frac{1}{8943}} = 236 \frac{\text{kN}}{\text{m}}$$

$$n_{cr}^{7.5m} = \frac{1}{\frac{1}{\left(\frac{D'_{11}\pi^2}{(\beta L)^2}\right)} + \frac{1}{S'_{55}}} = \frac{1}{\frac{1}{\left(\frac{614\pi^2}{(1.0\cdot 7.5)^2}\right)} + \frac{1}{8943}} = 106.5 \frac{\text{kN}}{\text{m}}$$

Based on these elastic critical force values the relative slenderness could be calculated with the aim of the given data:

The thickness of the different layers:

$$t_1 = t_2 = t_3 = t_4 = t_5 = 20\ \text{mm} = 0.020\ \text{m}$$

The characteristic compression strength in the grain and perpendicular to grain direction:

$$f_{c,0,k,1} = f_{c,0,k,2} = f_{c,0,k,3} = 21000\ \text{kPa}$$

$$f_{c,90,k,2} = f_{c,90,k,4} = 2500\ \text{kPa}$$

The generalized relative slenderness:

$$\lambda_{rel} = \sqrt{\frac{\sum_{i=1}^{N} t_i f_{c,\alpha,k,i}}{n_{cr}}} \quad ,$$

according to this equation the relative slenderness for the different height walls are as follows.

$$\lambda_{rel}^{1\text{m}} = \sqrt{\frac{\sum_{i=1}^{N} t_i f_{c,\alpha,k,i}}{n_{cr}^{1\text{m}}}} = \sqrt{\frac{3 \cdot 0.02 \cdot 21000 + 2 \cdot 0.02 \cdot 2500}{3612}} = 0.6136$$

Similarly with the other lengths:

$$\lambda_{rel}^{2\text{m}} = 1.024 \quad ; \quad \lambda_{rel}^{3\text{m}} = 1.474 \quad ; \quad \lambda_{rel}^{5\text{m}} = 2.401 \quad ; \quad \lambda_{rel}^{7.5\text{m}} = 3.574$$

According to the used Ayrton-Perry formula in EC5 the reduction factor calculation:

$$k = 0.5\left(1 + \beta_c\left(\lambda_{rel} - 0.3\right) + \lambda_{rel}^2\right)$$

$$k_c = \frac{1}{k + \sqrt{k^2 - \lambda_{rel}^2}}$$

In these examples we used here $\beta_c = 0.1$ straightness factor, but it is adjustable in FEM-Design.

The reduction factors are as follows according to the previous equations and data:

$$k_c^{1\text{m}} = 0.9534 \quad ; \quad k_c^{2\text{m}} = 0.7483 \quad ; \quad k_c^{3\text{m}} = 0.4210 \quad ; \quad k_c^{5\text{m}} = 0.1662 \quad ; \quad k_c^{7.5\text{m}} = 0.07617$$

The table below summarizes the calculated values by the different cases.

$\beta_c = 0.1$				
L [m]	n_{cr} [kN/m]	λ_{rel} [-]	k [-]	k_c [-]
1	3612	0.6136	0.7039	0.9534
2	1296	1.024	1.060	0.7483
3	626.2	1.474	1.645	0.4210
5	236.0	2.401	3.487	0.1662
7.5	106.5	3.574	7.050	0.07617

To calculate the buckling utilizations it is necessary to define the design values of the compressive and bending strengths. The design values of the compressive and bending strength of the layers:

$$f_{c,0,d} = k_{mod} \frac{f_{c,0,k}}{\gamma_M} = 0.6 \frac{21}{1.25} = 10.08\,\text{MPa}$$

$$f_{m,0,d} = k_{mod} \frac{f_{m,0,k}}{\gamma_M} = 0.6 \frac{24}{1.25} = 11.52\,\text{MPa}$$

Figure 46 – The relevant normal stress distribution in the five different height walls with the given centric loads

Fig. 46. shows the relevant stress results layer-by-layer from FEM-Design in the CLT panels with the given heights and loads.

With the calculated reduction factors the buckling utilization of the different height walls with the different centric loads (based on Fig. 46) in the grain directions of the relevant layers:

$$Util^{1m}=\frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}}+\frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}}=\frac{\left|-8.866\right|}{0.9534\cdot 10.08}+\frac{0}{11.52}=0.9226$$

$$Util^{2m}=\frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}}+\frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}}=\frac{\left|-6.799\right|}{0.7483\cdot 10.08}+\frac{0}{11.52}=0.9014$$

$$Util^{3m}=\frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}}+\frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}}=\frac{\left|-4.481\right|}{0.4210\cdot 10.08}+\frac{0}{11.52}=1.056$$

$$Util^{5m}=\frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}}+\frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}}=\frac{\left|-2.006\right|}{0.1662\cdot 10.08}+\frac{0}{11.52}=1.197$$

$$Util^{7.5m}=\frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}}+\frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}}=\frac{\left|-0.9783\right|}{0.07617\cdot 10.08}+\frac{0}{11.52}=1.274$$

After we have these utilization values we have made GMNIA analysis with an other software to make a comparison between the proposed practically usable effective length reduction factor method what was implemented in FEM-Design and one of the most precise numerical simulation method.

In the GMNIA the used finite elements were solid elements and the material model was orthotropic with the given nonlinear stress-strain models in Fig. 47 layer-by-layer.

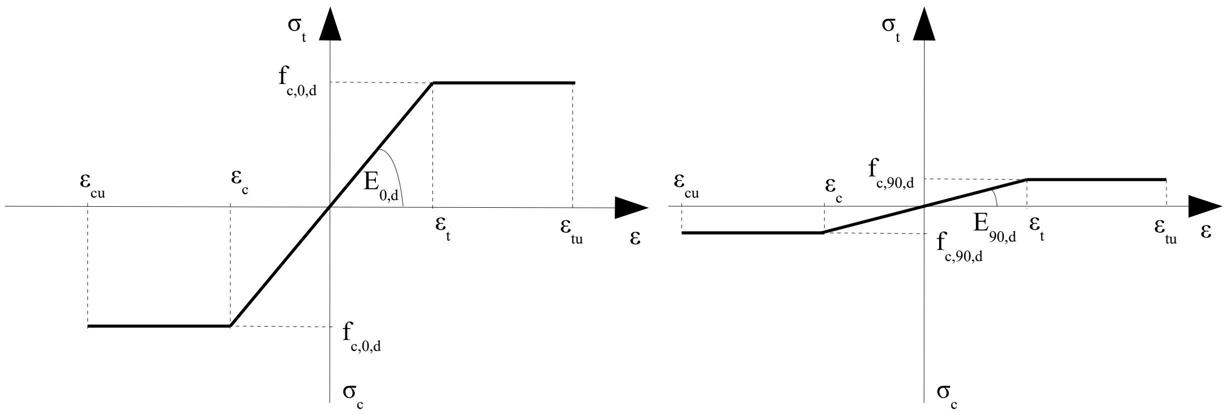


Figure 47 – The considered non-linear stress-strain model by GMNIA

In the GMNIA the adjusted elastic moduli were the following according to Ref. [41]:

$$E_{0,d} = 0.8\frac{E_{x,05}}{\gamma_M} = 0.8\frac{0.84 \cdot 11000}{1.25} = 5914\,\text{MPa}$$

$$E_{90,d} = 0.8\frac{E_{y,05}}{\gamma_M} = 0.8\frac{0.84 \cdot 370}{1.25} = 198.9\,\text{MPa}$$

$$G_{xy,d} = 0.8\frac{G_{xy,05}}{\gamma_M} = 0.8\frac{0.84 \cdot 690}{1.25} = 370.9\,\text{MPa}$$

$$G_{xz,d} = 0.8\frac{G_{xz,05}}{\gamma_M} = 0.8\frac{0.84 \cdot 690}{1.25} = 370.9\,\text{MPa}$$

$$G_{yz,d} = 0.8\frac{G_{yz,05}}{\gamma_M} = 0.8\frac{0.84 \cdot 69}{1.25} = 37.09\,\text{MPa}$$

The plastic limit values can be seen in Fig. 47, despite of the behaviour is symmetric in tension and compression the calculation is adequate because the analyzed cases are mainly under compression and on the tensioned side the values were not above the tension plastic limit at the failure load of GMNIA. It is important to note because it is very well-known that the timber material shows brittle fracture under pure tension, see Ref. [41].

By GMNIA another important question is the right consideration of the imperfections.

First of all a global initial sway was considered, see Fig. 45 right side. The initial sway according to the Eurocode was:

$\phi=\alpha_h \frac{1}{200}$; where $\alpha_h=\frac{2}{\sqrt{L}}$ but $2/3 \le \alpha_h \le 1.0$ where L is the wall height.

Secondly it is necessary to consider an initial bow imperfection with its design value. This value is very important because it should contain not only the geometrical bow imperfection of the member but also other imperfections such as the material and load (eccentricity). According to the EC5 and Ref. [46-47] an adequate design value of the initial bow imperfection is:

$e_{imp}=\frac{L}{300}$, where L is the wall height.

With these given values we performed the GMNIA with an independent software based on solid finite elements.

The given centric specific loads in case a) were the load-bearing capacity of the different height walls according to the GMNIA. Fig. 48 shows the back-calculated reduction factor values based on GMNIA.

Based on the fact that the reduction factor curves in Eurocode comes from stochastic and empiric calculation we can say that the results of the proposed FEM-Desing buckling check method are very close to the GMNIA. The results are closer to each other if we consider that the practically relevant relative slenderness is up to around 2.0 here.

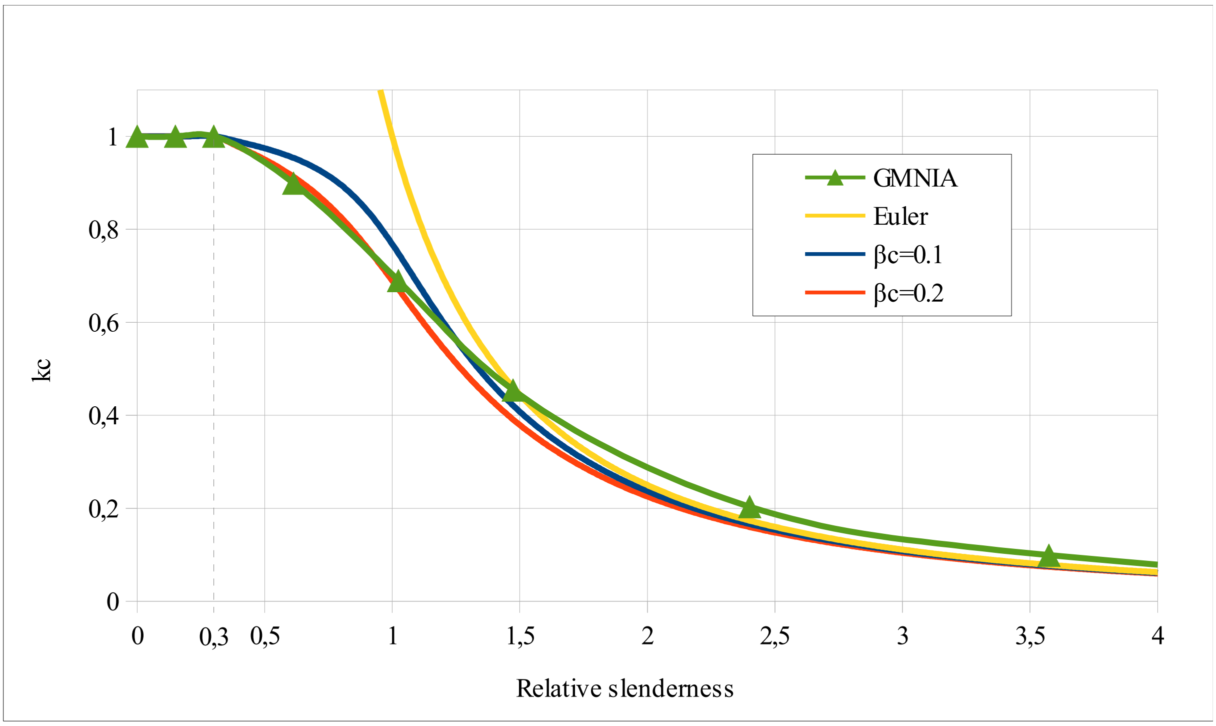


Figure 48 – The results of GMNIA in centic compression case compared to the proposed FEM-Design method

Fig. 49 shows the load-displacement curves from GMNIA. It is worth noting that by the first two short wall height (1 and 2 m) the load-displacement curves are almost straight, which means that the failure is close to the strength failure as expected. By the L = 3 m according to the results the failure is a mixture between the stability and strength failure. By the last two cases (5 and 7.5 m) the failure mode is stability failure which is obvious from Fig. 49.

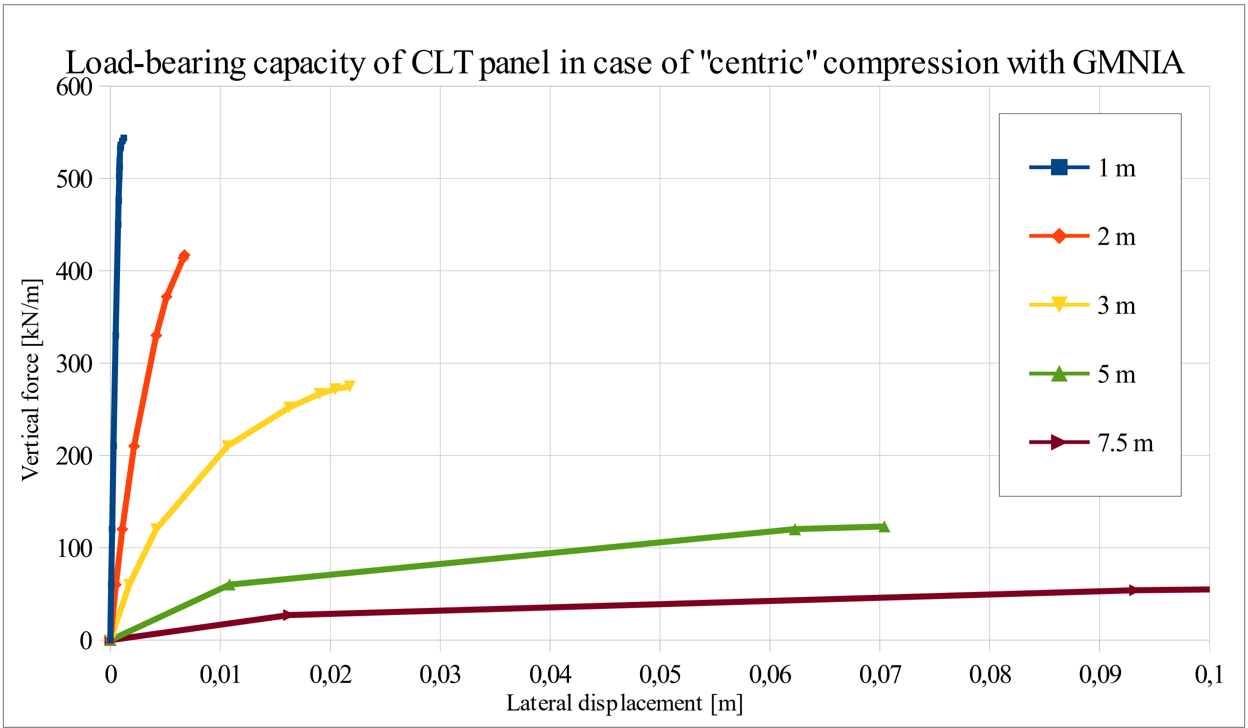


Figure 49 – The results of GMNIA – The applied vertical force versus the lateral displacement of the mid-section of the wall

Case b)

In this case the height of the wall was constant, but we increased the lateral loads on the wall as well.

The analyzed cases can be seen in the table below where we indicated the applied height of the wall and furthermore the applied vertical and later loads (see Fig. 44-45).

L [m]	F [kN/m]	q [kN/m^2]
3	274.8	0.0
3	252.0	0.840
3	235.0	1.567
3	208.3	2.778
3	171.5	4.574
3	130.5	6.960

Fig. 50 shows the relavant stress distribution in the walls under the different load conditions at the mid-section of the walls.

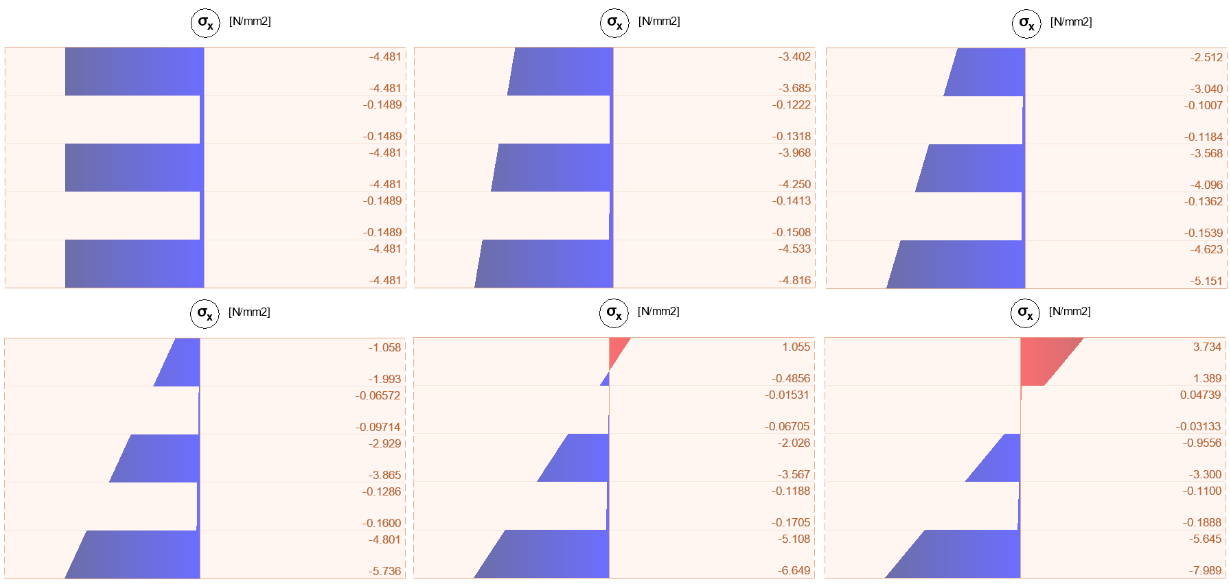


Figure 50 – The relevant normal stress distribution in the six different cases with the given centric and lateral loads

In case a) we calculated based on the proposed FEM-Design method the reduction factor to the 3 m height wall, thus we used that value by these calculations as well. The buckling utilizations of the walls with the different vertical and lateral loads by the six different load situations are as follows based on Fig. 45 and the mentioned stress separation in Chapter 5.8.5.4:

$$Util^1 = \frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}} + \frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}} = \frac{\left|-4.481\right|}{0.4210 \cdot 10.08} + \frac{0}{11.52} = 1.056$$

$$Util^2 = \frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}} + \frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}} = \frac{\left|-4.109\right|}{0.4210 \cdot 10.08} + \frac{0.707}{11.52} = 1.030$$

$$Util^3 = \frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}} + \frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}} = \frac{\left|-3.832\right|}{0.4210 \cdot 10.08} + \frac{1.319}{11.52} = 1.017$$

$$Util^4 = \frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}} + \frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}} = \frac{\left|-3.397\right|}{0.4210 \cdot 10.08} + \frac{2.339}{11.52} = 1.004$$

$$Util^5 = \frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}} + \frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}} = \frac{\left|-2.797\right|}{0.4210 \cdot 10.08} + \frac{3.853}{11.52} = 0.9936$$

$$Util^6 = \frac{\left|\sigma_{c,0,d}^{buckling}\right|}{k_c f_{c,0,d}} + \frac{\left|\sigma_{m,0,d}^{buckling}\right|}{f_{m,0,d}} = \frac{\left|-2.128\right|}{0.4210 \cdot 10.08} + \frac{5.861}{11.52} = 1.010$$

To verify these utilizations we performed GMNIA with the same input parameters what were used in case a) but we modified the loads. The given load situations in case b) were the load-bearing capacities of the analyzed walls based on GMNIA. Fig. 51 shows the load-displacement curves of the analyzed walls. Do not forget that by these vertical loads there were simultaneously increasing lateral loads as well on the walls during the calculation.

According to these information we can say that the utilizations of FEM-Design proposed calculation are very close to GMNIA results. We can conclude that FEM-Design provides an adequate stability calculation method in its new CLT modul design calculation.

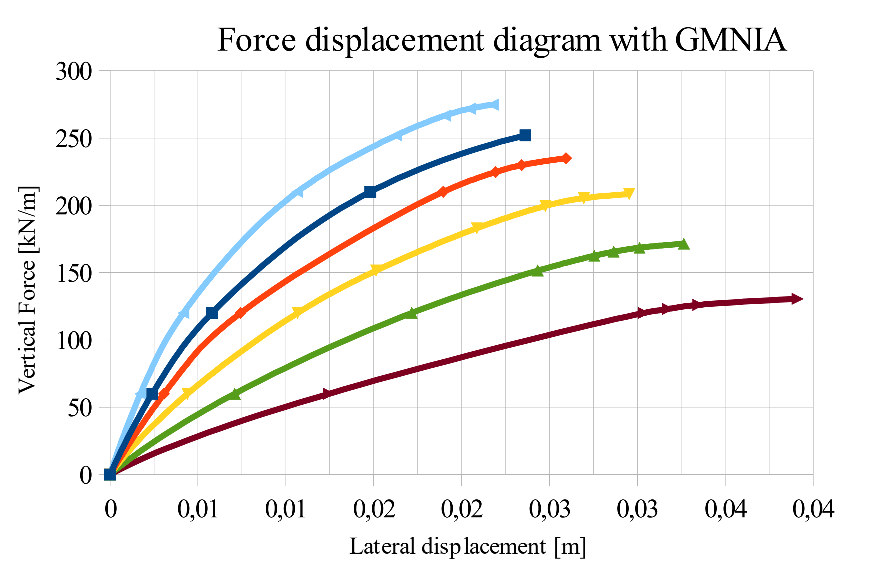


Figure 51 – The results of GMNIA – The applied vertical force versus the lateral displacement of the mid-section of the wall

Download link to the example file:

Case a):

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.5 Buckling check of a CLT wall case a.str

Case b):

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.5 Buckling check of a CLT wall case b.str

9.5 Automatic calculation of flexural buckling length

9.5.1 Concrete frame building

In this example we will calculate the buckling lengths of the indicated isolated columns (C.1; C.4, see Fig. 9.5.1.1) according to EN 1992-1-1:2004 Chapter 5.8.3.2. After the hand calculation we will compare the results with the FEM-Design automatic buckling length calculation results.

The geometry is shown in Fig. 9.5.1.1. The material is C25/30 the columns have 300/300 mm, the beams have 300/500 mm cross-sections. We will calculate the buckling lengths of the middle isolated columns at the ground floor and at the first floor. <u>The supports are fixed at the bottom of the ground floor columns.</u>

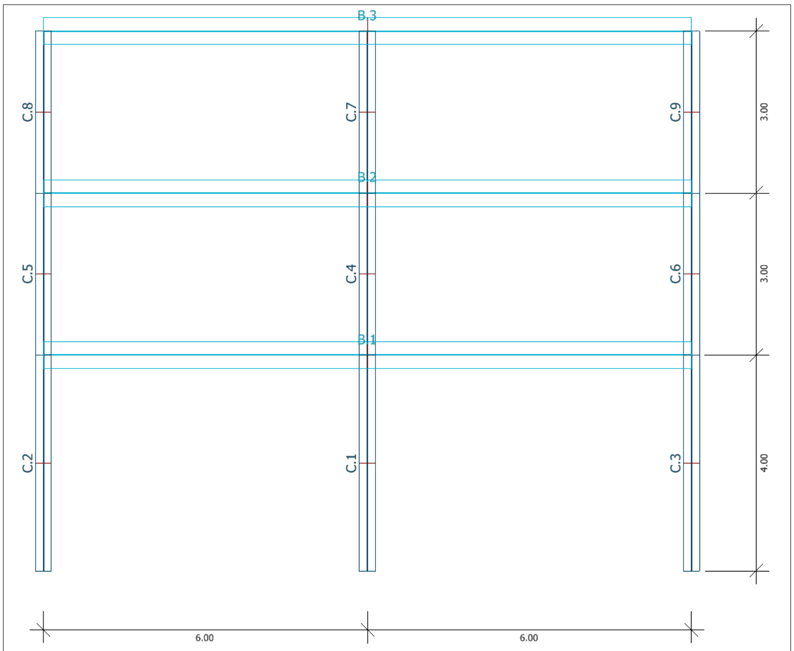


Figure 9.5.1.1 – The concrete planar frame geometry

9.5.1.1 Non-sway case

If the frame is a non-sway frame the method according to EN 1992-1-1:2004 Chapter 5.8.3.2. is the following:

The bending stiffness of the columns:

$$EI_c = 31000000 \cdot \frac{0.3^4}{12} = 20925\,\text{kNm}^2$$

The bending stiffness of the beams in the relevant direction:

$$EI_c = 31000000 \cdot \frac{0.3 \cdot 0.5^3}{12} = 96875\,\text{kNm}^2$$

C.1 column (see Fig. 9.5.1.1):

The distribution factors:

At bottom:

$k_1 = 0$ (fixed support);

At top:

$$k_2 = \frac{(EI_c / L_{c4})_{\text{above}} + (EI_c / L_{c1})_{\text{below}}}{\sum c\, EI_b / L_b} = \frac{\frac{20925}{3} + \frac{20925}{4}}{2\frac{96875}{6} + 2\frac{96875}{6}} = 0.189$$

By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

The beta factor of the buckling length:

$$\beta_1 = \frac{L_{cr1}}{L_{c1}} = 0.5 \cdot \sqrt{\left(1 + \frac{k_1}{0.45 + k_1}\right)\left(1 + \frac{k_2}{0.45 + k_2}\right)} = 0.5 \cdot \sqrt{\left(1 + \frac{0}{0.45 + 0}\right)\left(1 + \frac{0.1896}{0.45 + 0.189}\right)} = 0.569$$

C.4 column (see Fig. 9.5.1.1):

The distribution factors:

At bottom:

$$k_1 = \frac{(EI_c / L_{c4})_{\text{above}} + (EI_c / L_{c1})_{\text{below}}}{\sum c\, EI_b / L_b} = \frac{\frac{20925}{3} + \frac{20925}{4}}{2\frac{96875}{6} + 2\frac{96875}{6}} = 0.189$$

By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

At top:

$$k_2 = \frac{(EI_c / L_{c7})_{\text{above}} + (EI_c / L_{c4})_{\text{below}}}{\sum c\, EI_b / L_b} = \frac{\frac{20925}{3} + \frac{20925}{3}}{2\frac{96875}{6} + 2\frac{96875}{6}} = 0.216$$

By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

The beta factor of the buckling length:

$$\beta_4 = \frac{L_{cr4}}{L_{c4}} = 0.5 \cdot \sqrt{\left(1 + \frac{0.189}{0.45 + 0.189}\right)\left(1 + \frac{0.216}{0.45 + 0.216}\right)} = 0.655$$

Based on FEM-Design auto buckling length calculation method the results are:

$\beta_{1FEM} = 0.576$

$\beta_{4FEM} = 0.663$

The difference between the calculations is less than 1.5%. Fig. 9.5.1.2 shows the results based on FEM-Design.

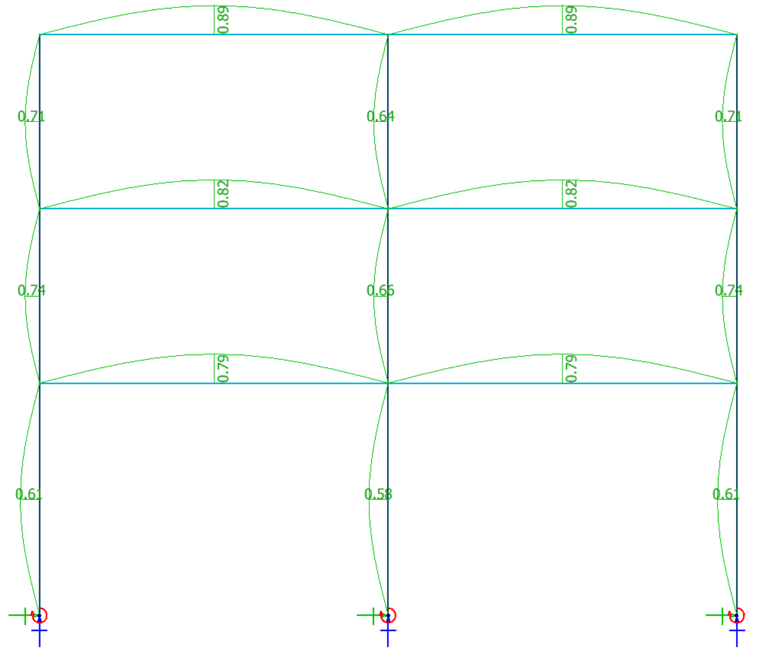


Figure 9.5.1.2 – The buckling lengths of the columns in non-sway case

9.5.1.2 Sway case

If the frame is a sway frame the method according to EN 1992-1-1:2004 Chapter 5.8.3.2. is the following:

C.1 column (see Fig. 9.5.1.1):

The distribution factors:

At bottom:

$k_1=0$ (fixed support);

At top:

$$k_2=\frac{(EI_c/L_{c4})_{\text{above}}+(EI_c/L_{c1})_{\text{below}}}{\sum c\,EI_b/L_b}=\frac{\frac{20925}{3}+\frac{20925}{4}}{6\frac{96875}{6}+6\frac{96875}{6}}=0.063$$

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

The beta factor of the buckling length:

$$\beta_1=\frac{L_{cr1}}{L_{c1}}=\max\left[\frac{\sqrt{1+10\frac{k_1\cdot k_2}{k_1+k_2}}}{\left(1+\frac{k_1}{1+k_1}\right)\cdot\left(1+\frac{k_2}{1+k_2}\right)}\right]=\max\left[\frac{\sqrt{1+10\frac{0\cdot 0.063}{0+0.063}}}{\left(1+\frac{0}{1+0}\right)\cdot\left(1+\frac{0.063}{1+0.063}\right)}\right]=1.06$$

C.4 column (see Fig. 9.5.1.1):

The distribution factors:

At bottom:

$$k_1=\frac{(EI_c/L_{c4})_{\text{above}}+(EI_c/L_{c1})_{\text{below}}}{\sum c\,EI_b/L_b}=\frac{\frac{20925}{3}+\frac{20925}{4}}{6\frac{96875}{6}+6\frac{96875}{6}}=0.063$$

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

At top:

$$k_2=\frac{(EI_c/L_{c7})_{\text{above}}+(EI_c/L_{c4})_{\text{below}}}{\sum c\,EI_b/L_b}=\frac{\frac{20925}{3}+\frac{20925}{3}}{6\frac{96875}{6}+6\frac{96875}{6}}=0.072$$

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

The beta factor of the buckling length:

$$\beta_4 = \frac{L_{cr4}}{L_4} = \max\left[\frac{\sqrt{1+10\frac{0.063\cdot 0.072}{0.063+0.072}}}{\left(1+\frac{0.063}{1+0.063}\right)\cdot\left(1+\frac{0.072}{1+0.072}\right)}\right] = 1.156$$

Based on FEM-Design auto buckling length calculation method the results are:

$\beta_{1FEM} = 1.07$

$\beta_{4FEM} = 1.15$

The difference between the calculations is less than 1%. Fig. 9.5.1.3 shows the results based on FEM-Design.

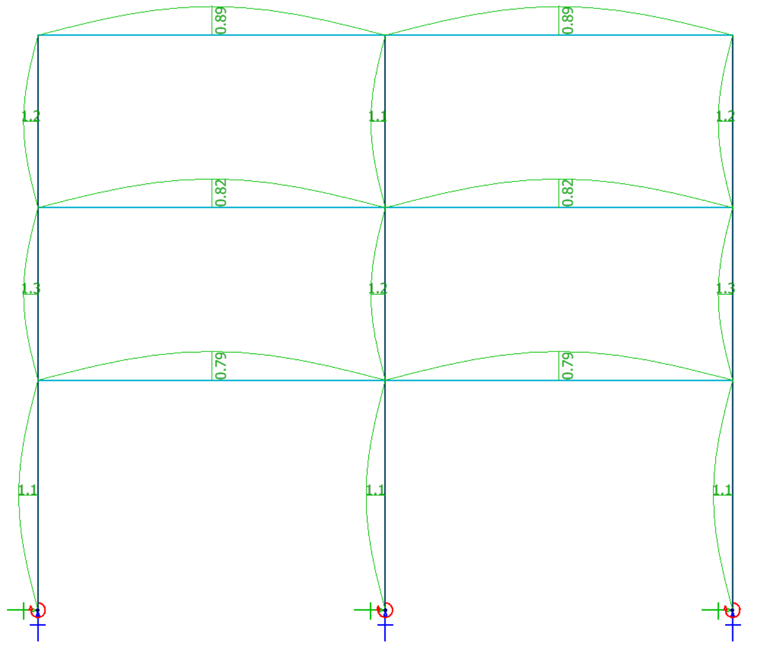


Figure 9.5.1.3 – The buckling lengths of the columns in non-sway case

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.5.1 Auto Buckling length concrete building.str

9.5.2 Steel frame building

In this example we will calculate the buckling lengths of the indicated isolated columns (C.2; C.6, see Fig. 9.5.2.1) according to the method in Ref. [17] which is basically indentical with the given method in the former ENV 1993-1-1:1992 Annex E. After the hand calculation we will compare the results with FEM-Design automatic buckling length calculation results.

The geometry is shown in Fig. 9.5.2.1. The material is S235, the outer columns have HEB220, the inner columns have HEB260, the beams have IPE450 and the beams at the roof have IPE360 cross-sections. We will calculate the buckling lengths of the middle isolated columns at the ground floor and at the first floor. The supports are hinged at the bottom of the ground floor columns.

9.5.2.1 Non-sway case

If the frame is a non-sway frame the method according to Ref. [17] is the following:

The bending stiffness of the columns (HEB260):

$$EI_c = 210000000 \cdot 0.0001492 = 31332\,\text{kNm}^2$$

The bending stiffness of the beams (IPE450):

$$EI_b = 210000000 \cdot 0.0003374 = 70854\,\text{kNm}^2$$

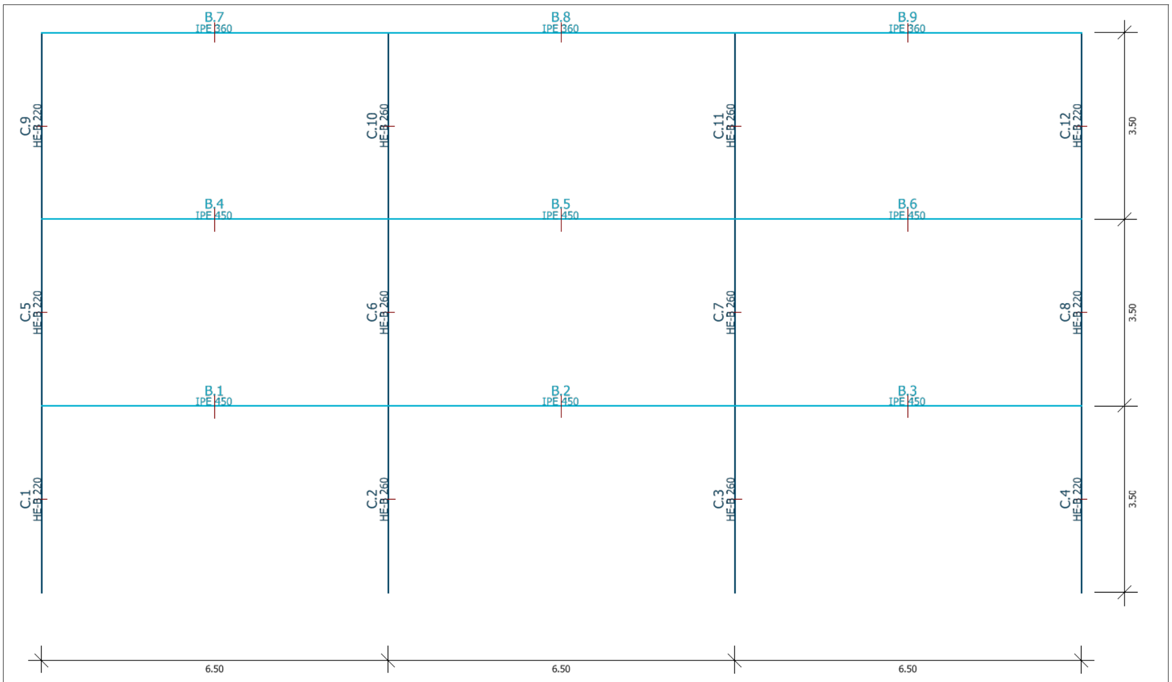


Figure 9.5.2.1 – The steel planar frame geometry

The rotational stiffness coefficient of the columns being analyzed (C.2 and C.6):

$$K_c = 4\frac{EI_c}{L_c} = 4\frac{31332}{3.5} = 35808\,\text{kNm}$$

C.2 column (see Fig. 9.5.2.1):

The distribution factors:

At bottom:

$$\eta_1=\frac{K_c}{K_c}=1.0 \quad \text{(hinged support)}$$

At top:

$$\eta_2=\frac{K_c+K_2}{K_c+K_2+K_{21}+K_{22}}=\frac{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}}{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}+2\frac{EI_b}{L_b}+2\frac{EI_b}{L_b}}=\frac{35808+35808}{35808+35808+2\frac{70854}{6.5}+2\frac{70854}{6.5}}$$

$$\eta_2=0.622$$

By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

The beta factor of the buckling length:

$$\beta_2=\frac{1+0.145(\eta_1+\eta_2)-0.265\eta_1\eta_2}{2-0.364(\eta_1+\eta_2)-0.247\eta_1\eta_2}=\frac{1+0.145(1.0+0.622)-0.265\cdot1.0\cdot0.622}{2-0.364(1.0+0.622)-0.247\cdot1.0\cdot0.622}$$

$$\beta_2=0.852$$

C.6 column (see Fig. 9.5.2.1):

The distribution factors:

At bottom:

$$\eta_1=\frac{K_c+K_1}{K_c+K_1+K_{11}+K_{12}}=\frac{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}}{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}+2\frac{EI_b}{L_b}+2\frac{EI_b}{L_b}}=\frac{35808+35808}{35808+35808+2\frac{70854}{6.5}+2\frac{70854}{6.5}}$$

$$\eta_1=0.622$$

By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

At top:

$$\eta_2=\frac{K_c+K_2}{K_c+K_2+K_{21}+K_{22}}=\frac{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}}{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}+2\frac{EI_b}{L_b}+2\frac{EI_b}{L_b}}=\frac{35808+35808}{35808+35808+2\frac{70854}{6.5}+2\frac{70854}{6.5}}$$

$$\eta_2=0.622$$

By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

The beta factor of the buckling length:

$$\beta_6=\frac{1+0.145(\eta_1+\eta_2)-0.265\eta_1\eta_2}{2-0.364(\eta_1+\eta_2)-0.247\eta_1\eta_2}=\frac{1+0.145(0.622+0.622)-0.265\cdot0.622\cdot0.622}{2-0.364(0.622+0.622)-0.247\cdot0.622\cdot0.622}$$

$\beta_6=0.743$

Based on FEM-Design auto buckling length calculation method the results are:

$\beta_{2FEM}=0.852$

$\beta_{6FEM}=0.742$

The calculations are identical to each other (see Fig. 9.5.2.2).

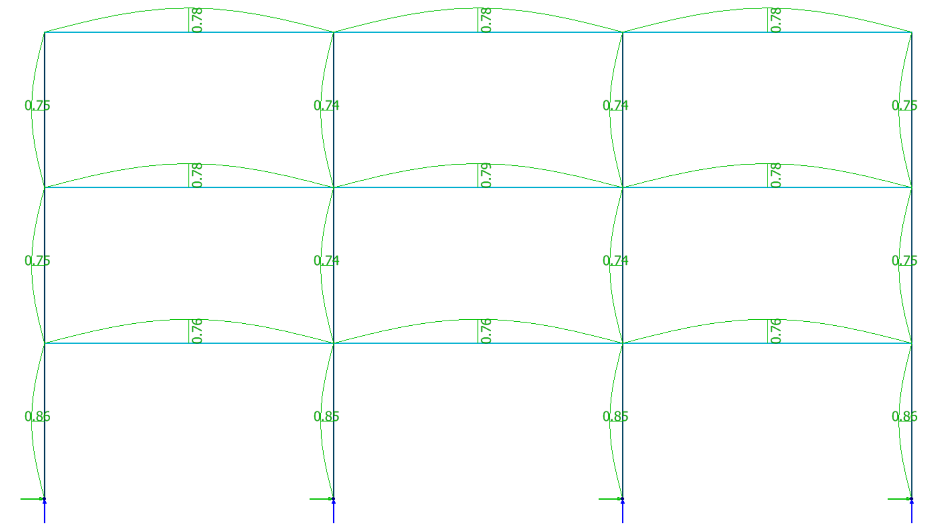


Figure 9.5.2.2 – The buckling lengths of the columns in non-sway case

9.5.2.2 Sway case

If the frame is a sway frame the method according to Ref. [17] is the following:

C.2 column (see Fig. 9.5.2.1):

The distribution factors:

At bottom:

$$\eta_1=\frac{K_c}{K_c}=1.0 \quad \text{(hinged support)}$$

At top:

$$\eta_2=\frac{K_c+K_2}{K_c+K_2+K_{21}+K_{22}}=\frac{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}}{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}+6\frac{EI_b}{L_b}+6\frac{EI_b}{L_b}}=\frac{35808+35808}{35808+35808+6\frac{70854}{6.5}+6\frac{70854}{6.5}}$$

$$\eta_2=0.354$$

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

The beta factor of the buckling length:

$$\beta_2=\sqrt{\frac{1-0.2(1.0+0.354)-0.12\cdot1.0\cdot0.354}{1-0.8(1.0+0.354)+0.6\cdot1.0\cdot0.354}}$$

$$\beta_2=2.305$$

C.6 column (see Fig. 9.5.2.1):

The distribution factors:

At bottom:

$$\eta_1=\frac{K_c+K_1}{K_c+K_1+K_{11}+K_{12}}=\frac{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}}{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}+6\frac{EI_b}{L_b}+6\frac{EI_b}{L_b}}=\frac{35808+35808}{35808+35808+6\frac{70854}{6.5}+6\frac{70854}{6.5}}$$

$$\eta_1=0.354$$

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

At top:

$$\eta_2=\frac{K_c+K_2}{K_c+K_2+K_{21}+K_{22}}=\frac{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}}{4\frac{EI_c}{L_c}+4\frac{EI_c}{L_c}+6\frac{EI_b}{L_b}+6\frac{EI_b}{L_b}}=\frac{35808+35808}{35808+35808+6\frac{70854}{6.5}+6\frac{70854}{6.5}}$$

$$\eta_2=0.354$$

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

The beta factor of the buckling length:

$$\beta_6=\sqrt{\frac{1-0.2(0.354+0.354)-0.12\cdot0.354\cdot0.354}{1-0.8(0.354+0.354)+0.6\cdot0.354\cdot0.354}}$$

$$\beta_6=1.287$$

Based on FEM-Design auto buckling length calculation method the results are:

$\beta_{2FEM} = 2.31$

$\beta_{6FEM} = 1.29$

The calculations are identical to each other (see Fig. 9.5.2.3).

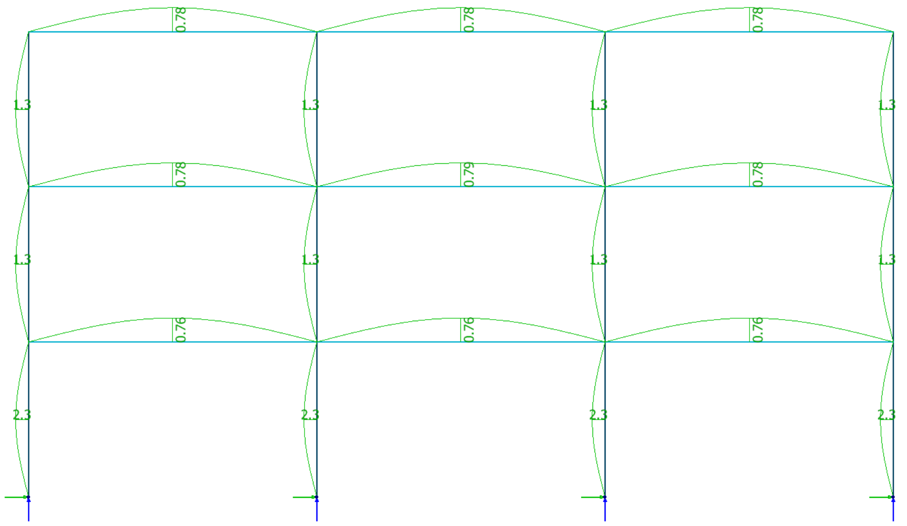


Figure 9.5.2.3 – The buckling lengths of the columns in sway case

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.5.2 Auto Buckling length steel building.str

9.5.3 A column and a supporting beam with various angles

Fig. 9.5.3.1 shows the analyzed problem. The vertical column is HEB260 and the bottom is fixed. The horizontal supporting beam is IPE360 and connected to the upper end of the column (the other end of the beam is simply supported). The angle of the connecting beam is varied between 0º-90º. The plain of the various angle beams is perpendicular to the column (see Fig. 9.5.3.1) thus the supporting beam is always horizontal.

The connecting beam rigidity has effect on the stiff and the weak buckling lengths of the column. After the calculation of the buckling lengths of the column based on the solution of the stability eigenvalue problem (stability calculation) we compared the beta factors with the FEM-Design automatic flexural buckling calculation results.

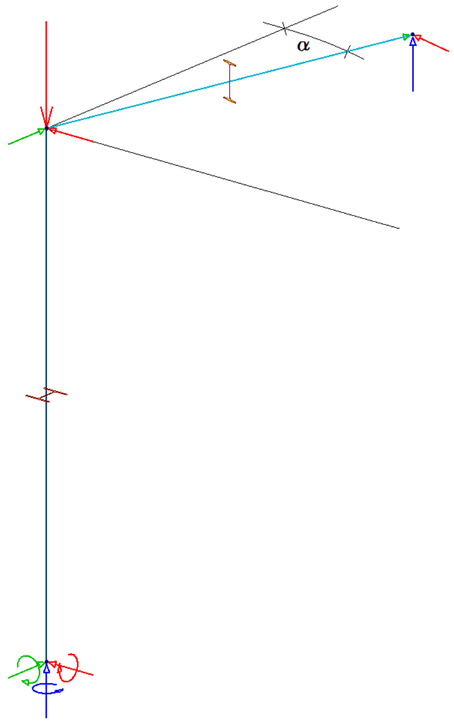


Figure 9.5.3.1 – The fixed column with various angle supporting beam

Around the stiff direction the buckling length is increasing because the supporting effect of the connecting beam is decreasing. Around the weak direction the buckling length is decreasing because the supporting effect of the connecting beam is increasing (see Fig. 9.5.3.1, Table 9.5.3.1-2).

The critical forces of the hinged-hinged column (so-called Euler force) based on the stability calculation in FEM-Design are:

$F_{cr1} = 11672\,\mathrm{kN}$ around stiff direction,

$F_{cr2} = 4230\,\mathrm{kN}$ around weak direction.

Be careful, these values contain the shear deformations and not only the deformation from bending because in FEM-Design the beam modell is the Timoshenko modell.

For example the beta factor around stiff direction based on the solution of the stability eigenvalue problem when the α angle is equal to 76°:

$$\beta_{76^o}^{stiff}=\sqrt{\frac{F_{cr1}}{F_{76^o}^{stiff}}}=\sqrt{\frac{11672}{22877}}=0.714$$

Angle	Critical load	Beta factor	Beta factor	Difference
[degree]	[kN]	Eigenvalue	AutoBucklingLength	[-]
0	30373	0,620	0,589	-0,0499
14	29948	0,624	0,592	-0,0517
27	28899	0,636	0,600	-0,0559
37	27665	0,650	0,611	-0,0593
45	26552	0,663	0,623	-0,0604
53	25410	0,678	0,638	-0,0587
63	24076	0,696	0,660	-0,0521
76	22877	0,714	0,686	-0,0396
90	22372	0,722	0,700	-0,0309

Table 9.5.3.1 – The beta factor around the stiff direction in the function of the given angle

Angle	Critical load	Beta factor	Beta factor	Difference
[degree]	[kN]	Eigenvalue	AutoBucklingLength	[-]
0	8600	0,701	0,700	-0,0019
14	8963	0,687	0,665	-0,0320
27	9811	0,657	0,616	-0,0619
37	10737	0,628	0,587	-0,0648
45	11523	0,606	0,571	-0,0576
53	12295	0,587	0,560	-0,0453
63	13171	0,567	0,551	-0,0277
76	13952	0,551	0,545	-0,0102
90	14284	0,544	0,543	-0,0022

Table 9.5.3.2 – The beta factor around the weak direction in the function of the given angle

The differences between the two calculation methods are less than 6% (see Table 9.5.3.1-2). In FEM-Design by the automatic beta factor calculation the column was assumed as a non-sway column according to the original supporting condition (see Fig. 9.5.3.1).

Fig. 9.5.3.2 shows the tendency of the beta factors in function of the supporting beam angle.

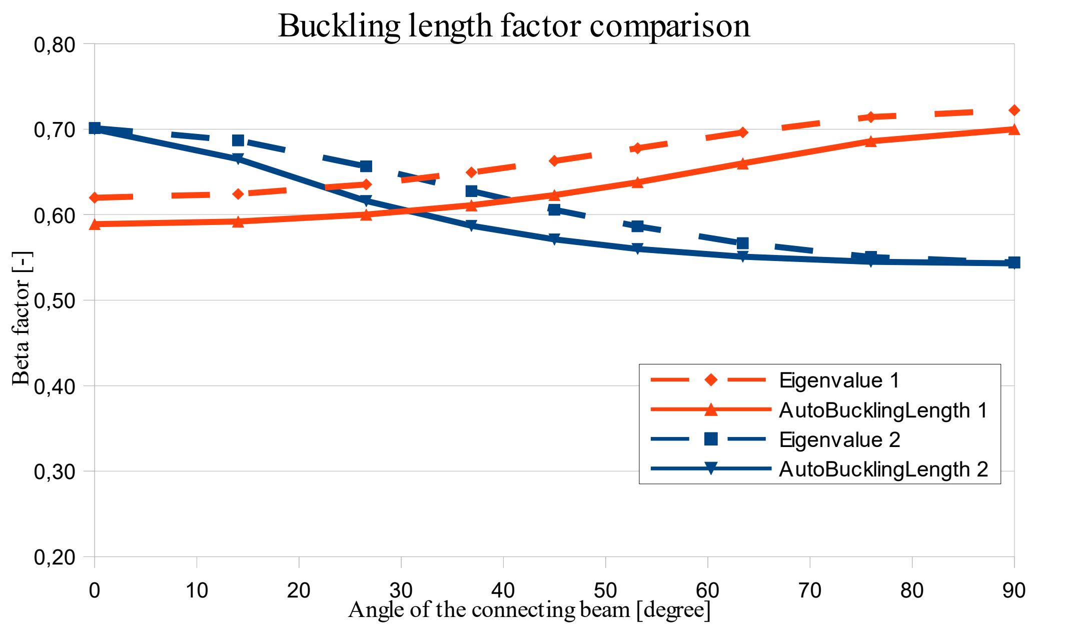


Figure 9.5.3.2 – The tendency of the beta factors (stiff 1, weak 2) in the function of the various angle

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.5.3 A column and a supporting beam with various angles.str

10 Cross section editor

10.1 Calculation of a compound cross section

An example for compound cross section is taken from [7] where the authors calculated the cross sectional properties with the *assumption of thin-walled simplifications*. The welded cross section is consisting of U300 and L160x80x12 (DIN) profiles. In the **Section Editor** the *exact cold rolled geometry* was analyzed as it is seen in Figure 8.1.1.

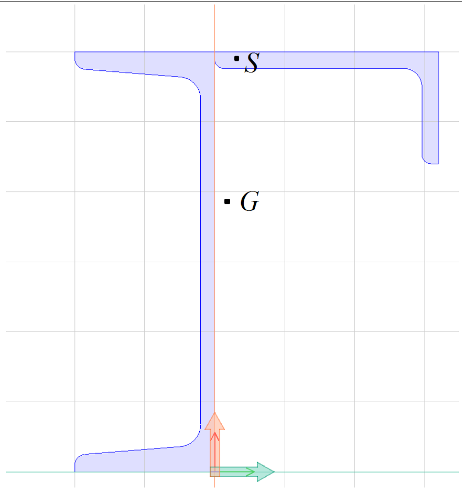


Figure 8.1.1 – The analyzed cross section

The following table contains the results of the two independent calculations with several cross sectional properties.

Notation	Ref. [1]	**Section Editor**
A [cm^2]	86.76	86.30
y_G [cm]	1.210	1.442
z_G [cm]	19.20	19.22
y'_S [cm]	1.39	0.7230
z'_S [cm]	10.06	10.36
I_y [cm^4]	11379.9	11431.2
I_z [cm^4]	4513.3	4372.9
I_{yz} [cm^4]	3013.2	3053.5
I_t [cm^4]	48.83	52.11
I_w [cm^6]	-	203082.0

Table 8.1.1 – The results of the example

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/8.1 Calculation of a compound cross section.sec

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Notes