Motto:

„There is singularity between linear and nonlinear world.”

(Dr. Imre Bojtár)
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Edited by
Zoltán I. Bocskai, Ph.D.
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In this verification handbook we highlighted the analytical results with green and the finite element results with blue background for better comparison. The analytical closed formulas are highlighted with a black frame. The comparisons between the hand calculations and FEM-Design calculations are highlighted with yellow.

If the finite element mesh is not mentioned during the example it means that the automatically generated mesh was used.

WARNING:

We are continuously developing this verification book therefore some discrepancy in the numbering of the chapters or some missing examples can occur.
1 Linear static calculations

1.1 Beam with two point loading at one-third of its span

Fig. 1.1.1 left side shows the simple supported problem. The loads, the geometry and material properties are as follows:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force</td>
<td>$F = 150$ kN</td>
</tr>
<tr>
<td>Length</td>
<td>$L = 6$ m</td>
</tr>
<tr>
<td>Cross section</td>
<td>Steel I beam HEA 300</td>
</tr>
<tr>
<td>The second moment of inertia in</td>
<td>$I_1 = 1.8264 \times 10^{-4}$ m$^4$</td>
</tr>
<tr>
<td>the relevant direction</td>
<td></td>
</tr>
<tr>
<td>The shear correction factor in</td>
<td>$\rho_2 = 0.21597$</td>
</tr>
<tr>
<td>the relevant direction</td>
<td></td>
</tr>
<tr>
<td>The area of the cross section</td>
<td>$A = 112.53$ cm$^2$</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>$E = 210$ GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>$G = 80.769$ GPa</td>
</tr>
</tbody>
</table>

The deflection of the mid-span based on the hand calculation (based on virtual force theorem [1], see Fig. 1.1.1 right side also):

$$e = \frac{2M}{EI} \left[ \frac{L}{3} \cdot \frac{2}{3} \cdot \frac{L}{4} + \frac{L}{6} \cdot \frac{2}{3} \cdot \frac{L}{4} + \frac{L}{6} \cdot \frac{1}{3} \cdot \frac{L}{4} \right] + \frac{2F}{\rho GA} \left[ 0.5 \cdot \frac{L}{3} \right] = \frac{23}{648} \frac{FL^3}{EI} + \frac{FL}{3 \rho GA}$$

$$e = \frac{23}{648} \cdot \frac{150 \cdot 6^3}{210000000 \cdot 1.8264 \times 10^{-4}} + \frac{150 \cdot 6}{3 \cdot 0.21597 \cdot 80769000 \cdot 0.011253} = 0.03151\ m = 31.51\ mm$$
The first part of this equation comes from the bending deformation and the second part comes from the consideration of the shear deformation as well, because FEM-Design is using Timoshenko beam theory (see the Scientific Manual).

The deflection and the bending moment at the mid-span based on the linear static calculation with three 2-noded beam elements (Fig. 1.1.2 and Fig. 1.1.3):

\[ e_{FEM} = 31.51 \text{ mm} \quad \text{and the bending moment} \quad M_{FEM} = 300 \text{ kNm} \]

The theoretical solution in this case (three 2-noded beam elements) must be equal to the finite element solution because with three beam elements the shape functions order coincides with the order of the theoretical function of the deflection (the solution of the differential equations).

Therefore the difference between the results of the two calculations is zero.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/1.1 Beam with two point loading at one-third of its span.str
1.2 Calculation of a circular plate with concentrated force at its center

In this chapter a circular steel plate with a concentrated force at its center will be analyzed. First of all the maximum deflection (translation) of the plate will be calculated at its center and then the bending moments in the plate will be presented.

Two different boundary conditions will be applied at the edge of the plate. In the first case the edge is clamped (Case I.) and in the second case is simply supported (Case II.), see Fig. 1.2.1.

The input parameters are as follows:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>The concentrated force</td>
<td>$P = 10 \text{ kN}$</td>
</tr>
<tr>
<td>The thickness of the plate</td>
<td>$h = 0.05 \text{ m}$</td>
</tr>
<tr>
<td>The radius of the circular plate</td>
<td>$R = 5 \text{ m}$</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>$E = 210 \text{ GPa}$</td>
</tr>
<tr>
<td>The Poisson's ratio</td>
<td>$\nu = 0.3$</td>
</tr>
</tbody>
</table>

The ratio between the diameter and the thickness is $2R/h = 200$. It means that based on the geometry the shear deformation only have negligible effects on the maximum deflections. It is important because FEM-Design uses the Mindlin plate theory (considering the shear deformation, see Scientific Manual for more details), but in this case the solution of Kirchhoff's plate theory and the finite element result must be close to each other based on the mentioned ratio.

The analytical solution of Kirchhoff's plate theory is given in a closed form [2][3].
Case I.:
For the clamped case the maximum deflection at the center is:

\[ w_{cl} = \frac{PR^2}{16\pi} \left( \frac{Eh^3}{12(1-\nu^2)} \right) \]

The reaction force at the edge:

\[ Q_r = \frac{P}{2\pi R} \]

And the bending moment in the tangential direction at the edge:

\[ M_{cl} = \frac{P}{4\pi} \]

With the given input parameters the results based on the analytical and the finite element results (with the default finite element mesh size, see Fig. 1.2.2) are:

\[ w_{cl} = \frac{10.5^2}{16\pi} \left( \frac{210000000 \cdot 0.05^3}{12(1-0.3^2)} \right) = 0.002069 \text{ m} = 2.069 \text{ mm} \]

\[ Q_r = \frac{10}{2\pi 5} = 0.318 \text{ kN/m} \]

\[ M_{cl} = \frac{10}{4\pi} = 0.796 \text{ kNm/m} \]

\[ w_{clFEM} = 2.04 \text{ mm} \]

\[ Q_{rFEM} = 0.318 \text{ kN/m} \]

\[ M_{clFEM} = 0.796 \text{ kNm/m} \]

Figure 1.2.2 – The clamped (Case I.) and the simply supported (Case II.) plate with the default mesh
Case II.:

For the simply supported case the maximum deflection in the center is:

$$w_{ss} = \frac{PR^2}{16\pi} \left( \frac{Eh^3}{12(1-\nu^2)} \right) \left( \frac{3+\nu}{1+\nu} \right)$$

The reaction force at the edge:

$$Q_r = \frac{P}{2\pi R}$$

With the given input parameters the results based on the analytical and the finite element results (with the default finite element mesh size, see Fig. 1.2.2) are:

$$w_{ss} = \frac{10\cdot 5^2}{16\pi} \left( \frac{21000000\cdot 0.05^3}{12(1-0.3^2)} \right) \left( \frac{3+0.3}{1+0.3} \right) = 0.005252 \text{ m} = 5.252 \text{ mm}$$

$$Q_r = \frac{10}{2\pi 5} = 0.318 \text{ kN/m}$$

$$w_{ssFEM} = 5.00 \text{ mm}$$

$$Q_{rFEM} = 0.318 \text{ kN/m}$$

Fig. 1.2.3 shows the two deflected shape in side view. The different boundary conditions are obvious based on the two different displacement shape. The differences between the analytical solutions and finite element solutions are less than 5% but the results could be more accurate if the applied mesh is more dense than the default size.

Based on the analytical solution the bending moments in plates under the concentrated loads are infinite. It means that if more and more dense mesh will be applied the bending moment under the concentrated load will be greater and greater. Thus the following diagram and table (Fig. 1.2.4 and Table 1.2.1) shows the convergence analysis of Case I. respect to the deflection and bending moment. The deflection converges to the analytical solution ($w_{cl} = 2.07 \text{ mm}$) and the bending moment converges to infinite.
Download links to the example files:

Clamped:
http://download.strusoft.com/FEM-Design/inst170x/models/1.2 Calculation of a circular plate with concentrated force at its center clamped.str

Simply supported:
http://download.strusoft.com/FEM-Design/inst170x/models/1.2 Calculation of a circular plate with concentrated force at its center simplysup.str

Figure 1.2.4 – Convergence analysis regarding to deflection and bending moment

Table 1.2.1 – The convergence analysis

<table>
<thead>
<tr>
<th>Number of elements [pcs.]</th>
<th>Deflection [mm]</th>
<th>Bending moment [kNm/m]</th>
<th>Average element size [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>341</td>
<td>2.034</td>
<td>2.73</td>
<td>0.5</td>
</tr>
<tr>
<td>533</td>
<td>2.057</td>
<td>3.91</td>
<td>0.4</td>
</tr>
<tr>
<td>957</td>
<td>2.060</td>
<td>3.62</td>
<td>0.3</td>
</tr>
<tr>
<td>2035</td>
<td>2.068</td>
<td>4.33</td>
<td>0.2</td>
</tr>
<tr>
<td>997</td>
<td>2.072</td>
<td>5.49</td>
<td>0.1</td>
</tr>
<tr>
<td>31719</td>
<td>2.073</td>
<td>6.40</td>
<td>0.05</td>
</tr>
<tr>
<td>31772</td>
<td>2.075</td>
<td>10.70</td>
<td>Local refinement 1</td>
</tr>
<tr>
<td>31812</td>
<td>2.076</td>
<td>14.30</td>
<td>Local refinement 2</td>
</tr>
</tbody>
</table>
1.3 A simply supported square plate with uniform load

In this example a simply supported concrete square plate will be analyzed. The external load is a uniform distributed load (see Fig. 1.3.1). We compare the maximum displacements and maximum bending moments of the analytical solution of Kirchhoff's plate theory and finite element results.

The input parameters are in this table:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>The intensity of the uniform load</td>
<td>$p = 40 \text{kN/m}^2$</td>
</tr>
<tr>
<td>The thickness of the plate</td>
<td>$h = 0.25 \text{ m}$</td>
</tr>
<tr>
<td>The edge of the square plate</td>
<td>$a = 5 \text{ m}$</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>$E = 30 \text{ GPa}$</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.2$</td>
</tr>
</tbody>
</table>

The ratio between the span and the thickness is $a/h = 20$. It means that based on the geometry the shear deformation may have effect on the maximum deflection. It is important because FEM-Design uses the Mindlin plate theory (considering the shear deformation, see Scientific Manual for more details), therefore in this case the results of Kirchhoff's theory and the finite element result could be different from each other due to the effect of shear deformations.

Based on Kirchhoff's plate theory [2][3] the maximum deflection is in the center of the simply supported square plate and its intensity can be given with the following closed form:

$$w_{max} = 0.00416 \frac{pa^4}{Eh^3 \left(1 - \nu^2\right)}$$
The maximum bending moment in the plate if the Poisson's ratio $v = 0.2$:

$$M_{\text{max}} = 0.0469 \ p \ a^2$$

According to the input parameters and the analytical solutions the results of this problem are the following:

The deflection at the center of the plate:

$$w_{\text{max}} = 0.00416 \left( \frac{40 \cdot 5^4}{3000000 \cdot 0.25^2} \right) = 0.002556 \text{ m} = 2.556 \text{ mm}$$  \[ w_{\text{max}} = 0.002556 \text{ m} = 2.556 \text{ mm} \]

$$w_{\text{maxFEM}} = 2.632 \text{ mm}$$

The bending moment at the center of the plate:

$$M_{\text{max}} = 0.0469 \cdot 40 \cdot 5^2 = 46.9 \text{ kNm}$$  \[ M_{\text{max}} = 46.9 \text{ kNm} \]

$$M_{\text{maxFEM}} = 45.97 \text{ kNm}$$

Next to the analytical solutions the results of the FE calculations are also indicated (see Fig. 1.3.2 and 1.3.3). The difference is less than 3% and it also comes from the fact that FEM-Design considers the shear deformation (Mindlin plate theory).

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/1.3 A simply supported square plate with uniform load.str

Figure 1.3.2 – The deflected shape [mm] and the reaction forces [kN/m] with the default mesh
Figure 1.3.3 – The internal forces: $m_x - m_y - m_{xy}$ [kNm/m]
1.4 Peak smoothing of the bending moments in a flat slab

Let's consider a flat slab with 8m \cdot 8m raster for the supporting columns (see Fig. 1.4.1). With the aim of Ref. [16] if the flat slab field assumed to infinite (with the proper consideration of the boundary conditions (line supports on the edges)) we can get “precise” results for the bending moment with three different consideration of the supporting effect of the columns. The load is a constant distributed load (p=20 kN/m²). The thickness of the slab is 40 cm, the columns are 80cm/80cm, the Young's modulus is \( E_{cm} = 31 \) GPa, the Poisson's ratio \( \nu = 0.167 \). We neglect the creep effect.

According to Ref. [16] the first modelling condition (Type a)) for a supporting column is a vertical point support (see Fig. 1.4.2). The reaction at this point support is:

\[
R_a = p \cdot L \cdot L = 20 \cdot 8 \cdot 8 = 1280 \text{kN}
\]

According to Ref. [16] the second modelling condition (Type b)) for a supporting column is a constant distributed reaction along the cross-section of the column (see Fig. 1.4.3).
According to Ref. [16] the third modelling condition (Type c)) for a supporting column is a constant distributed reaction (along the cross-section of the column) for half of the resultant reaction force and concentrated reactions at the corner of the column with half of the resultant reaction (one concentrated reaction represent the quarter of the half of the resultant reaction) (see Fig. 1.4.4).

\[
r_c = \frac{r_b}{2} = \frac{2000}{2} = 1000 \text{ kN/m}^2 \quad ; \quad R_c = \frac{R_a}{2 \cdot 4} = \frac{1280}{8} = 160 \text{ kN}
\]
In FEM-Design we used the following settings:

Calculation was performed with “fine” finite element group. The element size was 0.25 m on the slab (we didn't use any refinement, see Fig. 1.4.1 and 1.4.6). By the peak smoothing consideration we used the following settings (see Fig. 1.4.7).
Fig. 1.4.8 shows the $m_y$ bending moment results at the section at the middle of one slab field along the indicated light blue line (see Fig. 1.4.5).

![Positive bending moment along middle section](image)

Fig. 1.4.9 shows the $m_y$ bending moment results at the section above the columns along the indicated black line (see Fig. 1.4.5).

![Negative bending moment along column section](image)
We can say that Ref. [16] results are identical with the FEM-Design results without using the peak smoothing functions in the program (see Fig. 1.4.8-9). From Fig. 1.4.8 it is obvious that the bending moment results at the section at the middle of one slab field is almost independent from the support condition type.

From Fig. 1.4.9 it is obvious that the negative moment results at the section above the columns is highly depend on the supporting condition type. The theoretical solution above the support by Type a) would be infinite if we would use infinitely small element size (see also the bending moment result in Chapter 1.2 under concentrated point load).

If we check the Higher Order Smoothing results in FEM-Design with Type a) support condition we can say that these results are close to the results from Ref. [16] support condition Type b) (see Fig. 1.4.9).

If we check the Constant Smoothing results in FEM-Design with Type a) support condition we can say that these result are very close to the results from Ref. [16] support condition Type c).

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst180x/models/1.4 Peak smoothing.str
2 Second order analysis

2.1 A column with vertical and horizontal loads

We would like to analyze the following column (see Fig. 2.1.1) with second order theory. First of all we make a hand calculation with third order theory according to Ref. [6] and [8] with stability functions. After this step we compare the results with FEM-Design. In this moment we need to consider that in FEM-Design second order analysis is implemented and the hand calculation will be based on third order theory therefore the final results won't be exactly the same. By FEM-Design calculation we splitted the column into three bar elements thus the finite element number of the bars was three for more precise results.

The input parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus</td>
<td>E = 30 GPa</td>
</tr>
<tr>
<td>Normal force</td>
<td>P = 2468 kN</td>
</tr>
<tr>
<td>Horizontal load</td>
<td>q = 10 kN/m</td>
</tr>
<tr>
<td>Cross section</td>
<td>0.2 m x 0.4 m (rectangle)</td>
</tr>
<tr>
<td>Second moment of inertia in the relevant direction</td>
<td>I₂ = 0.0002667 m⁴</td>
</tr>
<tr>
<td>Column length</td>
<td>L = 4 m</td>
</tr>
</tbody>
</table>

Figure 2.1.1 – The column with vertical and horizontal loads

According to Ref. [6] and [8] first of all we need to calculate the following assistant quantities:

\[
\rho = \frac{P}{P_E} = \frac{P}{\left(\frac{\pi^2 EI}{L^2}\right)} = \frac{2468}{\left(\frac{\pi^2 \times 30000000 \times 0.0002667}{4^2}\right)} = 0.500
\]
The constants based on this value for the appropriate stability functions:

\[ s = 3.294 \quad ; \quad c = 0.666 \quad ; \quad f = 1.104 \]

With these values the bending moments and the shear forces based on third order theory and FEM-Design calculation:

**Clamped:**

- \[ M_{\text{clamped}} = f(1+c)\frac{qL^2}{12} = 1.104(1+0.666)\frac{10\cdot4^2}{12} = 24.52 \text{kNm} \]
- \[ M_{\text{2ndFEMclamped}} = 25.55 \text{kNm} \]
- \[ M_{\text{roller}} = 0.0 \text{kNm} \]
- \[ M_{\text{2ndFEMroller}} = 0.0 \text{kNm} \]

**Roller:**

- \[ V_{\text{clamped}} = \left[1 + \frac{f(1+c)}{6}\right]\left(\frac{qL}{2}\right) = \left[1 + \frac{1.104(1+0.666)}{6}\right]\left(\frac{10\cdot4}{2}\right) = 26.13 \text{kN} \]
- \[ V_{\text{2ndFEMclamped}} = 26.38 \text{kN} \]
- \[ V_{\text{roller}} = \left[1 - \frac{f(1+c)}{6}\right]\left(\frac{qL}{2}\right) = \left[1 - \frac{1.104(1+0.666)}{6}\right]\left(\frac{10\cdot4}{2}\right) = 13.87 \text{kN} \]
- \[ V_{\text{2ndFEMroller}} = 13.61 \text{kN} \]

The differences are less than 5% between the hand and FE calculations.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/2.1 A column with vertical and horizontal loads.str
Figure 2.1.3 – The lateral translations [mm] with 1st and 2nd order theory
2.2 A plate with in-plane and out-of-plane loads

In this chapter we will analyze a rectangular plate with single supported four edges. The load is a specific normal force at the shorter edge and a lateral distributed total load perpendicular to the plate (see Fig. 2.2.1). The displacement and the bending moment are the question based on a 2nd order analysis. First of all we calculate the results with analytical solution and then we compare the results with FE calculations.

In this case the material and the geometric properties are the following:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>The thickness of the plate</td>
<td>( h = 0.05 \text{ m} )</td>
</tr>
<tr>
<td>The dimensions of the plate</td>
<td>( a = 8 \text{ m}; b = 6 \text{ m} )</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>( E = 210 \text{ GPa} )</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>( \nu = 0.3 )</td>
</tr>
<tr>
<td>The specific normal force</td>
<td>( n_x = 1000 \text{ kN/m} )</td>
</tr>
<tr>
<td>The lateral distributed load</td>
<td>( q_x = 10 \text{ kN/m}^2 )</td>
</tr>
</tbody>
</table>

\[ w_{max} = 35.38 \text{ mm} , \quad m_{x, \text{max}} = 18.05 \frac{\text{kNm}}{\text{m}} , \quad m_{y, \text{max}} = 25.62 \frac{\text{kNm}}{\text{m}} , \quad m_{xy, \text{max}} = 13.68 \frac{\text{kNm}}{\text{m}} \]

Based on Chapter 3.2 the critical specific normal force for this example is:

\[ n_{cr} = 2860 \frac{\text{kN}}{\text{m}} \]

If the applied specific normal force is not so close to the critical value (now it is lower than the...
half of the critical value) we can assume the second order displacements and internal forces based on the linear solutions with the following formulas (with blue highlight we indicated the results of the FE calculation):

\[
\begin{align*}
W_{\text{max}, \text{2nd}} &= W_{\text{max}} \left(1 - \frac{n_x}{n_{cr}}\right) = 35.38\left(1 - \frac{1000}{2860}\right) = 54.41 \text{ mm}, \\
W_{\text{max}, \text{2nd}, \text{FEM}} &= 54.69 \text{ mm}
\end{align*}
\]

\[
\begin{align*}
M_{x, \text{max}, \text{2nd}} &= M_{x, \text{max}} \left(1 - \frac{n_x}{n_{cr}}\right) = 18.05\left(1 - \frac{1000}{2860}\right) = 27.76 \text{ kN} \cdot \text{m} \\
M_{x, \text{max}, \text{FEM}} &= 28.50 \text{ kN} \cdot \text{m}
\end{align*}
\]

\[
\begin{align*}
M_{y, \text{max}, \text{2nd}} &= M_{y, \text{max}} \left(1 - \frac{n_x}{n_{cr}}\right) = 25.62\left(1 - \frac{1000}{2860}\right) = 39.40 \text{ kN} \cdot \text{m} \\
M_{y, \text{max}, \text{FEM}} &= 40.30 \text{ kN} \cdot \text{m}
\end{align*}
\]

\[
\begin{align*}
M_{xy, \text{max}, \text{2nd}} &= M_{xy, \text{max}} \left(1 - \frac{n_x}{n_{cr}}\right) = 13.68\left(1 - \frac{1000}{2860}\right) = 21.04 \text{ kN} \cdot \text{m} \\
M_{xy, \text{max}, \text{FEM}} &= 20.53 \text{ kN} \cdot \text{m}
\end{align*}
\]

The differences are less than 3% between the hand and FEM-Design calculations.

Figure 2.2.2 shows the problem in FEM-Design with the default mesh.

The following figures show the moment distribution in the plate and the displacements with FEM-Design according to 1st and 2nd order theory.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/2.2 A plate with in-plane and out-of-plane loads.str
Figure 2.2.3 – The $m_x$ [kNm/m] moment with 1st and 2nd order analysis

Figure 2.2.4 – The $m_y$ [kNm/m] moment with 1st and 2nd order analysis

Figure 2.2.5 – The $m_{xy}$ [kNm/m] moment with 1st and 2nd order analysis
Figure 2.2.6 – The vertical translations [mm] with 1st and 2nd order analysis
3 Stability analysis

3.1 Flexural buckling analysis of a beam model with different boundary conditions

| The cross section is a rectangular section | see Fig. 3.1.1 |
| The material | C 20/25 concrete |
| The elastic modulus | $E = 30$ GPa |
| The second moment of inertia about the weak axis | $I_2 = 2.667 \times 10^{-4}$ m$^4$ |
| The length of the column | $L = 4$ m |
| The boundary conditions | see Fig. 3.1.1 |

The critical load parameters according to the Euler's theory are as follows and next to the analytical solutions [1] the relevant results of the FEM-Design calculation can be seen. By the calculation we splitted the beams to five finite elements to get more accurate buckling mode shapes (see Fig. 3.1.2).

Pinned-pinned boundary condition:

$$F_{cr1} = \frac{\pi^2 E I_2}{L^2} = 4934.8 \text{ kN}$$

$$F_{crFEM1} = 4910.9 \text{ kN}$$
Verification Examples

Fixed-pinned boundary condition:

\[ F_{cr2} = \frac{\pi^2 E I_2}{(0.6992 L)^2} = 10094.1 \text{kN} \quad F_{crFEM2} = 9974.6 \text{kN} \]

Fixed-fixed boundary condition:

\[ F_{cr3} = \frac{\pi^2 E I_2}{(0.5 L)^2} = 19739.2 \text{kN} \quad F_{crFEM3} = 19318.7 \text{kN} \]

Fixed-free boundary condition:

\[ F_{cr4} = \frac{\pi^2 E I_2}{(2 L)^2} = 1233.7 \text{kN} \quad F_{crFEM4} = 1233.1 \text{kN} \]

The differences between the two calculations are less than 3% but keep in mind that FEM-Design considers the shear deformation therefore we can be sure that the Euler's results give a bit higher critical values in these cases. Fig. 3.1.2 shows the first mode shapes of the problems with the different boundary conditions.

![Figure 3.1.2 – The buckling mode shapes for different boundary conditions: pinned-pinned; fixed-pinned; fixed-fixed; fixed-free](image-url)
Download links to the example files:

http://download.strusoft.com/FEM-Design/inst170x/models/3.1 Flexural buckling analysis of a beam modell with different boundary conditions fixed-free.str

http://download.strusoft.com/FEM-Design/inst170x/models/3.1 Flexural buckling analysis of a beam modell with different boundary conditions pinned-pinned.str
3.2 Buckling analysis of a plate with shell model

In this chapter we will analyze a rectangular plate with simply supported four edges. The load is a specific normal force at the shorter edge (see Fig. 3.2.1). The critical force parameters are the questions due to this edge load, therefore it is a stability problem of a plate.

In this case the material and the geometric properties are the following:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>The thickness of the plate</td>
<td>$h = 0.05$ m</td>
</tr>
<tr>
<td>The dimensions of the plate</td>
<td>$a = 8$ m; $b = 6$ m</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>$E = 210$ GPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.3$</td>
</tr>
</tbody>
</table>

The solutions of the differential equation of the plate buckling problem are as follows [6]:

$$n_{cr} = \left( \frac{mb}{a} + \frac{n^2 a}{mb} \right)^2 \frac{\pi^2}{\left( 12 \left( 1 - \nu^2 \right) \right)} \frac{E h^3}{b^2}$$

$m = 1, 2, 3 ... \quad n = 1, 2, 3 ...$

Figure 3.2.2 shows the problem in FEM-Design with the default mesh.
According to the analytical solution the first five critical load parameters are:

\( m = 1 \), \( n = 1 \)

\[ n_{cr1} = \left( \frac{1 \cdot 1 + 1^2}{8} \right)^2 \pi^2 \left( \frac{210000000 \cdot 0.05^3}{12 \left( 1 - 0.3^2 \right)} \right) \frac{6^2}{2860.36 \text{ kN/m}} \]

\( m = 2 \), \( n = 1 \)

\[ n_{cr2} = \left( \frac{2 \cdot 1}{8} + \frac{1^2}{2} \right)^2 \pi^2 \left( \frac{210000000 \cdot 0.05^3}{12 \left( 1 - 0.3^2 \right)} \right) \frac{6^2}{3093.77 \text{ kN/m}} \]

\( m = 3 \), \( n = 1 \)

\[ n_{cr3} = \left( \frac{3 \cdot 1}{8} + \frac{1^2}{3} \right)^2 \pi^2 \left( \frac{210000000 \cdot 0.05^3}{12 \left( 1 - 0.3^2 \right)} \right) \frac{6^2}{4784.56 \text{ kN/m}} \]

\( m = 4 \), \( n = 1 \)

\[ n_{cr4} = \left( \frac{4 \cdot 1}{8} + \frac{1^2}{4} \right)^2 \pi^2 \left( \frac{210000000 \cdot 0.05^3}{12 \left( 1 - 0.3^2 \right)} \right) \frac{6^2}{7322.53 \text{ kN/m}} \]

\( m = 3 \), \( n = 2 \)

\[ n_{cr5} = \left( \frac{3 \cdot 1}{8} + \frac{2^2}{3} \right)^2 \pi^2 \left( \frac{210000000 \cdot 0.05^3}{12 \left( 1 - 0.3^2 \right)} \right) \frac{6^2}{10691.41 \text{ kN/m}} \]

Next to these values we indicated the critical load parameters what were calculated with FEM-Design.
The difference between the calculations less than 5%.

Figure 3.2.3 shows the first five stability mode shapes of the rectangular simply supported plate.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/3.2_Buckling_analysis_of_a_plate_with_shell_modell.str
3.3 Lateral torsional buckling of an I section with shell modell

The purpose of this example is calculate the lateral torsional critical moment of the following simply supported beam (see Fig. 3.3.1).

<table>
<thead>
<tr>
<th>The length of the beam</th>
<th>L = 10 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>The cross section</td>
<td>see Fig. 3.3.2</td>
</tr>
<tr>
<td>The warping constant of the section</td>
<td>$I_w = 125841 \text{ cm}^6$</td>
</tr>
<tr>
<td>The St. Venant torsional inertia</td>
<td>$I_t = 15.34 \text{ cm}^4$</td>
</tr>
<tr>
<td>The minor axis second moment of area</td>
<td>$I_z = 602.7 \text{ cm}^4$</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>$E = 210 \text{ GPa}$</td>
</tr>
<tr>
<td>The shear modulus</td>
<td>$G = 80.77 \text{ GPa}$</td>
</tr>
</tbody>
</table>

In this case the critical moment can be calculated with the following formula based on the analytical solution [6]:

$$M_{cr} = \frac{\pi^2 E I_z}{L^2} \sqrt{\frac{I_w}{I_z} + \frac{L^2 G I_z}{\pi^2 E I_z}}$$

$$M_{cr} = \frac{\pi^2 \cdot 21000 \cdot 602.7}{1000^2} \sqrt{\frac{125841}{602.7} + \frac{1000^2 \cdot 8077 \cdot 15.34}{\pi^2 \cdot 21000 \cdot 602.7}} = 4328 \text{ kNcm}$$
In FEM-Design a shell model was built to analyze this problem. The moment loads in the shell model were considered with line loads at the end of the flanges (see Fig. 3.3.3).

The supports provide the simple supported beam effects with a fork support for the shell model (see Fig. 3.3.3).

From the FEM-Design stability calculation the critical moment value for this lateral torsional buckling problem is:

\[ M_{cr,FEM} = 4363 \text{kNm} \]

The critical shape is in Fig 3.3.4. The finite element mesh size was provided based on the automatic mesh generator of FEM-Design.

The difference between the two calculated critical moments is less than 1\%. 

\[ M_{cr,FEM} = 4363 \text{kNm} \]
Download link to the example file:
FEM-Design file:
http://download.strusoft.com/FEM-Design/inst170x/models/3.3 Lateral torsional buckling of an I section with shell modell.str
Section Editor file for cross-sectional properties:
http://download.strusoft.com/FEM-Design/inst170x/models/3.3 Lateral torsional buckling of an I section with shell modell.sec
3.4 Lateral torsional buckling of a cantilever with elongated rectangle section

The purpose of this example is calculate the critical force at the end of a cantilever beam (see Fig. 3.4.1). If the load is increasing the state of the cantilever will be unstable due to lateral torsional buckling.

The input parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>The length of the beam</td>
<td>( L = 10 \text{ m} )</td>
</tr>
<tr>
<td>The cross section</td>
<td>( t = 40 \text{ mm}; h = 438 \text{ mm}; ) see Fig. 3.4.1</td>
</tr>
<tr>
<td>The St. Venant torsional inertia</td>
<td>( I_t = 8806246 \text{ mm}^4 )</td>
</tr>
<tr>
<td>The minor axis second moment of area</td>
<td>( I_z = 2336000 \text{ mm}^4 )</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>( E = 210 \text{ GPa} )</td>
</tr>
<tr>
<td>The shear modulus</td>
<td>( G = 80.77 \text{ GPa} )</td>
</tr>
</tbody>
</table>

In this case (elongated rectangle cross section with cantilever boundary condition) the critical concentrated force at the end can be calculated with the following formula based on analytical solution [ask for the reference from Support team]:

\[
P_{cr} = \frac{4.01 E I_z}{L^2 \sqrt{G I_t E I_z}}
\]

\[
P_{cr} = \frac{4.01 \times 210000 \times 2336000}{10000^2 \sqrt{80770 \times 8806246 \times 210000 \times 2336000}} = 23687 \text{ N} = 23.69 \text{ kN}
\]
In FEM-Design a shell model was built to analyze this problem. The concentrated load at the end of the cantilever was considered at the top of the beam (see Fig. 3.4.2).

With the FEM-Design stability calculation the critical concentrated force value for this lateral torsional buckling problem is:

\[ P_{cr}^{FEM} = 24.00 \text{kN} \]

The critical shape is in Fig 3.4.3. The finite element mesh size was provided based on the automatic mesh generator of FEM-Design.

The difference between the two calculated critical load parameters is less than 2%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/3.4 Lateral torsional buckling of a cantilever with elongated rectangle section.str
4 Calculation of eigenfrequencies with linear dynamic theory

4.1 Continuous mass distribution on a cantilever column

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column height</td>
<td>H = 4 m</td>
</tr>
<tr>
<td>The cross section</td>
<td>square with 0.4 m edge</td>
</tr>
<tr>
<td>The second moment of inertia</td>
<td>I = 0.002133 m$^4$</td>
</tr>
<tr>
<td>The area of the cross section</td>
<td>A = 0.16 m$^2$</td>
</tr>
<tr>
<td>The shear correction factor</td>
<td>$\rho = 5/6 = 0.8333$</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>E = 30 GPa</td>
</tr>
<tr>
<td>The shear modulus</td>
<td>G = 12.5 GPa</td>
</tr>
<tr>
<td>The specific self-weight of the column</td>
<td>$\gamma = 25$ kN/m$^3$</td>
</tr>
<tr>
<td>The mass of the column</td>
<td>m = 1.631 t</td>
</tr>
</tbody>
</table>

Based on the analytical solution [4] the angular frequencies for this case is:

$$\omega_{B_i} = \mu_{B_i} \sqrt{\frac{EI}{mH^3}}$$

if only the bending deformations are considered.

The angular frequencies are [4]:
\[
\omega_S = \mu_{si} \sqrt{\frac{\rho GA}{mH}} \; ; \; \mu_{s1} = 0.5 \pi; \mu_{s2} = 1.5 \pi; \mu_{s3} = 2.5 \pi
\]

if only the shear deformations are considered.

Based on these two equations (considering bending and shear deformation) using the Föppl theorem the angular frequency for a continuous mass distribution column is:

\[
\frac{1}{\omega_n^2} = \frac{1}{\omega_B^2} + \frac{1}{\omega_S^2}
\]

Based on the given equations the first three angular frequencies separately for bending and shear deformations are:

\[
\begin{align*}
\omega_B1 &= 3.52 \sqrt{\frac{30000000 \cdot 0.002133}{1.631 \cdot 4^3}} = 87.16 \frac{1}{s} \\
\omega_B2 &= 22.03 \sqrt{\frac{30000000 \cdot 0.002133}{1.631 \cdot 4^3}} = 545.4 \frac{1}{s} \\
\omega_B3 &= 61.7 \sqrt{\frac{30000000 \cdot 0.002133}{1.631 \cdot 4^3}} = 1527.7 \frac{1}{s} \\
\omega_S1 &= 0.5 \pi \sqrt{\frac{0.8333 \cdot 12500000 \cdot 0.16}{1.631 \cdot 4}} = 793.9 \frac{1}{s} \\
\omega_S2 &= 1.5 \pi \sqrt{\frac{0.8333 \cdot 12500000 \cdot 0.16}{1.631 \cdot 4}} = 2381.8 \frac{1}{s} \\
\omega_S3 &= 2.5 \pi \sqrt{\frac{0.8333 \cdot 12500000 \cdot 0.16}{1.631 \cdot 4}} = 3969.6 \frac{1}{s}
\end{align*}
\]

According to the Föppl theorem the resultant first three angular frequencies of the problem are:

\[
\omega_{n1} = 86.639 \frac{1}{s} \; , \; \omega_{n2} = 531.64 \frac{1}{s} \; , \; \omega_{n3} = 1425.8 \frac{1}{s}
\]

And based on these results the first three eigenfrequencies are \((f = \omega/(2\pi))\):

\[
\begin{align*}
f_{n1} &= 13.789 \frac{1}{s} \; , \; f_{n2} = 84.613 \frac{1}{s} \; , \; f_{n3} = 226.923 \frac{1}{s}
\end{align*}
\]
In FEM-Design to consider the continuous mass distribution 200 beam elements were used for the cantilever column. The first three planar mode shapes are as follows according to the FE calculation:

\[
\begin{align*}
    f_{FEM1} &= 13.780 \frac{1}{s}, \\
    f_{FEM2} &= 83.636 \frac{1}{s}, \\
    f_{FEM3} &= 223.326 \frac{1}{s}
\end{align*}
\]

The first three mode shapes can be seen in Fig. 4.1.2.

The differences between the analytical and FE solutions are less than 2%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/4.1 Continuous mass distribution on a cantilever column.str

![Figure 4.1.2 – The first three mode shapes for the cantilever with continuous mass distribution](image)
4.2 Free vibration shapes of a clamped circular plate due to its self-weight

In the next example we will analyze a circular clamped plate. The eigenfrequencies are the question due to the self-weight of the slab.

In this case the material and the geometric properties are the following:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>The thickness of the plate</td>
<td>h = 0.05 m</td>
</tr>
<tr>
<td>The radius of the circular plate</td>
<td>R = 5 m</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>E = 210 GPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>ν = 0.3</td>
</tr>
<tr>
<td>The density</td>
<td>ρ = 7.85 t/m³</td>
</tr>
</tbody>
</table>

The solution of the dynamic differential equation for the first two angular frequencies of a clamped circular plate are [5]:

\[
\omega_{nm} = \frac{\pi^2}{R^2} \beta_{nm}^2 \sqrt{\frac{E h^3}{12 \left(1-\nu^2\right)}} \rho h , \quad \beta_{10} = 1.015 , \quad \beta_{11} = 1.468
\]

Figure 4.2.1 shows the problem in FEM-Design with the clamped edges and with the default mesh.

Figure 4.2.1 – The clamped circular plate and the default finite element mesh
According to the analytical solution the first two angular frequencies are:

\[
\omega_{10} = \frac{\pi^2}{5^2} 1.015^2 \left[ \frac{210000000 \cdot 0.05^3}{12(1-0.3^2)} \right]^{\frac{1}{2}} \cdot \frac{7.85 \cdot 0.05}{7.85 \cdot 0.05} = 31.83 \frac{1}{s}, \quad f_{10} = 5.066 \frac{1}{s}, \quad f_{10FEM} = 5.129 \frac{1}{s}
\]

\[
\omega_{11} = \frac{\pi^2}{5^2} 1.468^2 \left[ \frac{210000000 \cdot 0.05^3}{12(1-0.3^3)} \right]^{\frac{1}{2}} \cdot \frac{7.85 \cdot 0.05}{7.85 \cdot 0.05} = 66.58 \frac{1}{s}, \quad f_{11} = 10.60 \frac{1}{s}, \quad f_{11FEM} = 10.731 \frac{1}{s}
\]

Based on the angular frequencies we can calculate the eigenfrequencies in a very easy way. Next to these values we indicated the eigenfrequencies which were calculated with FEM-Design.

**The differences between the calculations are less than 2%.**

Figure 4.2.2 shows the first two vibration mode shapes of the circular clamped plate.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/4.2 Free vibration shapes of a clamped circular plate due to its self-weight.str
5 Seismic calculation

5.1 Lateral force method with linear shape distribution on a cantilever

Inputs:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Column height</td>
<td>H = 10 m</td>
</tr>
<tr>
<td>The cross section</td>
<td>square with 0.4 m edge</td>
</tr>
<tr>
<td>The second moment of inertia</td>
<td>I = 0.002133 m⁴</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>E = 31 GPa</td>
</tr>
<tr>
<td>The concentrated mass points</td>
<td>10 pieces of 1.0 t (see Fig. 5.1.1)</td>
</tr>
<tr>
<td>The total mass</td>
<td>m = 10.0 t</td>
</tr>
</tbody>
</table>

First of all based on a hand calculation we determine the first fundamental period:

The first fundamental period of a cantilever column (length \( H \)) with a concentrated mass at the end (\( m \) mass) and \( EI \) bending stiffness [4]:

\[
T_i = \frac{2 \pi}{\sqrt{\frac{3 EI}{m_i H_i^3}}} \]

Figure 5.1.1 – The cantilever column with the concentrated mass points, the first vibration shape \([T=0.765 \text{ s}]\), the equivalent forces \([\text{kN}]\), the shear force diagram \([\text{kN}]\) and the bending moment diagram \([\text{kNm}]\) with FEM-Design
The fundamental period separately for the mass points from bottom to top:

\[ T_1 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.1^3}} \cdot 0.00213} = 0.0141 \text{ s}; \]
\[ T_2 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.2^3}} \cdot 0.00213} = 0.0399 \text{ s}; \]
\[ T_3 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.3^3}} \cdot 0.00213} = 0.0733 \text{ s}; \]
\[ T_4 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.4^3}} \cdot 0.00213} = 0.1129 \text{ s}; \]
\[ T_5 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.5^3}} \cdot 0.00213} = 0.1577 \text{ s}; \]
\[ T_6 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.6^3}} \cdot 0.00213} = 0.2073 \text{ s}; \]
\[ T_7 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.7^3}} \cdot 0.00213} = 0.2613 \text{ s}; \]
\[ T_8 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.8^3}} \cdot 0.00213} = 0.3192 \text{ s}; \]
\[ T_9 = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.9^3}} \cdot 0.00213} = 0.3809 \text{ s}; \]
\[ T_{10} = \frac{2\pi}{\sqrt{\frac{3.31 \times 10^8}{1.10^3}} \cdot 0.00213} = 0.4461 \text{ s}. \]

The approximated period based on these values according to the Dunkerley summary and the result of FE calculation:

\[ T_{HC} = \sqrt[10]{\sum_{i=1}^{10} T_i^2} = 0.7758 \text{ s}; \]
\[ T_{FEM} = 0.765 \text{ s}. \]

The difference between the hand calculation and FEM-Design calculation is less than 2%, for further information on the fundamental period calculation see Chapter 4.

The base shear force according to the fundamental period of vibration (see Fig. 5.1.1) and the response spectrum (see Fig. 5.1.2):

\[ F_b = S_d(T_1) \lambda m = 0.658 \cdot 10 \cdot 1.0 = 6.588 \text{ kN}. \]

We considered the response acceleration based on the period from FE calculation to get a more comparable results at the end. Thus the equivalent forces on the different point masses are:

\[ F_i = F_b \frac{z_im_i}{\sum z_jm_j} = 6.588 \frac{z_im_i}{1.1 + 2.1 + 3.1 + 4.1 + 5.1 + 6.1 + 7.1 + 8.1 + 9.1 + 10.1} = 6.588 \frac{z_im_i}{55}. \]
Verification Examples

The equivalent forces from the bottom to the top on each point mass:

\[
F_1 = 0.120 \text{kN} ; \quad F_2 = 0.240 \text{kN} ; \quad F_3 = 0.359 \text{kN} ; \quad F_4 = 0.479 \text{kN} ; \quad F_5 = 0.599 \text{kN} ;
\]

\[
F_6 = 0.719 \text{kN} ; \quad F_7 = 0.838 \text{kN} ; \quad F_8 = 0.958 \text{kN} ; \quad F_9 = 1.078 \text{kN} ; \quad F_{10} = 1.198 \text{kN}
\]

These forces are identical with the FEM-Design calculation and the shear force and bending moment diagrams are also identical with the hand calculation.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/5.1 Lateral force method with linear shape distribution on a cantilever.str
5.2 Lateral force method with fundamental mode shape distribution on a cantilever

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column height</td>
<td>H = 10 m</td>
</tr>
<tr>
<td>The cross section</td>
<td>square with 0.4 m edge</td>
</tr>
<tr>
<td>The second moment of inertia</td>
<td>I = 0.002133 m$^4$</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>E = 31 GPa</td>
</tr>
<tr>
<td>The concentrated mass points</td>
<td>10 pieces of 1.0 t (see Fig. 5.1.1)</td>
</tr>
<tr>
<td>The total mass</td>
<td>m = 10.0 t</td>
</tr>
</tbody>
</table>

The base shear force according to the fundamental period of vibration (see Fig. 5.2.1) and the response spectrum (see Fig. 5.2.2):

\[
F_b = S_d(T_1) m \lambda = 0.6588 \times 10 \times 1.0 = 6.588 \text{kN}
\]

We considered the response acceleration based on the period from FE calculation to get a more comparable results at the end. Thus the equivalent forces on the different point masses are:
The equivalent forces from the bottom to the top on each point mass:

\[ F_i = F_b \sum \frac{s_i m_i}{s_j m_j} \]

\[ = 6.588 \frac{s_i m_i}{9.5 \cdot 1 + 36.2 \cdot 1 + 77.4 \cdot 1 + 130.6 \cdot 1 + 193.4 \cdot 1 + 263.3 \cdot 1 + 338.4 \cdot 1 + 416.8 \cdot 1 + 496.9 \cdot 1 + 577.6 \cdot 1} = 6.588 \frac{s_i m_i}{2540.1} \]

These forces are identical with the FEM-Design calculation and the shear force and bending moment diagrams are also identical with the hand calculation.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/5.2 Lateral force method with fundamental mode shape distribution on a cantilever.str

\[ F_1 = 0.0246 \text{kN} \quad F_2 = 0.0939 \text{kN} \quad F_3 = 0.201 \text{kN} \quad F_4 = 0.339 \text{kN} \quad F_5 = 0.502 \text{kN} \quad F_6 = 0.683 \text{kN} \quad F_7 = 0.878 \text{kN} \quad F_8 = 1.081 \text{kN} \quad F_9 = 1.289 \text{kN} \quad F_{10} = 1.498 \text{kN} \]
5.3 Modal analysis of a concrete frame building

In this chapter we show a worked example for modal analysis on a concrete frame building according to EN 1998-1:2008 with hand calculation and compare the results with FEM-Design. This example is partly based on [4]. The geometry, the dimensions, the material and the bracing system are in Fig. 5.3.1-3 and in the following table. As a bracing system we used trusses with very large normal stiffness \((EA)\) to reach pure eigenvectors by the fundamental period calculation (see Fig. 5.3.2 and Fig. 5.3.5).

### Inputs:

<table>
<thead>
<tr>
<th>Column height/Total height</th>
<th>(h = 3.2) m; (H = 2 \cdot 3.2 = 6.4) m</th>
</tr>
</thead>
<tbody>
<tr>
<td>The cross sections</td>
<td>Columns: 30/30 cm; Beams: 30/50 cm</td>
</tr>
<tr>
<td>The second moment of inertia</td>
<td>(I_c = 0.000675) m(^4); (I_b = 0.003125) m(^4)</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>(E = 28.80) GPa</td>
</tr>
<tr>
<td>The concentrated mass points</td>
<td>12 pieces of 13.358 t on 1(^{st}) storey and 12 pieces of 11.268 t on 2(^{nd}) storey (see Fig. 5.3.2)</td>
</tr>
</tbody>
</table>
| The total mass            | 1\(^{st}\) storey: \(m_1 = 160.3\) t  
                            | 2\(^{nd}\) storey: \(m_2 = 135.2\) t  
                            | total mass: \(M = 295.5\) t |
| Reduction factor for elastic modulus considering the cracking according to EN 1998-1:2008 | \(\alpha = 0.5\) |
| Behaviour factors         | \(q = 1.5\), \(q_d = 1.5\) |
| Accidental torsional effect was not considered | \(\xi = 0.05\) (damping factor) |

### Figure 5.3.1 – The concrete frame building with the columns and beams
The first exercise is the determination of the fundamental periods and mode shapes. There are several hand calculation modes to get these values but in this chapter the details of the modal analysis are important therefore we considered the first two fundamental periods based on FEM-Design calculation (see Fig. 5.3.5). See the details and example on the eigenfrequency calculation in Chapter 4.

The dead loads and the live loads are considered in the mass points (see Fig. 5.3.2 and the input table).
According to the fundamental periods in Fig. 5.3.5 the response accelerations from Fig. 5.3.4 are:

\[ T_1 = 0.704 \, \text{s} \quad S_{d1} = 1.115 \, \frac{\text{m}}{\text{s}^2} \; ; \]

\[ T_2 = 0.252 \, \text{s} \quad S_{d2} = 1.57 \, \frac{\text{m}}{\text{s}^2} \; . \]

The second step is to calculate the effective modal masses based on this formula:

\[
m_i = \frac{\left( \Phi_i^T \mathbf{m} \Phi_i \right)^2}{\Phi_i^T \mathbf{m} \Phi_i}
\]
During the hand calculation we assume that the structure is a two degrees of freedom system in the x direction with the two storeys, because the first two modal shapes are in the same plane see Fig. 5.3.5. Thus we only consider the seismic loads in one direction because in this way the hand calculation is more comprehensible.

\[
m_1^* = \begin{bmatrix} 40.9 & 73.6 \\ 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{2} = 272.3 \text{ t} ; \\
m_1^* M = 272.3 \frac{295.5}{295.5} = 92.1 \%
\]

\[
m_2^* = \begin{bmatrix} 67.6 & -44.5 \\ 67.6 & -44.5 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{2} = 23.23 \text{ t} ; \\
m_2^* M = 23.23 \frac{295.5}{295.5} = 7.9 \%
\]

According to the assumption of a two degrees of freedom system the sum of the effective modal masses is equal to the total mass:

\[
\frac{m_1^*}{M} + \frac{m_2^*}{M} = \frac{272.3}{295.5} + \frac{23.23}{295.5} = 100.0 \%
\]

Calculation of the base shear forces:

\[
F_{b1} = S_{d1} m_1^* = 1.115 \cdot 272.3 = 303.6 \text{ kN} ; \\
F_{b2} = S_{d2} m_2^* = 1.570 \cdot 23.23 = 36.5 \text{ kN}
\]

The equivalent forces come from this formula:

\[
p_i = m \Phi_i \Phi_i^T m \Phi_i S_{d1}
\]

The equivalent forces at the storeys respect to the mode shapes considering the mentioned two degrees of freedom model:

\[
p_1 = \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 40.9 \\ 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{1.115} = \begin{bmatrix} 120.6 \\ 183.0 \end{bmatrix} \text{ kN}
\]
The equivalent forces on one frame from the six (see Fig. 5.3.1):

\[ p_f = \begin{bmatrix} 120.6/6 \\ 183.0/6 \end{bmatrix} = \begin{bmatrix} 20.10 \\ 30.50 \end{bmatrix} \text{kN} \]

\[ p_n = \begin{bmatrix} 81.98/6 \\ -44.52/6 \end{bmatrix} = \begin{bmatrix} 13.66 \\ -7.420 \end{bmatrix} \text{kN} \]

The shear forces between the storeys respect to the two different mode shapes:

\[ V_1 = \begin{bmatrix} 20.1 + 30.5 \\ 30.5 \end{bmatrix} = \begin{bmatrix} 50.6 \\ 30.5 \end{bmatrix} \text{kN} \quad V_2 = \begin{bmatrix} 13.66 - 7.42 \\ -7.42 \end{bmatrix} = \begin{bmatrix} 6.24 \\ -7.42 \end{bmatrix} \text{kN} \]

The shear forces in the columns respect to the two different mode shapes:

\[ V_{c1} = \begin{bmatrix} 50.6/2 \\ 30.5/2 \end{bmatrix} = \begin{bmatrix} 25.30 \\ 15.25 \end{bmatrix} \text{kN} \quad V_{c2} = \begin{bmatrix} 6.24/2 \\ -7.42/2 \end{bmatrix} = \begin{bmatrix} 3.13 \\ -3.71 \end{bmatrix} \text{kN} \]

The bending moments in the columns respect to the two different mode shapes from the relevant shear forces (by the hand calculation we assumed zero bending moment points in the middle of the columns between the storeys):

\[ M_{c1} = \begin{bmatrix} 25.30 \cdot 3.2/2 \\ 15.25 \cdot 3.2/2 \end{bmatrix} = \begin{bmatrix} 40.48 \\ 24.40 \end{bmatrix} \text{kNm} \quad M_{c2} = \begin{bmatrix} 3.13 \cdot 3.2/2 \\ -3.71 \cdot 3.2/2 \end{bmatrix} = \begin{bmatrix} 5.008 \\ -5.936 \end{bmatrix} \text{kNm} \]

The bending moments in the beams respect to the two different mode shapes:

\[ M_{b1} = \begin{bmatrix} 40.48 + 24.40 \\ 24.40 \end{bmatrix} = \begin{bmatrix} 64.88 \\ 24.40 \end{bmatrix} \text{kNm} \quad M_{b2} = \begin{bmatrix} 5.008 - 5.936 \\ -5.936 \end{bmatrix} = \begin{bmatrix} -0.928 \\ -5.936 \end{bmatrix} \text{kNm} \]

The SRSS summation on the internal forces:
Verification Examples

\[ V_c = \begin{bmatrix} \sqrt{25.30^2 + 3.13^2} \\ \sqrt{15.25^2 + (-3.71)^2} \end{bmatrix} = \begin{bmatrix} 25.49 \\ 15.69 \end{bmatrix} \text{kN} \]

\[ M_c = \begin{bmatrix} \sqrt{40.48^2 + 5.008^2} \\ \sqrt{24.40^2 + 5.936^2} \end{bmatrix} = \begin{bmatrix} 40.79 \\ 25.11 \end{bmatrix} \text{kNm} \]

\[ M_b = \begin{bmatrix} \sqrt{64.88^2 + (-0.928)^2} \\ \sqrt{24.40^2 + (-5.936)^2} \end{bmatrix} = \begin{bmatrix} 64.89 \\ 25.11 \end{bmatrix} \text{kNm} \]

The CQC summation on the internal forces:

\[ \alpha_{12} = \frac{T_2}{T_1} = 0.252 \]
\[ r_{12} = \frac{8 \xi^2 (1 + \alpha_{12}) \alpha_{12}^{3/2}}{(1 - \alpha_{12}^2)^2} + 4 \xi^2 \alpha_{12} (1 + \alpha_{12})^2 \frac{1 - 0.358^2}{(1 - 0.358^2)^2 + 4 \cdot 0.05^2 \cdot 0.358^2 (1 + 0.358)^2} = 0.007588 \]
\[ r = \begin{bmatrix} 1 \\ 0.007588 \end{bmatrix} \]

And based on these values the results of the CQC summation:

\[ V_c = \begin{bmatrix} \sqrt{25.30^2 + 3.13^2 + 2 \cdot 25.3 \cdot 3.13 \cdot 0.007588} \\ \sqrt{15.25^2 + (-3.71)^2 + 2 \cdot 15.25 \cdot (-3.71) \cdot 0.007588} \end{bmatrix} = \begin{bmatrix} 25.52 \\ 15.67 \end{bmatrix} \text{kN} \]

\[ M_c = \begin{bmatrix} \sqrt{40.48^2 + 5.008^2 + 2 \cdot 40.48 \cdot 5.008 \cdot 0.007588} \\ \sqrt{24.40^2 + 5.936^2 + 2 \cdot 24.40 \cdot 5.936 \cdot 0.007588} \end{bmatrix} = \begin{bmatrix} 40.83 \\ 25.16 \end{bmatrix} \text{kNm} \]

\[ M_b = \begin{bmatrix} \sqrt{64.88^2 + (-0.928)^2 + 2 \cdot 64.88 \cdot (-0.928) \cdot 0.007588} \\ \sqrt{24.40^2 + (-5.936)^2 + 2 \cdot 24.40 \cdot (-5.936) \cdot 0.007588} \end{bmatrix} = \begin{bmatrix} 64.88 \\ 25.07 \end{bmatrix} \text{kNm} \]

The following displacements come from the FEM-Design calculation on the complete frame structure to ensure the comprehensible final results on the P-\(\Delta\) effect.

The displacements at the storeys respect to the two different mode shapes considering the displacement behaviour factor:

\[ u_1 = q_d \begin{bmatrix} 9.54 \\ 17.15 \end{bmatrix} = 1.5 \begin{bmatrix} 9.54 \\ 17.15 \end{bmatrix} = \begin{bmatrix} 14.31 \\ 25.73 \end{bmatrix} \text{mm} \]
\[ u_2 = q_d \begin{bmatrix} 0.818 \\ -0.540 \end{bmatrix} = 1.5 \begin{bmatrix} 0.818 \\ -0.540 \end{bmatrix} = \begin{bmatrix} 1.227 \\ -0.810 \end{bmatrix} \text{mm} \]

Based on these values the storey drifting respect to the two different mode shapes:
Δ₁ = \[
\begin{bmatrix}
14.31 \\
25.73 - 14.31
\end{bmatrix}
\] \times \[
\begin{bmatrix}
14.31 \\
11.42
\end{bmatrix}
\] \times \[
\begin{bmatrix}
14.31 \\
11.42
\end{bmatrix}
\] = \[
\begin{bmatrix}
14.31 \\
25.73 - 14.31
\end{bmatrix}
\] \times \[
\begin{bmatrix}
14.31 \\
11.42
\end{bmatrix}
\] = \[
\begin{bmatrix}
14.31 \\
11.42
\end{bmatrix}
\] mm

Δ₂ = \[
\begin{bmatrix}
1.227 \\
-0.810 - 1.227
\end{bmatrix}
\] \times \[
\begin{bmatrix}
1.227 \\
-0.810 - 1.227
\end{bmatrix}
\] = \[
\begin{bmatrix}
1.227 \\
-0.810 - 1.227
\end{bmatrix}
\] mm

SRSS summation on the story drifting:

\[\Delta = \sqrt{14.31^2 + 1.227^2} \div \sqrt{11.42^2 + (-2.027)^2} \times \[
\begin{bmatrix}
14.31 \\
11.42
\end{bmatrix}
\] \times \[
\begin{bmatrix}
14.31 \\
11.42
\end{bmatrix}
\] = \[
\begin{bmatrix}
14.36 \\
11.60
\end{bmatrix}
\] mm

P-Δ effect checking on the total building:

\[P_{tot} = \left( m_1 + m_2 \right) g \times \left( \frac{160.3 + 135.2}{135.2} \right) \times \left( 9.81 \right) = \[
\begin{bmatrix}
2899 \\
1326
\end{bmatrix}
\] kN
\]

\[V_{tot} = \frac{6\sqrt{50.6^2 + 6.24^2}}{6\sqrt{30.5^2 + (-7.42)^2}} \times \[
\begin{bmatrix}
305.9 \\
188.3
\end{bmatrix}
\] \times \[
\begin{bmatrix}
305.9 \\
188.3
\end{bmatrix}
\] kN
\]

\[\theta_1 = \frac{P_{tot}\Delta_1}{V_{tot}h} = \frac{2899 \times 14.36}{305.9 \times 3200} = 0.0425 \]

\[\theta_2 = \frac{P_{tot}\Delta_2}{V_{tot}h} = \frac{1326 \times 11.60}{188.3 \times 3200} = 0.0255 \]

After the hand calculation let's see the results from the FEM-Design calculation and compare them to each other. Fig. 5.3.6 shows the effective modal masses from the FE calculation. Practically these values coincide with the hand calculation.

<table>
<thead>
<tr>
<th>Shape no.</th>
<th>T [s]</th>
<th>mx₁ [%]</th>
<th>mx₂ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.704</td>
<td>92.2</td>
<td>272.374</td>
</tr>
<tr>
<td>2</td>
<td>0.252</td>
<td>7.8</td>
<td>23.139</td>
</tr>
</tbody>
</table>

Fig. 5.3.6 and the following table shows the equivalent resultant shear forces and the base shear forces respect to the first two mode shapes. The differences between the two calculations are less than 2%.

<table>
<thead>
<tr>
<th></th>
<th>Storey 1 equivalent resultant [kN]</th>
<th>Storey 2 equivalent resultant [kN]</th>
<th>Base shear force [kN]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hand</td>
<td>FEM</td>
<td>Hand</td>
</tr>
<tr>
<td>Mode shape 1</td>
<td>120.6</td>
<td>121.9</td>
<td>81.98</td>
</tr>
<tr>
<td>Mode shape 2</td>
<td>183.0</td>
<td>185.0</td>
<td>–44.52</td>
</tr>
</tbody>
</table>
Fig. 5.3.8-9 and the following table shows the internal forces after the different summation methods (SRSS and CQC). The differences between the two calculation methods are less than 2%. Here by the moments the difference between the hand and FEM calculations (10%) comes from the simplified moment hand calculation.

<table>
<thead>
<tr>
<th></th>
<th>Column shear force [kN]</th>
<th>Column bending moment [kNm]</th>
<th>Beam bending moment [kNm]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Storey 1</td>
<td>Storey 2</td>
<td>Storey 1</td>
</tr>
<tr>
<td>SRSS Hand</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Storey 1</td>
<td>25.49</td>
<td>15.69</td>
<td>40.79</td>
</tr>
<tr>
<td>SRSS FEM</td>
<td>25.78</td>
<td>15.89</td>
<td>(37.18+45.32)/2=41.25</td>
</tr>
<tr>
<td>CQC Hand</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Storey 1</td>
<td>25.50</td>
<td>15.67</td>
<td>40.83</td>
</tr>
<tr>
<td>CQC FEM</td>
<td>25.80</td>
<td>15.86</td>
<td>(37.21+45.36)/2=41.29</td>
</tr>
</tbody>
</table>

Figure 5.3.7 – The equivalent forces respect to the storeys and the base shear forces for the first two mode shapes [kN]

Figure 5.3.8 – The shear force [kN] and bending moment diagram [kNm] after the SRSS summation rule
Fig. 5.3.10 shows the $\Theta$ values from FEM-Design. The differences between the hand calculation and FEM-Design calculations are less than 3%.

<table>
<thead>
<tr>
<th>Storey</th>
<th>$\Theta$ x</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0420</td>
</tr>
<tr>
<td>2</td>
<td>0.0253</td>
</tr>
</tbody>
</table>

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/5.3 Modal analysis of a concrete frame building.str
6 Calculation considering diaphragms

6.1. A simple calculation with diaphragms

If we apply two diaphragms on the two storeys of the building from Chapter 5.3 then the eigenfrequencies and the periods will be the same what we indicated in Chapter 5.3 with the bracing system.

6.2. The calculation of the shear center

In this example we show that how can we calculate the shear center of a storey based on the FEM-Design calculation. We analyzed a bottom fixed cantilever structure made of three concrete shear walls which are connected to each other at the edges (see Fig. 6.2.1). The diaphragm is applied at the top plane of the structure (see also Fig. 6.2.1 right side). If the height of the structure is high enough then the shear center will be on the same geometry point where it should be when we consider the complete cross section of the shear walls as a “thin-walled” “C” cross section (see Fig. 6.2.1 left side). Therefore we calculate by hand the shear center of the “C” profile assumed to be a thin-walled cross section then compare the results what we can get from FEM-Design calculation with diaphragms.

Secondly we calculate the idealized bending stiffnesses in the principal rigidity directions by hand and compare the results what we can calculate with FEM-Design results.

Inputs:

<table>
<thead>
<tr>
<th>Height of the walls</th>
<th>H = 63 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>The thickness of the walls</td>
<td>t = 20 cm</td>
</tr>
<tr>
<td>The width of wall number 1 and 3</td>
<td>w₁ = w₃ = 4.0 m</td>
</tr>
<tr>
<td>The width of wall number 2</td>
<td>w₂ = 6.0 m</td>
</tr>
<tr>
<td>The applied Young's modulus of concrete</td>
<td>E = 9.396 GPa</td>
</tr>
</tbody>
</table>

Figure 6.2.1 – The geometry of the bracing core and the height of the bottom fixed structure (the diaphragm is lying on the top plane, see the red line and hatch)
First of all let's see Fig. 6.2.2. The applied cross section is a symmetric cross section. In the web the shear stress distribution comes from the shear formula regarding bending with shear (see Fig. 6.2.2) therefore it is a second order polynom. In the flanges the shear stress distribution is linear according to the thin walled theory. With the resultant of these shear stress distribution (see Fig. 6.2.2, $V_1$, $V_2$ and $V_3$) the position of the shear center can be calculated based on the statical (equilibrium) equations.

The shear stress values (see Fig. 6.2.2):

\[
\tau = \frac{V S}{I t} = \frac{1 \cdot (0.2 \cdot 4 \cdot 3)}{\left(\frac{0.2 \cdot 6^3}{12} + \frac{4 \cdot 6.2^3}{12} - \frac{4 \cdot 5.8^3}{12}\right) \cdot 0.2} = 0.6665 \text{ kN/m}^2
\]

\[
\tau_{\text{max}} = \frac{V S_{\text{max}}}{I t} = \frac{1 \cdot (0.2 \cdot 4 \cdot 3 + 0.2 \cdot 3 \cdot 1.5)}{\left(\frac{0.2 \cdot 6^3}{12} + \frac{4 \cdot 6.2^3}{12} - \frac{4 \cdot 5.8^3}{12}\right) \cdot 0.2} = 0.9164 \text{ kN/m}^2
\]

Based on these stresses the resultant in the flanges and in the web:

\[
V_1 = V_3 = \frac{\tau w_i}{2} = \frac{0.6665 \cdot 0.2 \cdot 4}{2} = 0.2667 \text{ kN}
\]

\[
V_2 = \frac{2}{3} (\tau_{\text{max}} - \tau) w_2 t + \tau \cdot w_2 t = \frac{2}{3} (0.9164 - 0.6665) 6 \cdot 0.2 + 0.6665 \cdot 6 \cdot 0.2 = 0.9997 \text{ kN}
\]

Respect to the equilibrium (sum of the forces):

\[
V = 1 \text{ kN} \approx V_2 = 0.9997 \text{ kN}
\]

And also respect to the equilibrium (if the external load is acting on the shear center, see Fig.
6.2.2) the sum of the moments:

\[ V_1w_2 = V_2x_S \]

\[ x_S = \frac{V_1w_2}{V_2} = \frac{0.2667 \cdot 6}{0.9997} = 1.601 \text{ m} \]

Thus the shear center is lying on the symmetry axis and it is \( x_S = 1.601 \text{ m} \) from the web (see Fig. 6.2.2). In FEM-Design the global coordinate system does not coincide with the symmetry axis of the structure (see Fig. 6.2.1). Therefore we need no transform the results.

Let's be a selected key node at the diaphragm in the global coordinate system (see Fig. 6.2.1):

\[ x_m = 0 \text{ m} \quad ; \quad y_m = 0 \text{ m} \]

Based on the unit forces (1 kN) and moment (1 kNm) on the key node the displacements of the key node are as follows based on the FEM-Design calculation (see the Scientific Manual Calculation considering diaphragm chapter also):

According to unit force on key node in X direction:

\[ u_{xx} = 1.5852 \text{ mm} \quad u_{yx} = 0.72166 \text{ mm} \quad \phi_{zx} = 0.29744 \cdot 10^{-4} \text{ rad} \]

According to unit force on key node in Y direction:

\[ u_{xy} = 0.72166 \text{ mm} \quad u_{yy} = 7.3314 \text{ mm} \quad \phi_{zy} = 0.10328 \cdot 10^{-2} \text{ rad} \]

According to unit moment on key node around Z direction:

\[ \phi_{zz} = 0.16283 \cdot 10^{-3} \text{ rad} \]

Based on these finite element results the global coordinates of the shear center of the diaphragm are:

\[ x_S = x_m - \phi_{zy} \frac{\phi_{zx}}{\phi_{zz}} = 0 - \frac{0.10328 \cdot 10^{-2}}{0.16283 \cdot 10^{-3}} = -6.343 \text{ m} \]

\[ y_S = y_m + \phi_{zx} \frac{\phi_{zy}}{\phi_{zz}} = 0 + \frac{0.29744 \cdot 10^{-4}}{0.16283 \cdot 10^{-3}} = +0.1827 \text{ m} \]

In FEM-Design the coordinates of the middle point of the web are (see Fig. 6.2.1):

\[ x_{mid} = -4.919 \text{ m} \quad ; \quad y_{mid} = +0.894 \text{ m} \]

With the distance between these two points we get a comparable solution with the hand calculation.
The difference between FEM and hand calculation is less than 1%.

The gravity center of the cross section (Fig. 6.2.2) can be calculated based on the statical moments. And of course the gravity center lying on the symmetry axis. The distance of the gravity center from the web is:

\[ x'_{G} = \frac{S_{y}}{A} = \frac{2(0.2 \cdot 4 \cdot 2)}{0.2(4+6+4)} = 1.143 \text{ m} \]

With the input Young's modulus and with the second moments of inertia the idealized bending stiffnesses in the principal directions can be calculated by hand.

\[ E I_{1} = 9396 \cdot 10^{3} \left( \frac{0.2 \cdot 6^{3}}{12} + \frac{4 \cdot 6.2^{3}}{12} - \frac{4 \cdot 5.8^{3}}{12} \right) = 1.692 \cdot 10^{8} \text{ kNm}^{2} \]

\[ E I_{2} = 9396 \cdot 10^{3} \left( 2 \cdot \frac{0.2 \cdot 4^{3}}{12} + 2(0.2 \cdot 4(2-1.143)^{2}) + \frac{6 \cdot 0.2^{4}}{12} + 0.2 \cdot 6(1.143)^{2} \right) = 4.585 \cdot 10^{7} \text{ kNm}^{2} \]

With the finite element results we can calculate the translations of the shear center according to the unit forces and moment on the key node (see the former calculation method).

The distances between the shear center and the selected key node are:

\[ \Delta x = x_{S} - x_{m} = -6.343 - 0 = -6343 \text{ mm} \]

\[ \Delta y = y_{S} - y_{m} = +0.1827 - 0 = +0.1827 \text{ m} = 182.7 \text{ mm} \]

The translations of the shear center are as follows:

\[ u_{Sxx} = u_{x} - q_{x} \Delta y = 1.5852 - 0.29744 \cdot 10^{-4} \cdot 182.7 = 1.5798 \text{ mm} \]

\[ u_{Syy} = u_{y} + q_{y} \Delta x = 0.72166 + 0.29744 \cdot 10^{-4} \cdot (-6343) = 0.5330 \text{ mm} \]

\[ u_{Sxy} = u_{xy} + q_{z} \Delta y = 0.72166 - 0.10328 \cdot 10^{-2} \cdot 182.7 = 0.5330 \text{ mm} \]

\[ u_{Syy} = u_{xy} + q_{z} \Delta x = 7.3314 + 0.10328 \cdot 10^{-2} \cdot (-6343) = 0.7803 \text{ mm} \]

Based on these values the translations of the shear center in the principal directions:

\[ u_{1} = \frac{u_{Sxx} + u_{Syy}}{2} + \sqrt{\left( \frac{u_{Sxx} - u_{Syy}}{2} \right)^{2} + u_{Sxy}^{2}} = \sqrt{\left( \frac{1.5798 + 0.7803}{2} \right)^{2} + 0.5330^{2}} = 1.8463 \text{ mm} \]
According to these values the angles of the principal rigidity directions:

\[ \alpha_{1FEM} = \arctan \frac{u_1 - u_{Sxx}}{u_{Sxy}} = \arctan \frac{1.8463 - 1.5798}{0.533} = 26.57^\circ \]

\[ \alpha_{2FEM} = \arctan \frac{u_2 - u_{Sxx}}{u_{Sxy}} = \arctan \frac{0.5138 - 1.5798}{0.533} = -63.43^\circ \]

The directions coincide with the axes of symmetries (see Fig. 6.2.1-2) which is one of the principal rigidity direction in this case.

Then with FEM-Design results we can calculate the idealized bending stiffnesses of the structure:

\[ EI_{1FEM} = \frac{H^3}{3u_2} = \frac{63^3}{3 \cdot (0.5138/1000)} = 6.22 \cdot 10^6 \text{ kNm}^2 \]

\[ EI_{2FEM} = \frac{H^3}{3u_1} = \frac{63^3}{3 \cdot (1.8463/1000)} = 4.514 \cdot 10^7 \text{ kNm}^2 \]

The difference between FEM and hand calculation is less than 4%.

Download link to the example file:
FEM-Design file:
http://download.strusoft.com/FEM-Design/inst170x/models/6.1.1 A simple calculation with diaphragms.str
Section Editor file:
http://download.strusoft.com/FEM-Design/inst170x/models/6.1.2 The calculation of the shear center.sec
7 Calculations considering nonlinear effects

7.1 Uplift calculation

7.1.1 A trusses with limited compression members

In this example a truss will be analyzed. First of all we calculate the normal forces in the truss members and the maximum deflection for the given concentrated loads. After this step we calculate the load multiplier when the vertical truss members reaches its limit compression bearing capacity what we set. See the inputs in the following table. After the hand calculation we compare the results with the FEM-Design nonlinear calculation results.

Inputs:

<table>
<thead>
<tr>
<th>Column height/Span length</th>
<th>H = 2.0 m; L = 8.0 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>The cross sections</td>
<td>KKR 80x80x6</td>
</tr>
<tr>
<td>The area of the cross sections</td>
<td>A = 1652 mm²</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>E = 210 GPa, structural steel</td>
</tr>
<tr>
<td>The concentrated loads</td>
<td>F = 40 kN</td>
</tr>
<tr>
<td>Limited compression of the vertical truss members</td>
<td>P&lt;sub&gt;cr&lt;/sub&gt; = 700 kN</td>
</tr>
</tbody>
</table>

![Figure 7.1.1.1 – The truss with the concentrated loads and with the virtual loads for the translation calculation](image)
The normal forces in the truss members based on the hand calculation (without further details) are:

\[ N_1 = N_{17} = -100 \text{kN} \quad \text{and} \quad N_2 = N_{14} = 0 \text{kN} \quad \text{and} \quad N_3 = N_{15} = +84.85 \text{kN} \]

\[ N_4 = N_5 = N_{13} = N_{16} = -60.00 \text{kN} \quad \text{and} \quad N_6 = N_{10} = +60.00 \text{kN} \]

\[ N_7 = N_{11} = +28.28 \text{kN} \quad \text{and} \quad N_8 = N_{12} = -80.00 \text{kN} \quad \text{and} \quad N_9 = -40.00 \text{kN} \]

The normal forces in the truss members according to the vertical virtual force (see Fig. 7.1.1.1):

\[ N_{1,1} = N_{1,17} = -0.5 \text{kN} \quad \text{and} \quad N_{1,2} = N_{1,14} = N_{1,9} = 0 \text{kN} \]

\[ N_{1,3} = N_{1,15} = N_{1,7} = N_{1,11} = +0.7071 \text{kN} \]

\[ N_{1,4} = N_{1,5} = N_{1,13} = N_{1,16} = -0.5 \text{kN} \quad \text{and} \quad N_{1,6} = N_{1,10} = +0.5 \text{kN} \]

\[ N_{1,8} = N_{1,12} = -1.0 \text{kN} \]

The normal forces in the truss members according to the horizontal virtual force (see Fig. 7.1.1.1):

\[ N_{2,2} = N_{2,6} = N_{2,10} = N_{2,14} = +1.0 \text{kN} \]

\[ N_{2,3} = N_{2,4} = N_{2,5} = N_{2,7} = N_{2,8} = N_{2,9} = N_{2,11} = N_{2,12} = N_{2,13} = N_{2,15} = N_{2,16} = N_{2,17} = 0 \text{kN} \]

The hand calculation of the vertical translation at the mid-span with the virtual force method:

\[ e_z = \frac{1}{EA} \sum_{i=1}^{17} N_i \delta N_{i,i} l_i = 0.003841 \text{m} = 3.841 \text{mm} \]

The hand calculation of the horizontal translation at right roller with the virtual force method:

\[ e_x = \frac{1}{EA} \sum_{i=1}^{17} N_i \delta N_{2,i} l_i = 0.0006918 \text{m} = 0.6918 \text{mm} \]
**Figure 7.1.1.3** – The reaction forces

**Figure 7.1.1.4** – The normal forces in the truss members

**Figure 7.1.1.5** – The vertical translation at the mid-span and the horizontal translation at the right roller [mm]
The translations and the normal forces in the truss members based on the hand calculation are identical with the FEM-Design calculation, see Fig. 7.1.1.2-5.

After this step we would like to know the maximum load multiplier when the vertical truss members reaches its limit compression bearing capacity what we set, $P_{cr} = 700$ kN. The maximum compression force arises in the side columns, see the hand calculation, $N_1 = (-)100$ kN. Therefore the load multiplier based on the hand calculation is $\lambda = 7.0$.

Let's see the FEM-Design uplift calculation considering the limit compression in the vertical members.

With $\lambda_{FEM} = 7.00$ multiplier the FEM-Design analysis gives the accurate result but with $\lambda_{FEM} = 7.01$ (see Fig. 7.1.1.7) large nodal displacements occurred, see Fig. 7.1.1.6. Thus by this structure if we neglect the effect of the side members the complete truss became a statically overdetermined structure. FEM-Design solve this problem with iterative solver due to the fact that these kind of problems are nonlinear.

Based on the FEM-Design calculation the load multiplier is identical with the hand calculation.

Download link to the example file:  
http://download.strusoft.com/FEM-Design/inst170x/models/7.1.1.A trusses with limited compression members.str
7.1.2 A continuous beam with three supports

In this example we analyse non-linear supports of a beam. Let's consider a continuous beam with three supports with the following parameters:

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span length</td>
<td>L = 2 m, total length = 2x2 = 4 m</td>
</tr>
<tr>
<td>The cross sections</td>
<td>Rectangle: 120x150 mm</td>
</tr>
<tr>
<td>The elastic modulus</td>
<td>E = 30 GPa, concrete C20/25</td>
</tr>
<tr>
<td>Intensity of distributed load (total, partial)</td>
<td>p = 10 kN/m</td>
</tr>
</tbody>
</table>

In Case I. the distribution of the external load and the nonlinearity of the supports differ from Case II. See the further details below (Fig. 7.1.2.1 and Fig. 7.1.2.8).

a) Case I.

In this case the distributed load is a total load (Fig. 7.1.2.1). In the first part all of the three supports behave the same way for compression and tension. In the second part of this case the middle support only bears tension. We calculated in both cases the deflections, shear forces and bending moments by hand and compared the results with FEM-Design uplift (nonlinear) calculations.

\[
e_{\text{max}} = \frac{2.1 \cdot p \cdot L^4}{384 \cdot EI} = \frac{2.1 \cdot 10 \cdot 2^4}{384 \cdot 30000000 \cdot 0.12 \cdot 0.15^3 / 12} = 0.0008642 \text{ m} = 0.8642 \text{ mm}
\]

The relevant results with FEM-Design:

\[
\begin{align*}
&0.8834 \\
&0.8834
\end{align*}
\]

Figure 7.1.2.1 – The beam with three supports and uniform distributed load

In first part of this case the maximum deflection comes from the following formula considering only the bending deformations in the beam:
The extremums of the shear force without signs:

\[ V_1 = \frac{3}{8} pL = \frac{3}{8} \cdot 10 \cdot 2 = 7.5 \text{kN} \]

\[ V_2 = \frac{5}{8} pL = \frac{5}{8} \cdot 10 \cdot 2 = 12.5 \text{kN} \]

The relevant results with FEM-Design:

\[ M_{\text{midspan}} = \frac{9}{128} pL^2 = \frac{9}{128} \cdot 10 \cdot 2^2 = 2.812 \text{kNm} \]

\[ M_{\text{middle}} = \frac{1}{8} pL^2 = \frac{1}{8} \cdot 10 \cdot 2^2 = 5.0 \text{kNm} \]

When the middle support only bears tension (second part of this case) basically under the total vertical load (Fig. 7.1.2.1) the middle support is not active (support nonlinearity). Therefore it works as a simply supported beam with two supports. The deflection, the shear forces and the bending moments are the following:

The maximum deflection comes from the following formula considering only the bending deformations in the beam:

\[ e_{\text{max}} = \frac{5}{384} \frac{p(L+L)^3}{EI} = \frac{5}{384} \frac{10 \cdot (2+2)^3}{3000000 \cdot 0.12 \cdot 0.15^3 / 12} = 0.03292 \text{m} = 32.92 \text{mm} \]
The relevant results with FEM-Design:

The maximum of the shear force without sign:

\[ V = \frac{1}{2} p (L + L) = \frac{1}{2} 10 (2 + 2) = 20 \text{kN} \]

The extremum of the bending moment without sign:

\[ M_{\text{max}} = \frac{1}{8} p (L + L)^2 = \frac{1}{8} 10 \cdot (2 + 2)^2 = 20 \text{kNm} \]

The differences between the calculated results by hand and by FEM-Design are less than 2%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/7.1.2 A continuous beam with three supports case a.str
b) Case II.

In this case the distributed load is a partial load (Fig. 7.1.2.8). In the first part all of the three supports behave the same way for compression and tension. In the second part of this case the right side support only bears compression. We calculate in both cases the deflections, shear forces and bending moments by hand and compared the results with FEM-Design calculations.

The extremums of the deflection come from the following formulas considering only the bending deformations in the beam (without signs):

\[
e_{\text{max}} \approx \frac{2.1}{384} \left(\frac{p}{2}\right) L^4 \frac{1}{EI} + \frac{5}{384} \left(\frac{p}{2}\right) L^4 \frac{1}{EI} = \frac{2.1}{384} \frac{10/2 \cdot 2^4}{EI} + \frac{5}{384} \frac{10/2 \cdot 2^4}{EI} = 0.001461 \text{ m} = 1.461 \text{ mm}
\]

\[
e_{\text{min}} \approx \frac{5}{384} \left(\frac{p}{2}\right) L^4 \frac{1}{EI} - \frac{2}{384} \left(\frac{p}{2}\right) L^4 \frac{1}{EI} = \frac{5}{384} \frac{10/2 \cdot 2^4}{EI} - \frac{2}{384} \frac{10/2 \cdot 2^4}{EI} = 0.0006173 \text{ m} = 0.6173 \text{ mm}
\]

The relevant results with FEM-Design:

The extremums of the shear force without signs:

\[
V_1 = \frac{7}{16} pL = \frac{7}{16} \cdot 10 \cdot 2 = 8.75 \text{ kN} \quad ; \quad V_2 = \frac{9}{16} pL = \frac{9}{16} \cdot 10 \cdot 2 = 11.25 \text{ kN} \quad ; \\
V_3 = \frac{1}{16} pL = \frac{1}{16} \cdot 10 \cdot 2 = 1.25 \text{ kN}
\]
The relevant results with FEM-Design:

![Image of shear diagram](image1)

*Figure 7.1.2.10 – The shear diagram [kN] of the beam with three supports (partial load)*

The extremums of the bending moment without signs:

\[ M_{\text{midspan}} = \frac{\left( \frac{7}{16} p L \right)^2}{2 p} = \frac{\left( \frac{7}{16} \cdot 10 \cdot 2 \right)^2}{2 \cdot 10} = 3.828 \text{kNm} \]

\[ M_{\text{middle}} = \frac{1}{16} p L^2 = \frac{1}{16} \cdot 10 \cdot 2^2 = 2.5 \text{kNm} \]

The relevant results with FEM-Design:

![Image of bending moment diagram](image2)

*Figure 7.1.2.11 – The bending moment diagram [kNm] of the beam with three supports (partial load)*

When the right side support only bear compression (second part of this case) basically under the partial vertical load (Fig. 7.1.2.8) the right side support is not active (support nonlinearity). Therefore it works as a simply supported beam with two supports. The deflection, the shear forces and the bending moments are the following:

The maximum deflection comes from the following formula considering only the bending deformations in the beam:

\[ e_{\text{midspan}} = \frac{5}{384} \frac{p L^4}{EI} = \frac{5}{384} \frac{10 \cdot 2^4}{EI} = 0.002058 \text{m} = 2.058 \text{mm} \]

\[ e_{\text{right}} = \frac{1}{24} \frac{p L^4}{EI} = \frac{1}{24} \frac{10 \cdot 2^4}{EI} = 0.006584 \text{m} = 6.584 \text{mm} \]
The relevant results with FEM-Design:

The extremum of the shear force without sign:

\[ V = \frac{1}{2} p L = \frac{1}{2} \cdot 10 \cdot 2 = 10 \text{ kN} \]

The relevant results with FEM-Design:

The extremum of the bending moment without sign:

\[ M_{\text{max}} = \frac{1}{8} p L^2 = \frac{1}{8} \cdot 10 \cdot 2^2 = 5.0 \text{ kNm} \]

The differences between the calculated results by hand and by FEM-Design are less than 2%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.1.2 A continuous beam with three supports case b.str
### 7.2 Cracked section analysis by reinforced concrete elements

#### 7.2.1 Cracked deflection of a simply supported beam

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span length</td>
<td>$L_{\text{eff}} = 7.2 \text{ m}$</td>
</tr>
<tr>
<td>The cross section</td>
<td>Rectangle: $b = 300 \text{ mm}; h = 450 \text{ mm}$</td>
</tr>
<tr>
<td>The elastic modulus of concrete</td>
<td>$E_{\text{cm}} = 31.476 \text{ GPa, C25/30}$</td>
</tr>
<tr>
<td>The creep factor</td>
<td>$\phi_{28} = 2.35$</td>
</tr>
<tr>
<td>Effective elastic modulus of concrete</td>
<td>$E_{\text{eff}} = E_{\text{cm}}/(1+\phi_{28}) = 9.396 \text{ GPa}$</td>
</tr>
<tr>
<td>Mean tensile strength</td>
<td>$f_{\text{cm}} = 2.565 \text{ MPa}$</td>
</tr>
<tr>
<td>Elastic modulus of steel bars</td>
<td>$E_s = 200 \text{ GPa}$</td>
</tr>
<tr>
<td>Characteristic value of dead load</td>
<td>$g_k = 8.5 \text{ kN/m}$</td>
</tr>
<tr>
<td>Characteristic value of live load</td>
<td>$q_k = 12.0 \text{ kN/m}$</td>
</tr>
<tr>
<td>Live load combination factor</td>
<td>$\psi_2 = 0.6$</td>
</tr>
<tr>
<td>Diameter of the longitudinal reinforcement</td>
<td>$\phi_1 = 18 \text{ mm}$</td>
</tr>
<tr>
<td>Diameter of the stirrup reinforcement</td>
<td>$\phi_2 = 8 \text{ mm}$</td>
</tr>
<tr>
<td>Area of longitudinal reinforcement</td>
<td>$A_1 = 5\times18^2\pi/4 = 1272.3 \text{ m}^2$</td>
</tr>
<tr>
<td>Nominal concrete cover</td>
<td>$c_{\text{nom}} = 20 \text{ mm}$</td>
</tr>
<tr>
<td>Effective height</td>
<td>$d = h - c_{\text{nom}} - \phi_2 - \phi_1/2 = 413 \text{ mm}$</td>
</tr>
<tr>
<td>Shrinkage strain</td>
<td>$\varepsilon_{\text{cs}} = 0.5 %$</td>
</tr>
</tbody>
</table>

The cross sectional properties without calculation details (considering creep effect):

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. stress stadium second moment of inertia</td>
<td>$I_1 = 3.075\times10^9 \text{ mm}^4$</td>
</tr>
<tr>
<td>II. stress stadium second moment of inertia</td>
<td>$I_{\text{II}} = 2.028\times10^9 \text{ mm}^4$</td>
</tr>
<tr>
<td>I. stress stadium position of neutral axis</td>
<td>$x_1 = 256.4 \text{ mm}$</td>
</tr>
<tr>
<td>II. stress stadium position of neutral axis</td>
<td>$x_{\text{II}} = 197.3 \text{ mm}$</td>
</tr>
</tbody>
</table>

*Figure 7.2.1.1 – The simply supported RC beam*
The calculation of deflection according to EN 1992-1-1:
The load value for the quasi-permanent load combination:

\[ p_{qp} = g_k + \psi_2 q_k = 8.5 + 0.6 \times 12 = 15.7 \text{ kN/m} \]

The maximum deflection with cross sectional properties in Stadium I. (uncracked):

\[ w_{k.I} = \frac{5}{384} \frac{p_{qp} L_{eff}^4}{E_{eff} I_{I}} = \frac{5}{384} \frac{15.7 \times 7.2^4}{9396000 \times 0.003075} = 0.01901 \text{ m} = 19.01 \text{ mm} \]

The maximum deflection with cross sectional properties in Stadium II. (cracked):

\[ w_{k.II} = \frac{5}{384} \frac{p_{qp} L_{eff}^4}{E_{eff} I_{II}} = \frac{5}{384} \frac{15.7 \times 7.2^4}{9396000 \times 0.002028} = 0.02883 \text{ m} = 28.83 \text{ mm} \]

The maximum bending moment under the quasi-permanent load (see Fig. 7.2.1.1):

\[ M_{max} = \frac{1}{8} p_{qp} L_{eff}^2 = \frac{1}{8} 15.7 \times 7.2^2 = 101.74 \text{ kNm} \]

The cracking moment with the mean tensile strength:

\[ M_{cr} = f_{ctm} \frac{I_{I}}{h-x_I} = 2565 \frac{0.003075}{0.45-0.2564} = 40.74 \text{ kNm} \]

The interpolation factor considering the mixture of cracked and uncracked behaviour at the most unfavourable cross-section:

\[ \zeta = \max \left[ 1 - 0.5 \left( \frac{M_{cr}}{M_{max}} \right)^2, 0 \right] = \max \left[ 1 - 0.5 \left( \frac{40.74}{101.74} \right)^2, 0 \right] = 0.9198 \]

This value is almost 1.0, it means that the final deflection will be closer to the cracked deflection than to the uncracked one.

The final deflection with the aim of interpolation factor:

\[ w_k = (1-\zeta) w_{k.I} + \zeta w_{k.II} = (1-0.9198) \times 19.01 + 0.9198 \times 28.83 = 28.04 \text{ mm} \]
This deflection is even greater if we are considering the effect of shrinkage.

The curvatures in uncracked and cracked states due to shrinkage:

\[
\kappa_{I,cs} = \varepsilon_{cs} \frac{E_s}{E_{c,eff}} \frac{S_{e,l}}{I} = \frac{0.5 \times 200 \times 1272.3 \times (413 - 256.4)}{1000 \times 9.396 \times 3.075 \times 10^9} = 6.896 \times 10^{-4} \frac{1}{m}
\]

\[
\kappa_{II,cs} = \varepsilon_{cs} \frac{E_s}{E_{c,eff}} \frac{S_{e,II}}{I_{II}} = \frac{0.5 \times 200 \times 1272.3 \times (413 - 197.3)}{1000 \times 9.396 \times 2.028 \times 10^9} = 1.440 \times 10^{-3} \frac{1}{m}
\]

The additional deflection in the two different states due to shrinkage:

\[
w_{k,cs} = \frac{1}{8} L_{eff}^2 \kappa_{I,cs} = \frac{1}{8} 7.2^2 \times 6.896 \times 10^{-4} = 0.004469 \text{ m} = 4.469 \text{ mm}
\]

\[
w_{k,II,cs} = \frac{1}{8} L_{eff}^2 \kappa_{II,cs} = \frac{1}{8} 7.2^2 \times 1.440 \times 10^{-3} = 0.009331 \text{ m} = 9.331 \text{ mm}
\]

The total deflection considering cracking and the effect of shrinkage:

\[
w_{k,cs} = (1 - \zeta) (w_{k,I} + w_{k,cs}) + \zeta (w_{k,II} + w_{k,II,cs}) =
\]

\[
= (1 - 0.9198) (19.01 + 4.469) + 0.9198 (28.83 + 9.331) = 36.98 \text{ mm}
\]
First we modelled the beam with beam elements. In FEM-Design we increased the division number of the beam finite elements to ten to get the more accurate results.

Fig. 7.2.1.2 shows the applied cross section and reinforcement with the defined input parameters.

Fig. 7.2.1.3 shows the deflection after the cracked section analysis without and with considering shrinkage. The deflection of the beam model in FEM-Design:
Verification Examples

Cracked section analysis without shrinkage: \( w_{kFEM} = 27.39 \text{ mm} \)
Cracked section analysis with shrinkage: \( w_{kcsFEM} = 35.92 \text{ mm} \)

The difference between the hand and FEM-Design calculations is less than 3%, but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result.

Secondly we modelled the beam with shell finite elements. Fig. 7.2.1.4 shows the applied specific reinforcement with the defined input parameters with slab model.

Fig. 7.2.1.5 shows the deflection and the finite element mesh after the cracked section analysis

Figure 7.2.1.4 – The specific reinforcement with the shell model in FEM-Design

Figure 7.2.1.5 – The deflection [mm] of the RC shell model in FEM-Design with cracked section analysis (without shrinkage [above], with shrinkage [below])

Fig. 7.2.1.5 shows the deflection and the finite element mesh after the cracked section analysis
without and with considering shrinkage. The deflection of the shell model in FEM-Design:

Cracked section analysis without shrinkage:  \( w_{K_{\text{FEM}}} = 27.26 \text{ mm} \)

Cracked section analysis with shrinkage:  \( w_{K_{\text{FEM}}} = 35.50 \text{ mm} \)

The difference between the hand and FEM-Design calculations is less than 4% but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result.

Fig. 7.2.1.6 shows the effect of tension stiffening (without shrinkage effects) in FEM-Design at the relevant SLS load interval. We indicated the load level where the first crack occured.

![Effect of tension stiffening in FEM-Design](image)

**Figure 7.2.1.6 – The effect of tension stiffening by a simply supported beam**

Download link to the example files:

Beam model:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.1 Cracked deflection of a simply supported beam.beam.str

Shell model:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.1 Cracked deflection of a simply supported beam.shell.str
7.2.2 \textit{Cracked deflection of a statically indeterminate beam}

**Inputs:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span length</td>
<td>( L_{\text{eff}} = 7.2 \text{ m} )</td>
</tr>
<tr>
<td>The cross section</td>
<td>Rectangle: ( b = 300 \text{ mm}; h = 450 \text{ mm} )</td>
</tr>
<tr>
<td>The elastic modulus of concrete</td>
<td>( E_{\text{cm}} = 31.476 \text{ GPa}, \text{ C25/30} )</td>
</tr>
<tr>
<td>The creep factor</td>
<td>( \varphi_{28} = 2.35 )</td>
</tr>
<tr>
<td>Effective elastic modulus of concrete</td>
<td>( E_{\text{eff}} = E_{\text{cm}}/(1+\varphi_{28}) = 9.396 \text{ GPa} )</td>
</tr>
<tr>
<td>Mean tensile strength</td>
<td>( f_{\text{cm}} = 2.565 \text{ MPa} )</td>
</tr>
<tr>
<td>Elastic modulus of steel bars</td>
<td>( E_s = 200 \text{ GPa} )</td>
</tr>
<tr>
<td>Characteristic value of dead load</td>
<td>( g_k = 8.5 \text{ kN/m} )</td>
</tr>
<tr>
<td>Characteristic value of live load</td>
<td>( q_k = 12.0 \text{ kN/m} )</td>
</tr>
<tr>
<td>Live load combination factor</td>
<td>( \psi_2 = 0.6 )</td>
</tr>
<tr>
<td>Diameter of the longitudinal reinforcement</td>
<td>( \phi_l = 18 \text{ mm} )</td>
</tr>
<tr>
<td>Diameter of the stirrup reinforcement</td>
<td>( \phi_s = 8 \text{ mm} )</td>
</tr>
<tr>
<td>Area of longitudinal reinforcement</td>
<td>( A_l = 5 \times 18^2 \pi/4 = 1272.3 \text{ m}^2 )</td>
</tr>
<tr>
<td>Nominal concrete cover</td>
<td>( c_{\text{nom}} = 20 \text{ mm} )</td>
</tr>
<tr>
<td>Effective height</td>
<td>( d = h - c_{\text{nom}} - \phi_s - \phi_l/2 = 413 \text{ mm} )</td>
</tr>
<tr>
<td>Shrinkage strain</td>
<td>( \varepsilon_{cs} = 0.5 % )</td>
</tr>
</tbody>
</table>

The cross sectional properties without calculation details (considering creep effect):

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. stress stadium second moment of inertia</td>
<td>( I_1 = 3.075 \times 10^9 \text{ mm}^4 )</td>
</tr>
<tr>
<td>II. stress stadium second moment of inertia</td>
<td>( I_{II} = 2.028 \times 10^9 \text{ mm}^4 )</td>
</tr>
<tr>
<td>I. stress stadium position of neutral axis</td>
<td>( x_l = 256.4 \text{ mm} )</td>
</tr>
<tr>
<td>II. stress stadium position of neutral axis</td>
<td>( x_{II} = 197.3 \text{ mm} )</td>
</tr>
</tbody>
</table>

In this chapter we will calculate the cracked deflection of a statically indeterminate structure (see Fig. 7.2.2.1).
The maximum deflection with cross sectional properties in Stadium I. (uncracked):

\[
w_{k.I} = \frac{2.1}{384} \frac{P_{qp} L_{eff}^4}{E_{eff} I_I} = \frac{2.1}{384} \frac{15.7 \cdot 7.2^4}{9396000 \cdot 0.003075} = 0.007986 \text{ m} = 7.986 \text{ mm}
\]

The maximum deflection with cross sectional properties in Stadium II. (cracked):

\[
w_{k.II} = \frac{2.1}{384} \frac{P_{qp} L_{eff}^4}{E_{eff} I_{II}} = \frac{2.1}{384} \frac{15.7 \cdot 7.2^4}{9396000 \cdot 0.002028} = 0.01211 \text{ m} = 12.11 \text{ mm}
\]

The maximum bending moment under the quasi-permanent load at the fixed end (see Fig. 7.2.2.1):

\[
M_{max} = \frac{1}{8} P_{qp} L_{eff}^2 = \frac{1}{8} 15.7 \cdot 7.2^2 = 101.74 \text{ kNm}
\]

The cracking moment with the mean tensile strength:

\[
M_{cr} = f_{ctm} \frac{I_I}{h - x_I} = 2565 \frac{0.003075}{0.45 - 0.2564} = 40.74 \text{ kNm}
\]

The interpolation factor considering the mixture of cracked and uncracked behaviour at the most unfavourable cross-section:
\[ \zeta = \max \left[ 1 - 0.5 \left( \frac{M_{cr}}{M_{\text{max}}} \right)^2, 0 \right] = \max \left[ 1 - 0.5 \left( \frac{40.74}{101.74} \right)^2, 0 \right] = 0.9198 \]

This value is almost 1.0, it means that the final deflection will be closer to the cracked deflection than to the uncracked one with the hand calculation.

The final deflection with the aim of interpolation factor:
\[
 w_k = (1 - \zeta) w_{k.I} + \zeta w_{k.II} = (1 - 0.9198) \cdot 7.986 + 0.9198 \cdot 12.11 = 11.78 \text{ mm}
\]

This deflection is even greater if we are considering the effect of shrinkage.

The curvatures in uncracked and cracked states due to shrinkage:
\[
 \kappa_{I,cs} = \varepsilon_{cs} \frac{E_s S_{s, I}}{E_{c, eff} I} = 0.5 \frac{200}{1000} \frac{1272.3 \cdot (413 - 256.4)}{9.396 \cdot 3.075 \cdot 10^9} = 6.896 \cdot 10^{-4} \frac{1}{\text{m}}
\]
\[
 \kappa_{II,cs} = \varepsilon_{cs} \frac{E_s S_{s, II}}{E_{c, eff} I} = 0.5 \frac{200}{1000} \frac{1272.3 \cdot (413 - 197.3)}{9.396 \cdot 2.028 \cdot 10^9} = 1.440 \cdot 10^{-3} \frac{1}{\text{m}}
\]

The additional deflection due to shrinkage:
\[
 w_{k.I,cs} = \frac{1}{12} L_{eff}^2 \kappa_{I,cs} = \frac{1}{12} 7.2^2 \cdot 6.896 \cdot 10^{-4} = 0.002979 \text{ m} = 2.979 \text{ mm}
\]
\[
 w_{k.II,cs} = \frac{1}{12} L_{eff}^2 \kappa_{II,cs} = \frac{1}{12} 7.2^2 \cdot 1.440 \cdot 10^{-3} = 0.006221 \text{ m} = 6.221 \text{ mm}
\]

The total deflection considering cracking and the effect of shrinkage:
\[
 w_{k,cs} = (1 - \zeta) (w_{k.I} + w_{k.I,cs}) + \zeta (w_{k.II} + w_{k.II,cs}) = (1 - 0.9198) (7.986 + 2.979) + 0.9198 (12.11 + 6.221) = 17.74 \text{ mm}
\]

First we modelled the beam with beam elements. In FEM-Design we increased the division number of the beam finite elements to ten to get the more accurate results.
Fig. 7.2.2.2 shows the applied cross section and reinforcement with the defined input parameters.

![Figure 7.2.2.2 – The cross section of the RC beam in FEM-Design](image)

Fig. 7.2.2.3 shows the deflection after the cracked section analysis without and with considering shrinkage. The deflection of the beam model in FEM-Design:

Cracked section analysis without shrinkage: $w_{kFEM} = 10.62 \text{ mm}$

Cracked section analysis with shrinkage: $w_{kcsFEM} = 15.55 \text{ mm}$

The difference between the hand and FEM-Design calculations is around 10%, but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result and by a statically indeterminate structure it causes greater difference.
Secondly we modelled the beam with shell finite elements. Fig. 7.2.2.4 shows the applied specific reinforcement with the defined input parameters with slab model.

\[
\begin{align*}
\text{Cracked section analysis without shrinkage:} & & w_{k,FEM} &= 10.63 \text{ mm} \\
\text{Cracked section analysis with shrinkage:} & & w_{k,cs,FEM} &= 15.14 \text{ mm}
\end{align*}
\]

Fig. 7.2.2.5 shows the deflection and the finite element mesh after the cracked section analysis without and with considering shrinkage. The deflection of the shell model in FEM-Design:
The difference between the hand and FEM-Design calculations is around 10%, but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result and by a statically indeterminate structure it causes greater difference.

Fig. 7.2.2.6 shows the effect of tension stiffening (without shrinkage effects) in FEM-Design at the relevant SLS load interval. We indicated the load level where the first crack occurred at the fixed end and at the mid-span.

Download link to the example files:
Beam model:
http://download.strusoft.com/FEM-Design/inst170x/models/7.2.2 Cracked deflection of a statically indeterminate beam.beam.str

Shell model:
http://download.strusoft.com/FEM-Design/inst170x/models/7.2.2 Cracked deflection of a statically indeterminate beam.shell.str
### 7.2.3 Cracked deflection of a cantilever beam

**Inputs:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span length</td>
<td>$L_{\text{eff}} = 4 \text{ m}$</td>
</tr>
<tr>
<td>The cross section</td>
<td>Rectangle: $b = 300 \text{ mm}; h = 450 \text{ mm}$</td>
</tr>
<tr>
<td>The elastic modulus of concrete</td>
<td>$E_{\text{cm}} = 31.476 \text{ GPa}, \text{C25/30}$</td>
</tr>
<tr>
<td>The creep factor</td>
<td>$\phi_{28} = 2.35$</td>
</tr>
<tr>
<td>Effective elastic modulus of concrete</td>
<td>$E_{\text{eff}} = \frac{E_{\text{cm}}}{1+\phi_{28}} = 9.396 \text{ GPa}$</td>
</tr>
<tr>
<td>Mean tensile strength</td>
<td>$f_{\text{cm}} = 2.565 \text{ MPa}$</td>
</tr>
<tr>
<td>Elastic modulus of steel bars</td>
<td>$E_s = 200 \text{ GPa}$</td>
</tr>
<tr>
<td>Characteristic value of dead load</td>
<td>$g_k = 8.5 \text{ kN/m}$</td>
</tr>
<tr>
<td>Characteristic value of live load</td>
<td>$q_k = 12.0 \text{ kN/m}$</td>
</tr>
<tr>
<td>Live load combination factor</td>
<td>$\psi_2 = 0.6$</td>
</tr>
<tr>
<td>Diameter of the longitudinal reinforcement</td>
<td>$\phi_l = 18 \text{ mm}$</td>
</tr>
<tr>
<td>Diameter of the stirrup reinforcement</td>
<td>$\phi_s = 8 \text{ mm}$</td>
</tr>
<tr>
<td>Area of longitudinal reinforcement</td>
<td>$A_l = 5 \times 18^2 \pi/4 = 1272.3 \text{ m}^2$</td>
</tr>
<tr>
<td>Nominal concrete cover</td>
<td>$c_{\text{nom}} = 20 \text{ mm}$</td>
</tr>
<tr>
<td>Effective height</td>
<td>$d = h - c_{\text{nom}} - \phi_s - \phi_l/2 = 413 \text{ mm}$</td>
</tr>
<tr>
<td>Shrinkage</td>
<td>$\varepsilon_{\text{cs}} = 0.4 %$</td>
</tr>
</tbody>
</table>

The cross sectional properties without calculation details:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. stress stadium second moment of inertia</td>
<td>$I_I = 3.075 \times 10^9 \text{ mm}^4$</td>
</tr>
<tr>
<td>II. stress stadium second moment of inertia</td>
<td>$I_{II} = 2.028 \times 10^9 \text{ mm}^4$</td>
</tr>
<tr>
<td>I. stress stadium position of neutral axis</td>
<td>$x_I = 256.4 \text{ mm}$</td>
</tr>
<tr>
<td>II. stress stadium position of neutral axis</td>
<td>$x_{II} = 197.3 \text{ mm}$</td>
</tr>
</tbody>
</table>

*Figure 7.2.3.1 – The cantilever RC beam*
The calculation of deflection according to EN 1992-1-1:
The load value for the quasi-permanent load combination:
\[ p_{qp} = g_k + \psi_2 q_k = 8.5 + 0.6 \cdot 12 = 15.7 \text{ kN/m} \]

The maximum deflection with cross sectional properties in Stadium I. (uncracked):
\[ w_{k,I} = \frac{1}{8} p_{qp} L_{eff}^4 = \frac{1}{8} \frac{15.7 \cdot 4^4}{9396000 \cdot 0.003075} = 0.01739 \text{ m} = 17.39 \text{ mm} \]

The maximum deflection with cross sectional properties in Stadium II. (cracked):
\[ w_{k,II} = \frac{1}{8} p_{qp} L_{eff}^4 = \frac{1}{8} \frac{15.7 \cdot 4^4}{9396000 \cdot 0.002028} = 0.02637 \text{ m} = 26.37 \text{ mm} \]

The maximum bending moment under the quasi-permanent load:
\[ M_{max} = \frac{1}{2} p_{qp} L_{eff}^2 = \frac{1}{2} 15.7 \cdot 4^2 = 125.6 \text{ kNm} \]

The cracking moment with the mean tensile strength:
\[ M_{cs} = f_{ctm} \frac{I_I}{h-x_I} = 2565 \cdot \frac{0.003075}{0.45 - 0.2564} = 40.74 \text{ kNm} \]

The interpolation factor considering the mixture of cracked and uncracked behaviour:
\[ \xi = \max \left[ 1 - 0.5 \left( \frac{M_{cr}}{M_{max}} \right)^2 , 0 \right] = \max \left[ 1 - 0.5 \left( \frac{40.74}{125.6} \right)^2 , 0 \right] = 0.9474 \]

This value is almost 1.0, it means that the final deflection will be closer to the cracked deflection than to the uncracked one.

The final deflection with the aim of interpolation factor:
\[ w_k = (1 - \xi) w_{k,I} + \xi w_{k,II} = (1 - 0.9474) 17.39 + 0.9474 \cdot 26.37 = 25.90 \text{ mm} \]

This deflection is even greater if we are considering the effect of shrinkage.
The curvatures in uncracked and cracked states due to shrinkage:

\[
\kappa_{I,cs} = \varepsilon_{cs} \frac{E_s S_{s,I}}{E_{c,eff} I_I} = \frac{0.4}{1000} \frac{200}{9.396} \frac{1272.3 \cdot (413 - 256.4)}{3.075 \cdot 10^9} = 5.517 \cdot 10^{-4} \frac{1}{m}
\]

\[
\kappa_{II,cs} = \varepsilon_{cs} \frac{E_s S_{s,II}}{E_{c,eff} I_{II}} = \frac{0.4}{1000} \frac{200}{9.396} \frac{1272.3 \cdot (413 - 197.3)}{2.028 \cdot 10^9} = 1.152 \cdot 10^{-3} \frac{1}{m}
\]

The additional deflection due to shrinkage:

\[
w_{k,I,cs} = \frac{1}{2} L_{eff}^2 \kappa_{I,cs} = \frac{1}{2} 4^2 \cdot 5.517 \cdot 10^{-4} = 0.004414 \text{ m} = 4.414 \text{ mm}
\]

\[
w_{k,II,cs} = \frac{1}{2} L_{eff}^2 \kappa_{II,cs} = \frac{1}{2} 4^2 \cdot 1.152 \cdot 10^{-3} = 0.009216 \text{ m} = 9.216 \text{ mm}
\]

The total deflection considering cracking and the effect of shrinkage:

\[
w_{k,cs} = (1 - \zeta)(w_{k,I} + w_{k,I,cs}) + \zeta(w_{k,II} + w_{k,II,cs}) =
= (1 - 0.9474)(17.39 + 4.414) + 0.9474(26.37 + 9.216) = 34.86 \text{ mm}
\]
Verification Examples

We modelled the beam with beam finite elements. In FEM-Design we increased the division number of the beam finite elements to five to get the more accurate results.

Fig. 7.2.3.2 shows the applied cross section and reinforcement with the defined input parameters.

Fig. 7.2.3.2 – The cross section of the RC cantilever in FEM-Design

Fig. 7.2.3.3 – The deflection [mm] of the RC bar model in FEM-Design with cracked section analysis (without shrinkage [above], with shrinkage [below])
Fig. 7.2.2.3 shows the deflection after the cracked section analysis. The deflection of the beam model in FEM-Design:

Cracked section analysis without shrinkage: \( w_{AFEM} = 24.32 \text{ mm} \)
Cracked section analysis with shrinkage: \( w_{k csFEM} = 31.50 \text{ mm} \)

The difference between the hand and FEM-Design calculations is around 9%, but keep in mind that FEM-Design considers the interpolation factors individually in every finite elements one by one to get a more accurate result.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/7.2.3 Cracked deflection of a cantilever beam.str
7.2.4 Cracked deflection of a cantilever beam with compressed reinforcement bars

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span length</td>
<td>$L_{\text{eff}} = 3$ m</td>
</tr>
<tr>
<td>The cross section</td>
<td>$b = 200$ mm; $h = 400$ mm</td>
</tr>
<tr>
<td>The elastic modulus of concrete</td>
<td>$E_{\text{cm}} = 31.5$ GPa, C25/30</td>
</tr>
<tr>
<td>The creep factor</td>
<td>$\phi_{28} = 2.0$</td>
</tr>
<tr>
<td>Effective elastic modulus of concrete</td>
<td>$E_{\text{eff}} = E_{\text{cm}}/(1+\phi_{28}) = 10.5$ GPa</td>
</tr>
<tr>
<td>Mean tensile strength</td>
<td>$f_{\text{ctm}} = 2.6$ MPa</td>
</tr>
<tr>
<td>Elastic modulus of steel bars</td>
<td>$E_s = 200$ GPa</td>
</tr>
<tr>
<td>Characteristic value of the point moment at the end</td>
<td>$M_k = 40$ kNm</td>
</tr>
<tr>
<td>Diameter of the longitudinal reinforcement</td>
<td>$\phi_l = 12$ mm</td>
</tr>
<tr>
<td>Diameter of the stirrup reinforcement</td>
<td>$\phi_s = 10$ mm</td>
</tr>
<tr>
<td>Area of longitudinal reinforcement (tension)</td>
<td>$A_l = 4 \times 12^2 \pi / 4 = 452.4$ m$^2$</td>
</tr>
<tr>
<td>Area of longitudinal reinforcement (compression)</td>
<td>$A_l' = 2 \times 12^2 \pi / 4 = 226.2$ m$^2$</td>
</tr>
<tr>
<td>Nominal concrete cover</td>
<td>$c_{\text{nom}} = 20$ mm</td>
</tr>
<tr>
<td>Effective height (tension)</td>
<td>$d = h - c_{\text{nom}} - \phi_s - \phi_l / 2 = 364$ mm</td>
</tr>
<tr>
<td>Effective height (compression)</td>
<td>$d' = c_{\text{nom}} + \phi_s + \phi_l / 2 = 36$ mm</td>
</tr>
<tr>
<td>Shrinkage</td>
<td>$\varepsilon_{\text{cs}} = 0.4$ %</td>
</tr>
</tbody>
</table>

The cross sectional properties without calculation details:

<table>
<thead>
<tr>
<th>Moment of Inertia</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. stress stadium second moment of inertia</td>
<td>$I_l = 1.409 \times 10^9$ mm$^4$</td>
</tr>
<tr>
<td>II. stress stadium second moment of inertia</td>
<td>$I_{ll} = 6.563 \times 10^9$ mm$^4$</td>
</tr>
<tr>
<td>I. stress stadium position of neutral axis</td>
<td>$x_l = 207.6$ mm</td>
</tr>
<tr>
<td>II. stress stadium position of neutral axis</td>
<td>$x_{ll} = 128.0$ mm</td>
</tr>
</tbody>
</table>

Figure 7.2.4.1 – The cantilever RC beam with compressed RC bars
The calculation of deflection according to EN 1992-1-1:

The load value for the quasi-permanent load combination (see Fig. 7.2.4.1):

\[ M_k = 40 \text{ kNm} \]

The maximum deflection with cross sectional properties in Stadium I. (uncracked):

\[
\frac{1}{2}\frac{M_k L_{\text{eff}}^2}{E_{\text{eff}} I_I} = \frac{1}{2}\frac{40 \cdot 3^2}{10500000 \cdot 0.001409} = 0.01217 \text{ m} = 12.17 \text{ mm}
\]

The maximum deflection with cross sectional properties in Stadium II. (cracked):

\[
\frac{1}{2}\frac{M_k L_{\text{eff}}^2}{E_{\text{eff}} I_{II}} = \frac{1}{2}\frac{40 \cdot 3^2}{10500000 \cdot 0.0006563} = 0.02612 \text{ m} = 26.12 \text{ mm}
\]

The maximum bending moment under the quasi-permanent load (see Fig. 7.2.4.1):

\[ M_{\text{max}} = M_k = 40 \text{ kNm} \]

The cracking moment with the mean tensile strength:

\[
M_{cr} = f_{ctm} I_I = \frac{2600}{0.001409} \cdot \frac{0.40 - 0.2076}{10.5} = 19.04 \text{ kNm}
\]

The interpolation factor considering the mixture of cracked and uncracked behaviour:

\[
\zeta = \max \left[ 1 - 0.5 \left( \frac{M_{cr}}{M_{\text{max}}} \right)^2, 0 \right] = \max \left[ 1 - 0.5 \left( \frac{19.04}{40} \right)^2, 0 \right] = 0.8867
\]

The final deflection with the aim of interpolation factor:

\[
w_k = (1 - \zeta) w_{k,I} + \zeta w_{k,II} = (1 - 0.8867) \cdot 12.17 + 0.8867 \cdot 26.12 = 24.54 \text{ mm}
\]

This deflection is even greater if we are considering the effect of shrinkage.

The curvatures in uncracked and cracked states due to shrinkage:

\[
\kappa_{I,cs} = \frac{E_s}{E_{s,\text{eff}}} \frac{S_{s,I}}{I_I} = \frac{0.4}{1000} \cdot \frac{200 \cdot 452.4 \cdot (364 - 207.6) - 226.2 \cdot (207.6 - 36)}{1.409 \cdot 10^9} = 1.727 \cdot 10^{-4} \text{ m}
\]

\[
\kappa_{II,cs} = \frac{E_s}{E_{s,\text{eff}}} \frac{S_{s,II}}{I_{II}} = \frac{0.4}{1000} \cdot \frac{200 \cdot 452.4 \cdot (364 - 128.0) - 226.2 \cdot (128.0 - 36)}{6.563 \cdot 10^8} = 9.979 \cdot 10^{-4} \text{ m}
\]

The additional deflection due to shrinkage:
Verification Examples

\[ w_{k,I,cs} = \frac{1}{2} L_{\text{eff}}^2 \kappa_{I,cs} = \frac{1}{2} 3^2 \cdot 1.727 \cdot 10^{-4} = 0.0007772 \text{ m} = 0.7772 \text{ mm} \]

\[ w_{k,II,cs} = \frac{1}{2} L_{\text{eff}}^2 \kappa_{II,cs} = \frac{1}{2} 3^2 \cdot 9.979 \cdot 10^{-4} = 0.004491 \text{ m} = 4.491 \text{ mm} \]

The total deflection considering cracking and the effect of shrinkage:

\[ w_{k,cs} = (1 - \zeta) (w_{k,I} + w_{k,I,cs}) + \zeta (w_{k,II} + w_{k,II,cs}) = (1 - 0.8867)(12.17 + 0.7772) + 0.8867(26.12 + 4.491) = 28.61 \text{ mm} \]

We modelled the beam with beam finite elements. In FEM-Design we increased the division number of the beam finite elements to five to get the more accurate results.

![Figure 7.2.4.2 – The cross section of the RC cantilever in FEM-Design](image)

Fig. 7.2.4.2 shows the applied cross section and reinforcement with the defined input parameters.
Fig. 7.2.4.3 shows the deflection after the cracked section analysis. The deflection of the beam model in FEM-Design:

Cracked section analysis without shrinkage: \( w_{kFEM} = 23.03 \text{ mm} \)

Cracked section analysis with shrinkage: \( w_{kcsFEM} = 27.13 \text{ mm} \)

The difference between the hand and FEM-Design calculations is around 7%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.4 Cracked deflection of a cantilever beam with compressed reinforcement bars
7.2.5 *Cracked deflection of a cantilever with bending moment and normal forces*

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span length</td>
<td>$L_{\text{eff}} = 2 \text{ m}$</td>
</tr>
<tr>
<td>The cross section</td>
<td>Rectangle: $b = 200 \text{ mm}; h = 400 \text{ mm}$</td>
</tr>
<tr>
<td>The elastic modulus of concrete</td>
<td>$E_{\text{cm}} = 31.5 \text{ GPa}$</td>
</tr>
<tr>
<td>The creep factor</td>
<td>$\varphi_{28} = 2.0$</td>
</tr>
<tr>
<td>Effective elastic modulus of concrete</td>
<td>$E_{\text{ceff}} = E_{\text{cm}}/(1 + \varphi_{28}) = 10.5 \text{ GPa}$</td>
</tr>
<tr>
<td>Ratio between the moduli</td>
<td>$\alpha = E_s/E_{\text{ceff}} = 19.05$</td>
</tr>
<tr>
<td>Mean tensile strength</td>
<td>$f_{\text{cm}} = 2.2 \text{ MPa}$</td>
</tr>
<tr>
<td>Elastic modulus of steel bars</td>
<td>$E_s = 200 \text{ GPa}$</td>
</tr>
<tr>
<td>Diameter of the longitudinal reinforcement</td>
<td>$\phi_l = 12 \text{ mm}$</td>
</tr>
<tr>
<td>Diameter of the stirrup reinforcement</td>
<td>$\phi_s = 10 \text{ mm}$</td>
</tr>
<tr>
<td>Area of longitudinal reinforcement (top)</td>
<td>$A_l = 4 \times 12 \pi \times 4 = 452.4 \text{ m}^2$</td>
</tr>
<tr>
<td>Area of longitudinal reinforcement (bottom)</td>
<td>$A_l' = 4 \times 12 \pi \times 4 = 452.4 \text{ m}^2$</td>
</tr>
<tr>
<td>Nominal concrete cover</td>
<td>$c_{\text{nom}} = 20 \text{ mm}$</td>
</tr>
<tr>
<td>Effective height (bottom)</td>
<td>$d = h - c_{\text{nom}} - \phi_s - \phi_l/2 = 364 \text{ mm}$</td>
</tr>
<tr>
<td>Effective height (top)</td>
<td>$d' = c_{\text{nom}} + \phi_s + \phi_l/2 = 36 \text{ mm}$</td>
</tr>
</tbody>
</table>

In this chapter we will calculate the deflection (vertical translation) of the end of a cantilever (see Fig. 7.2.5.1). According to the behaviour of the reinforced concrete material this deflections will depend on the amount of the applied normal force. At the free end of the cantilever we applied a constant concentrated bending moment ($M=50 \text{kNm}$) and we changed the intensity of the applied concentrated normal force at the end from $-500 \text{kN}$ compression to $+500 \text{kN}$ tension. The force is acting on the centroid of the uncracked RC section. Now it is in the middle.

**During the hand calculation (and by the FEM-Design calculation as well) we considered eccentricity caused by cracking in cracked section analysis.**

We are going to calculate the compressed concrete zone by hand at five notable compressed zone condition and based on these values the inhomogeneous inertias and areas are also can be calculated. The deflections depend on these inertias. After these we calculate the interpolation factors to get the accurate deflection considering the tension stiffening effect. At the end of this chapter we compare the results with FEM-Design solutions.
Case a)

\( M = 50 \text{ kNm} \); \( N = -500 \text{ kN} \) (compression):

In this case the complete section is uncracked, the total concrete zone is active. Therefore we only need to calculate an inhomogeneous cross-section.

\[
A_a = b \cdot h + \alpha (A_i + A_i') = 200 \cdot 400 + 19.05 \cdot (452.4 + 452.4) = 0.09724 \text{ m}^2
\]

\[
I_a = \frac{b \cdot h^3}{12} + \alpha A_i \left( d - \frac{h}{2} \right)^2 + \alpha A_i' \left( \frac{h}{2} - d' \right)^2 =
\]

\[
= \frac{200 \cdot 400^3}{12} + 19.05 \cdot 452.4 \left( 364 - \frac{400}{2} \right)^2 + 19.05 \cdot 452.4 \left( \frac{400}{2} - 36 \right)^2 = 0.0015303 \text{ m}^4
\]

The stresses at the extreme fibres are as follows:

\[
\sigma_{\text{top}} = \frac{N}{A_a} + \frac{M}{I_a} \frac{h}{2} = \frac{-500}{0.09724} + \frac{50}{0.0015303} \frac{0.4}{2} = 1.392 \text{ MPa} < f_{\text{ctm}} = 2.2 \text{ MPa} \quad \text{(tension)}
\]

\[
\sigma_{\text{bottom}} = \frac{N}{A_a} - \frac{M}{I_a} \frac{h}{2} - \frac{50}{0.0015303} \frac{0.4}{2} = -7.049 \text{ MPa} \quad \text{(compression)}
\]

Based on these equation it is trivial that the first crack occur at the following normal force:

\[
N_{\text{crack}} = \left( f_{\text{ctm}} - \frac{M}{I_a} \frac{h}{2} \right) A_a = \left( 2200 - \frac{50}{0.0015303} \frac{0.4}{2} \right) 0.09724 = -421.5 \text{ kN} \quad \text{(compression)}
\]

The deflection (vertical translation) of the free end of the cantilever:

\[
w_a = \frac{M L_{\text{eff}}^2}{2E_{\text{eff}} I_a} = \frac{50 \cdot 2^2}{2 \cdot 10500000 \cdot 0.0015303} = 6.223 \text{ mm}
\]

Case b)

\( M = 50 \text{ kNm} \); \( N = -200 \text{ kN} \) (compression):

The compressed zone (measured from the compressed side) in this case comes from the solution of a third order polynomial:

\[
\frac{M}{N} - \frac{\alpha A_i (d - x_b) (d - h/2) + \alpha A_i' (x_b - d') (h/2 - d') + \frac{b x_b^2}{2} (h/2 - \frac{1}{3} x_b)}{\alpha A_i (d - x_b) - \alpha A_i' (x_b - d') - \frac{b x_b^2}{2}} = 0
\]

The relevant solution of this equation:

\( x_b = 199.7 \text{ mm} \)
Based on this, the position of the centroid measured from the compressed side:

\[
x_{Sb} = \frac{b x_b^2}{2} + \alpha A_i d + \alpha A_i' d' - \frac{b x_b + \alpha (A_i + A_i')}{b x_b + \alpha (A_i + A_i')} = 130.0 \text{ mm}
\]

According to this the cracked section area and inertia:

\[
A_b = b x_b + \alpha (A_i + A_i') = 200 \cdot 199.7 + 19.05 \cdot (452.4 + 452.4) = 0.05718 \text{ m}^2
\]

\[
I_b = \frac{b x_b^3}{12} + b x_b \left( x_{Sb} - x_b / 2 \right)^2 + \alpha A_i \left( d - x_{Sb} \right)^2 + \alpha A_i' \left( x_{Sb} - d' \right)^2 = 0.0007171 \text{ m}^4
\]

The deflection (vertical translation) of the free end of the cantilever with this cracked condition considering the decreased moment at the cracked centroid from the additional eccentricities (because the original loads are acting on the centroid of the uncracked RC section):

\[
w_{II} = \frac{(M + N (h/2 - x_{Sb})) L_{eff}^2}{2 E_{eff} I_b} = \frac{(50 - 200 (0.2 - 0.130)) \cdot 2^2}{2 \cdot 10500000 \cdot 0.0007171} = 9.562 \text{ mm}
\]

But this is not the final deflection, because we need to consider the tension stiffening to get a more comparable result.

The stress in the tension reinforcement calculated on the basis of a cracked section and the additional moment due to cracked eccentricity:

\[
\sigma_{sb} = \alpha \left[ \frac{N}{A_b} + \frac{M + N (h/2 - x_{Sb})}{I_b} \left( d - x_{Sb} \right) \right] = 19.05 \left[ \frac{-200}{0.05718} + \frac{50 - 200(0.2 - 0.130)}{0.0007171} \left( 0.364 - 0.130 \right) \right] = 157.2 \text{ MPa}
\]

The load conditions causing first crack:

\[
\eta_h \left[ \frac{N}{A_a} + \frac{M}{I_a} \left( h/2 \right) \right] = \eta_h \left[ \frac{-200}{0.09724} + \frac{50}{0.0015303} \left( 0.4 / 2 \right) \right] = f_{cme} = 2.2 \text{ MPa} ; \text{ thus } \eta_h = 0.4913
\]

Thus the normal force:

\[
\eta_h N = 0.4913 \cdot (-200) = -98.26 \text{ kN} \text{ (compression)}
\]

And the bending moment:

\[
\eta_h M = 0.4913 \cdot 50 = 24.57 \text{ kNm}
\]
The stress in the tension reinforcement calculated on the basis of a cracked section under the loading conditions causing first cracking considering the additional moment due to cracked eccentricity:

\[
\sigma_{srb} = \alpha \left[ \frac{\eta_b N}{A_b} + \eta_b M + \eta_b N \left( \frac{h}{2} - x_{sb} \right) \left( d - x_{sb} \right) \right] = 19.05 \left[ -98.26 + \frac{24.57}{0.05718} - 98.26 \left( \frac{0.2}{0.0007171} \right) \left( \frac{0.364}{0.130} \right) \right] = 77.24 \text{ MPa}
\]

The interpolation factor considering the mixture of cracked and uncracked behaviour:

\[
\zeta_b = \max \left[ 1 - 0.5 \left( \frac{\sigma_{srb}}{\sigma_{sb}} \right)^2, 0 \right] = \max \left[ 1 - 0.5 \left( \frac{77.24}{157.2} \right)^2, 0 \right] = 0.8792
\]

The final deflection with the aim of interpolation factor:

\[
w_b = (1 - \zeta_b) w_a + \zeta_b w_{bII} = (1 - 0.8792) 6.223 + 0.8792 \times 9.562 = 9.159 \text{ mm}
\]

Case c)

\[M = 50 \text{ kNm}; N = 0 \text{ kN (pure bending)};\]

In this case the cross section is under pure bending. In this situation the following equation gives the position of the compressed zone (cracking occur on the tension side).

\[
\frac{1}{2} x_c^2 b + \alpha (x_c - d') A_{t'} + \alpha (x_c - d) A_t = 0
\]

The relevant solution of this equation:

\[x_c = 118.5 \text{ mm}\]

Based on this value the cracked cross-sectional area and inertia:

\[A_c = b \times x_c + \alpha (A_t + A_{t'}) = 200 \times 118.51 + 19.05 \times (452.4 + 452.4) = 0.04094 \text{ m}^2\]

\[I_c = \frac{b \times x_c^3}{3} + \alpha A_t \left[ (d - x_c)^2 + \alpha A_{t'} \left( x_c - d' \right)^2 \right] =
\]

\[= \frac{200 \times 118.5^3}{3} + 19.05 \times 452.4 \left[ 364 - 118.5 \right]^2 + 19.05 \times 452.4 \left[ 118.5 - 36 \right]^2 = 0.000689 \text{ m}^4\]

The deflection (vertical translation) of the free end of the cantilever with this cracked condition:

\[
w_{eff} = \frac{M L_{eff}^2}{2E_{eff} I_c} = \frac{50 \times 2^2}{2 \times 10500000 \times 0.000689} = 13.82 \text{ mm}\]
But this is not the final deflection, we need to consider the tension stiffening to get a more comparable solutions. The interpolation factor based on the pure bending condition:

The cracking moment with the mean tensile strength:

$$ M_{cr} = f_{ctm} \frac{I_d}{h/2} = 2200 \frac{0.0015303}{0.4/2} = 16.83 \text{kNm} $$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$ \zeta_c = \max \left[ 1 - 0.5 \left( \frac{M_{cr}}{M} \right)^2, 0 \right] = \max \left[ 1 - 0.5 \left( \frac{16.83}{50} \right)^2, 0 \right] = 0.9434 $$

The final deflection with the aim of interpolation factor:

$$ w_c = (1 - \zeta_c) w_a + \zeta_c w_{cII} = (1 - 0.9434) 6.223 + 0.9434 \cdot 13.82 = 13.39 \text{mm} $$

Case d)

$M = 50 \text{kNm}; \ N = 200 \text{kN (tension)}$

The compressed zone (measured from the compressed side) in this case comes from the solution of a third order polynomial:

$$ \frac{M}{N} - \frac{\alpha A_1 (d-x_d)(d-h/2) + \alpha A_1' (x_d-d')(h/2-d') + b \frac{x_d^2}{2} (h/2 - \frac{1}{3} x_d)}{\alpha A_1 (d-x_d) - \alpha A_1' (x_d-d') - b \frac{x_d^2}{2}} = 0 $$

The relevant solution of this equation:

$$ x_d = 58.38 \text{mm} $$

Based on this the position of the centroid measured from the compressed side:

$$ x_{sd} = \frac{b \frac{x_d^2}{2} + \alpha A_1 d + \alpha A_1' d'}{b x_d + \alpha (A_1 + A_1')} = 131.0 \text{mm} $$

According to this the cracked section area and inertia:

$$ A_d = b x_d + \alpha (A_1 + A_1') = 200 \cdot 58.38 + 19.05 \cdot (452.4 + 452.4) = 0.02891 \text{m}^2 $$

$$ I_d = \frac{b \cdot x_d^3}{12} + b x_d (x_{sd} - x_d/2)^2 + \alpha A_1 (d-x_{sd})^2 + \alpha A_1' (x_{sd} - d')^2 = 0.0006700 \text{m}^4 $$
The deflection (vertical translation) of the free end of the cantilever with this cracked condition considering the increased moment at the cracked centroid from the additional eccentricities (because the original loads are acting on the centroid of the uncracked RC section):

\[
w_{\text{dII}} = \frac{(M + N(h/2 - x_{sd}))L_{\text{eff}}^2}{2E_{\text{eff}}I_d} = \frac{(50 + 200(0.2 - 0.131)) \cdot 2^2}{2 \cdot 10500000 \cdot 0.00067} = 18.14 \text{mm}
\]

But this is not the final deflection, we need to consider the tension stiffening to get a more comparable solution.

The stress in the tension reinforcement calculated on the basis of a cracked section and the additional moment due to cracked eccentricity:

\[
\sigma_{sd} = \alpha \left[ \frac{N}{A_d} + \frac{M + N(h/2 - x_{sd}) \cdot (d - x_{sd})}{I_d} \right] = 19.05 \left[ \frac{200}{0.02891} + \frac{50 + 200(0.2 - 0.131)}{0.00067} \cdot (0.364 - 0.131) \right] = 554.5 \text{MPa}
\]

The load conditions causing first crack:

\[
\eta_{d} \left[ \frac{N}{A_a} + \frac{M(h/2)}{I_a} \right] = \eta_{d} \left[ \frac{200}{0.09724} + \frac{50}{0.0015303} \cdot (0.4/2) \right] = f_{\text{cm}} = 2.2 \text{MPa}; \text{ thus } \eta_{d} = 0.2560
\]

Thus the normal force:

\[
\eta_{d} N = 0.2560 \cdot 200 = 51.2 \text{kN kNm}
\]

And the bending moment:

\[
\eta_{d} M = 0.2560 \cdot 50 = 12.80 \text{kNm}
\]

The stress in the tension reinforcement calculated on the basis of a cracked section under the loading conditions causing first cracking considering the additional moment due to cracked eccentricity:

\[
\sigma_{srd} = \alpha \left[ \eta_{d} \frac{N}{A_d} + \eta_{d} M + \eta_{d} N(h/2 - x_{sd}) \cdot (d - x_{sd}) \right] = 19.05 \left[ \frac{51.2}{0.02891} + \frac{12.8 + 51.2(0.2 - 0.131)}{0.00067} \cdot (0.364 - 0.131) \right] = 141.9 \text{MPa}
\]

The interpolation factor considering the mixture of cracked and uncracked behaviour:

\[
\zeta_{d} = \max \left[ 1 - 0.5 \left( \frac{\sigma_{srd}}{\sigma_{sd}} \right)^2, 0 \right] = \max \left[ 1 - 0.5 \left( \frac{141.9}{554.5} \right)^2, 0 \right] = 0.9673
\]
The final deflection with the aim of interpolation factor:

\[ w_d = (1 - \zeta_d) w_a + \zeta_d w_{\text{eff}} = (1 - 0.9673)6.223 + 0.9673 \times 18.14 = 17.75 \text{ mm} \]

Case e)

M = 50 kNm; N= 500 kN (tension):

In this case the cross section is fully cracked (therefore this is only a theoretical solution). The whole concrete zone is cracked. Practically in means that only the reinforcement bars are active in the section, thus the cross-sectional properties:

\[ A_e = \alpha (A_l + A'_l) = 19.05 \times (452.4 + 452.4) = 0.01724 \text{ m}^2 \]

\[ I_e = \alpha A_l \left( d - h/2 \right)^2 + \alpha A'_l \left( h/2 - d' \right)^2 = 19.05 \times 452.4 \left( 364 - 200 \right)^2 + 19.05 \times 452.4 \left( 200 - 36 \right)^2 = 0.0004636 \text{ m}^4 \]

The deflection (vertical translation) of the free end of the cantilever with this fully cracked condition (only the RC bars are working):

\[ w_{\text{eff}} = \frac{M I_e^2}{2 E_{\text{eff}} I_e} = \frac{50 \times 2^2}{2 \times 10500000 \times 0.0004636} = 20.54 \text{ mm} \]

But this is not the final deflection, we need to consider the tension stiffening to get a more comparable solution.

The stress in the tension reinforcement calculated on the basis of a cracked section:

\[ \sigma_{se} = \alpha \left( \frac{N}{A_e} + \frac{M}{I_e} \left( d - h/2 \right) \right) = 19.05 \left[ \frac{500}{0.01724} + \frac{50}{0.0004636} \left( 0.364 - 0.2 \right) \right] = 889.4 \text{ MPa} \]

The load conditions causing first crack:

\[ \eta_e \left[ \frac{N}{A_e} + \frac{M}{I_e} \left( h/2 \right) \right] = \eta_e \left[ \frac{500}{0.09724} + \frac{50}{0.0015303} \left( 0.4/2 \right) \right] = f_{\text{cmt}} = 2.2 \text{ MPa} \quad \text{; thus} \quad \eta_e = 0.4279 \]

Thus the normal force:

\[ \eta_e N = 0.4279 \times 500 = 214.0 \text{ kN} \]

And the bending moment:

\[ \eta_e M = 0.4279 \times 50 = 21.40 \text{ kNm} \]

The stress in the tension reinforcement calculated on the basis of a cracked section under the loading conditions causing first cracking:
\[ \sigma_{sre} = \alpha \left[ \frac{\eta N}{A_e} + \frac{\eta M}{I_e} (d - h/2) \right] = 19.05 \left[ \frac{214.0}{0.01724} + \frac{21.40}{0.0004636} (0.364 - 0.2) \right] = 380.7 \text{MPa} \]

The interpolation factor considering the mixture of cracked and uncracked behaviour:

\[ \zeta_e = \max \left[ 0, 1 - 0.5 \left( \frac{\sigma_{sre}}{\sigma_{se}} \right)^2 \right] = \max \left[ 0, 1 - 0.5 \left( \frac{380.7}{889.4} \right)^2 \right] = 0.9084 \]

The final deflection with the aim of interpolation factor:

\[ w_e = (1 - \zeta_e) w_a + \zeta_e w_{e II} = (1 - 0.9084) \cdot 6.223 + 0.9084 \cdot 20.54 = 19.23 \text{mm} \]

Fig. 7.2.5.2 shows the FEM-Design model with bars. Fig. 7.2.5.3 shows the deflections (vertical translations) of the free end of cantilever under the constant bending moment regarding different normal forces (compression and tension as well). In Fig. 7.2.5.3 in addition to the hand calculation we indicated the FEM-Design results with beam model and with shell model also. The differences between the hand and FEM-Design calculations are less than 5\%, thus we can say that the results are identical to each other.
NOTE: The cracked section analysis in FEM-Design is based on a non-linear calculation but only accurate by SLS combinations because the used material model for the reinforcement is linear (both compression and tension) and for the concrete is a non-tension material (only compression, assumed to be linear) if the extreme fibres reached the mean tensile strength in the cross-section.

Download link to the example files:
Beam model:
http://download.strusoft.com/FEM-Design/inst170x/models/7.2.5 Cracked deflection of a cantilever with bending moment and different normal forces beam.str
Shell model:
http://download.strusoft.com/FEM-Design/inst170x/models/7.2.5 Cracked deflection of a cantilever with bending moment and different normal forces shell.str
### 7.2.6 Cracked deflection of a simply supported square slab

**Inputs:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span length</td>
<td>L&lt;sub&gt;eff&lt;/sub&gt; = 6 m (Fig. 6.2.3.1)</td>
</tr>
<tr>
<td>The thickness</td>
<td>h = 200 mm</td>
</tr>
<tr>
<td>The elastic modulus of concrete</td>
<td>E&lt;sub&gt;cm&lt;/sub&gt; = 30 GPa, C20/25</td>
</tr>
<tr>
<td>The Poisson's ratio of concrete</td>
<td>ν = 0.2</td>
</tr>
<tr>
<td>The creep factor</td>
<td>ϕ&lt;sub&gt;28&lt;/sub&gt; = 2.35</td>
</tr>
<tr>
<td>Effective elastic modulus of concrete</td>
<td>E&lt;sub&gt;eff&lt;/sub&gt; = E&lt;sub&gt;cm&lt;/sub&gt;/(1+ϕ&lt;sub&gt;28&lt;/sub&gt;) = 8.96 GPa</td>
</tr>
<tr>
<td>Mean tensile strength</td>
<td>f&lt;sub&gt;cm&lt;/sub&gt; = 2.2 MPa</td>
</tr>
<tr>
<td>Elastic modulus of steel bars</td>
<td>E&lt;sub&gt;s&lt;/sub&gt; = 200 GPa</td>
</tr>
<tr>
<td>Characteristic value of dead load</td>
<td>g&lt;sub&gt;k&lt;/sub&gt; = 6 kN/m</td>
</tr>
<tr>
<td>Characteristic value of live load</td>
<td>q&lt;sub&gt;k&lt;/sub&gt; = 10 kN/m</td>
</tr>
<tr>
<td>Live load combination factor</td>
<td>ψ&lt;sub&gt;2&lt;/sub&gt; = 0.6</td>
</tr>
<tr>
<td>Diameter of the longitudinal reinforcement</td>
<td>φ&lt;sub&gt;l&lt;/sub&gt; = 10 mm/200 mm</td>
</tr>
<tr>
<td>The specific reinforcement</td>
<td>A&lt;sub&gt;s&lt;/sub&gt; = 0.393 mm&lt;sup&gt;2&lt;/sup&gt;/mm</td>
</tr>
<tr>
<td>Nominal concrete cover</td>
<td>c&lt;sub&gt;x&lt;/sub&gt; = 30 mm; c&lt;sub&gt;y&lt;/sub&gt; = 40 mm</td>
</tr>
<tr>
<td>Average effective height</td>
<td>d = 160 mm</td>
</tr>
<tr>
<td>The ratio of the elastic modulus</td>
<td>α&lt;sub&gt;s&lt;/sub&gt; = E&lt;sub&gt;s&lt;/sub&gt;/E&lt;sub&gt;eff&lt;/sub&gt; = 22.32</td>
</tr>
</tbody>
</table>

The load value for the quasi-permanent load combination:

\[
p_{qp} = g_k + \psi q_k = 6 + 0.6 \times 10 = 12 \text{ kN/m}^2
\]
The deflection of an isotropic simply supported square slab under uniform load (see Chapter 1.3):

\[ w_{\text{max}} = 0.00416 \frac{p a^4}{E h^3} \left( \frac{12}{12(1-\nu^2)} \right) \]

The uncracked unreinforced specific inertia (only with the concrete):

\[ I_c = \frac{h^3}{12} = \frac{200^3}{12} = 6.667 \cdot 10^5 \text{ mm}^4 \text{ mm} = 0.0006667 \text{ m}^4 \text{ m} \]

The deflection based on this inertia:

\[ w_{\text{max}, c} = 0.00416 \frac{12 \cdot 6^4}{8960000 \cdot 0.0006667} \left( \frac{1}{1-0.2^2} \right) = 0.01040 \text{ m} = 10.4 \text{ mm} \]

The uncracked reinforced specific inertia (Stadium I.):

\[ I_I = \frac{h^2}{2} + \alpha_s a_s d = \frac{200^2}{2} + 22.32 \cdot 0.393 \cdot 160 = 102.52 \text{ mm} \]

\[ I_I = \frac{x_I^3}{3} + \frac{(h-x_I)^3}{3} + \alpha_s a_s (d-x_I)^2 = \frac{102.52^3}{3} + \frac{(200-102.52)^3}{3} + 22.32 \cdot 0.393 (160-102.52)^2 \]

\[ I_I = 6.969 \cdot 10^5 \text{ mm}^4 \text{ mm} = 0.0006969 \text{ m}^4 \text{ m} \]

The deflection based on this inertia:

\[ w_{\text{max}, I} = 0.00416 \frac{12 \cdot 6^4}{8960000 \cdot 0.0006969} \left( \frac{1}{1-0.2^2} \right) = 0.009947 \text{ m} = 9.947 \text{ mm} \]
The cracked reinforced specific inertia (Stadium II):

\[
x_{II} = \frac{x_{II}^2 + \alpha_s a_s d}{2} = \frac{x_{II}^2 + 22.32 \cdot 0.393 \cdot 160}{x_{II} + 22.32 \cdot 0.393}
\]

\[x_{II} = 44.93 \text{ mm}\]

\[
I_{II} = \frac{x_{II}^3}{3} + \alpha_s a_s (d - x_{II})^2 = \frac{44.93^3}{3} + 22.32 \cdot 0.393 (160 - 44.93)^2 = 1.4638 \cdot 10^4 \text{ mm}^4
\]

The deflection based on this inertia:

\[w_{\text{max}, II} = 0.00416 \frac{12 \cdot 6^4}{8960000 \cdot 0.00014638 \cdot (1 - 0.2^2)} = 0.04735 \text{ m} = 47.35 \text{ mm}\]

The cracking moment:

\[m_{\text{cr}} = f_{\text{ctm}} \frac{I_{\text{cr}}}{h - x_{\text{cr}}} = 2.2 \cdot \frac{6.969 \cdot 10^5}{200 - 102.52} = 15730 \text{ Nmm} = 15.73 \text{ kNm}\]

\[m_{\text{max}} = 0.0469 \text{ kNm}^2 = 0.0469 \cdot 12 \cdot 6^2 = 20.26 \text{ kNm}\]

The interpolation factor considering the mixture of cracked and uncracked behaviour:

\[\zeta = \max \left[ 1 - 0.5 \left( \frac{m_{\text{cr}}}{m_{\text{max}}} \right)^2, 0 \right] = \max \left[ 1 - 0.5 \left( \frac{15.73}{20.26} \right)^2, 0 \right] = 0.6986\]

The deflection based on this interpolation factor:

\[w_{\text{max, cr}} = (1 - \zeta) w_{\text{max, I}} + \zeta w_{\text{max, II}} = (1 - 0.6986) 9.947 + 0.6986 \cdot 47.35 = 36.08 \text{ mm}\]
Verification Examples

Figure 7.2.6.4 – The finite element mesh [average element size: 0.2 m]

Figure 7.2.6.5 – The reactions [kN/m] without reinforcement and with reinforced cracked section analysis

Figure 7.2.6.6 – The $m_x$ [kNm/m] without reinforcement and with reinforced cracked section analysis
Verification Examples

Figure 7.2.6.7 – The $m_y$ [kNm/m] without reinforcement and with reinforced cracked section analysis

Figure 7.2.6.8 – The $m_x$ [kNm/m] without reinforcement and with reinforced cracked section analysis
This hand calculation method is very very conservative, thus do not considering the realistic crack pattern and the torsional effects. In addition the shear deformations in the slab is also neglected.

Fig. 7.2.6.9 shows the deflection based on FEM-Design. Here the difference between the deflections is quite large but this comes from the mentioned very very conservative hand calculation method.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.2.3 Cracked deflection of a simply supported square slab.str

Figure 7.2.6.9 – The deflection [mm] without reinforcement and with reinforced cracked section analysis
7.3 Nonlinear soil calculation

This chapter goes beyond the scope of this document, therefore additional informations are located in:

FEM-Design – Geotechnical modul in 3D, Theoretical background and verification and validation handbook

7.4 Elasto-plastic calculations

7.4.1 Elasto-plastic point support in a beam

Inputs:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Span length</td>
<td>L = 6 m</td>
</tr>
<tr>
<td>Distributed load</td>
<td>q = 90 kN/m</td>
</tr>
<tr>
<td>Structural steel</td>
<td>S235, f_y = 235 MPa</td>
</tr>
<tr>
<td>Young's modulus of steel</td>
<td>E = 210 GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>G = 80.77 GPa</td>
</tr>
<tr>
<td>Cross section</td>
<td>IPE 400</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>A = 8446 mm²</td>
</tr>
<tr>
<td>The first principal inertia</td>
<td>I₁ = 231283781 mm⁴</td>
</tr>
<tr>
<td>The shear correction factor in</td>
<td>ρ₂ = 0.4</td>
</tr>
<tr>
<td>the relevant direction</td>
<td></td>
</tr>
<tr>
<td>The elastic cross sectional</td>
<td>W_el = 1156419 mm³</td>
</tr>
<tr>
<td>modulus</td>
<td></td>
</tr>
<tr>
<td>The first moment of the half</td>
<td>S₀ = 653575 mm³</td>
</tr>
<tr>
<td>of the cross sectional area</td>
<td></td>
</tr>
<tr>
<td>about the centroid</td>
<td></td>
</tr>
<tr>
<td>Plastic load-bearing moment</td>
<td>Mₚ₀ = 2S₀f_y = 307.2 kNm</td>
</tr>
<tr>
<td>capacity of the section</td>
<td></td>
</tr>
</tbody>
</table>

The problem is a beam with fixed end on the left side and a roller on the right side (see Fig. 7.4.1.1). The input parameters are in the table above. First of all we calculate the deflection and the internal forces according to the external total distributed load based on linear elastic theory (Case 1). For the second calculation we assume that the fixed support can only bear maximum bending moment (Case 2). Thus we assume a plastic hinge after a certain amount of load level when the beam reaches this plastic limit moment in the fixed support.

Based on the plastic hinge the distribution of the internal forces and the deflection will be different from the linear elastic calculation.

At the hand calculation we neglect the shear deformation (Euler-Bernoulli beam theory) to get a simple solution for this problem. In FEM-Design the applied beam theory considers the shear deformation (Timoshenko beam theory). It means that when we compare the solution of the hand calculation to the finite element solution the deflection will be a bit larger in case of the
FEM calculation. To avoid this difference in the FEM analysis of the beam then the cross-sectional area must be rewritten with a large number (e.g.: \(A = 10^6 \text{m}^2\)). Practically it means that the shear stiffness is almost infinite. In this case the solutions will be the same than the results of the hand calculations. At the end of this example we will show two finite element results. One of them will neglect and the other will consider the shear deformation.

Case 1:

The characteristic shear forces, bending moments, rotations and translations are indicated in Fig. 7.4.1.3 left side according to linear elastic calculation. Here are some hand calculation results without further details based on the classical theory of elasticity:

\[
V_{\text{max}} = \frac{5}{8} q L = \frac{5}{8} \times 90 \times 6 = 337.5 \text{ kN} \quad ; \quad M_{\text{max}} = \frac{q L^2}{8} = \frac{90 \times 6^2}{8} = 405 \text{ kNm}
\]

\[
\phi_{\text{roller}} = \frac{q L^3}{48 EI_i} = \frac{90 \times 6^3}{48 \times 210 \times 10^6 \times 0.000231283781} = 0.008339 \text{ rad}
\]

\[
e_{\text{midspan}} = \frac{2}{384} \times \frac{q L^4}{EI_i} = \frac{2}{384} \frac{90 \times 6^4}{210 \times 10^6 \times 0.000231283781} = 12.51 \text{ mm}
\]

Case 2:

To easily consider the bending moment limit (plastic analysis) at the fixed support the following simplification need to be declared at the hand calculation. Until the certain amount of load level \((q_1)\) which causes \(M_{pl}\) bending moment at the fixed support the behaviour is linear elastic:

\[
M_{i} = M_{pl} = \frac{q_1 L^2}{8} = 307.2 \text{ kNm} \quad \Rightarrow \quad q_1 = \frac{8 M_{pl}}{L^2} = 68.27 \text{ kN/m}
\]

After this load level the fixed end will be a plastic hinge therefore the statical system differs from the original one (see Fig. 7.4.1.2). The rest of the load level which will act on a simply supported beam (according to the plastic hinge):

\[
q_2 = q - q_1 = 90 - 68.27 = 21.73 \text{ kN/m}
\]
With this assumption we can superpose the two different results based on the two different statical systems. The specific shear forces, bending moments, rotations and translations are as follows without further details (Fig. 7.4.1.3 right side):

\[
V_{\text{max}} = \frac{5}{8} q_1 L + \frac{1}{2} q_2 L = \frac{5}{8} 68.27 \cdot 6 + \frac{1}{2} 21.73 \cdot 6 = 321.2 \text{ kN}
\]

\[
M_{\text{max}} = M_{\text{pl}} =\frac{q_1 L^2}{8} = \frac{68.27 \cdot 6^2}{8} = 307.2 \text{ kNm}
\]

\[
\phi_{\text{roller}} = \frac{q_1 L^3}{48 EI} + \frac{q_2 L^3}{24 EI} = \left(\frac{68.27}{48} + \frac{21.73}{24}\right) \frac{6^3}{210 \cdot 10^6 \cdot 0.000231283781} = 0.010352 \text{ rad}
\]

\[
\phi_{\text{pl hinge}} = \frac{q_2 L^3}{24 EI} = \frac{21.73}{24} \frac{6^3}{210 \cdot 10^6 \cdot 0.000231283781} = 0.004027 \text{ rad}
\]

\[
e_{\text{midspan}} = \frac{2}{384} \frac{q_1 L^4}{EI} + \frac{5}{384} \frac{q_2 L^4}{EI} = \left(\frac{2 \cdot 68.27}{384} + \frac{5 \cdot 21.73}{384}\right) \frac{6^4}{210 \cdot 10^6 \cdot 0.000231283781} = 17.04 \text{ mm}
\]

Figure 7.4.1.3 left side shows the linear elastic and right side shows the plastic results.

Figure 7.4.1.4 shows the FEM-Design model with the input data.

Figure 7.4.1.5 shows the FEM-Design results without shear deformation ($A = 10^4 \text{ mm}^2$). The left side shows the linear elastic calculation and the right side shows the plastic analysis.
We can say that the results are identical with the hand calculations.

7.4.1.4 – The model with the input data in FEM-Design

7.4.1.4 – The internal forces, the rotations and translations in the beam according to linear elastic (left) and plastic limit calculation (right) with FEM-Design without shear deformation ($A=10^6 \text{ mm}^2$)
Figure 7.4.1.5 shows the FEM-Design results with shear deformation \((A = 8446 \text{ mm}^2)\). The left side shows the linear elastic calculation and the right side shows the plastic analysis.

In this case the result differs a bit from the hand calculations according to the shear deformation (Timoshenko beam theory).

Download links to the example files:

Without shear deformation:
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.1 Elasto-plastic point supports in a beam_without_shear_def.str

With shear deformation:
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.1 Elasto-plastic point supports in a beam_with_shear_def.str
7.4.2 Elasto-plastic line support in a plate

Inputs:

<table>
<thead>
<tr>
<th>Input</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimensions of the slab</td>
<td>$L_x = 4 \text{ m} ; L_y = 8 \text{ m}$</td>
</tr>
<tr>
<td>Distributed load</td>
<td>$q_z = 10 \text{ kN/m}^2$</td>
</tr>
<tr>
<td>Concrete slab</td>
<td>$C 30/37$</td>
</tr>
<tr>
<td>Young's modulus of concrete</td>
<td>$E = 33 \text{ GPa}$</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.2$</td>
</tr>
<tr>
<td>Thickness</td>
<td>$t = 200 \text{ mm}$</td>
</tr>
<tr>
<td>Line support</td>
<td>$m_{pl} = m_e = 30 \text{ kNm/m}$ (around line)</td>
</tr>
</tbody>
</table>

The problem is a concrete slab with a fixed support on its edge and with a total distributed load (see Fig. 7.4.2.1).

Case 1:
In this case we assume that the line support has a plastic limit moment capacity ($m_{pl}$, around the edge) along the whole support (blue and red line equally, see Fig. 7.4.2.1). The slab behaves similarly as a cantilever beam. If the plastic limit moment capacity is valid along the whole line support the load bearing capacity can be assumed with the help of the following equation:

$$m_{pl} = \frac{q_z L_x^2}{2}$$

This is the approximated specific moment value at the fixed end.

It means that the maximum of the total distributed load which the structure can bear is:

$$q_z = \frac{2m_{pl}}{L_x^2} = \frac{2 \cdot 30}{4^2} = 3.75 \text{ kN/m}^2$$
Thus if we apply $q_z = 10 \text{kN/m}^2$ load on the slab there is no equilibrium on this final load level. The last converged load level must be around:

$$\eta = \frac{q_z}{q} = \frac{3.75}{10} = 0.375 = 37.5\%$$

We modelled this case in FEM-Design. Let's see the results based on the finite element analysis. Fig. 7.4.2.2 shows the model with the adjusted parameters.

After the elasto-plastic analysis the last converged load level based on the program is 37% (see Fig. 7.4.2.3). Fig. 7.4.2.3 also shows the reactions. We can see that the reaction moment around the line support is a constant **30.00 kNm/m** which is the expected value according to the adjusted limit moment value ($m_{pl} = m_x = 30 \text{kNm/m}$).

Thus we can say that the program gives the same solution as the hand calculation.
Case 2:
In this case we assume that the line support has a **plastic limit moment capacity** ($m_{pl}$, around the edge) along the red part (see Fig. 7.4.2.1). The blue parts behave in a **linear elastic** way. In this case the slab behaves also as a cantilever beam. The slab behaves linearly until load level $q_1=3.75 \text{ kN/m}^2$. Above this load level there will be a plastic hinge line along the red part of the fixed support (see Fig. 7.4.2.1). This remaining load is:

$$ q_2 = q_z - q_1 = 10 - 3.75 = 6.25 \text{ kN/m}^2 $$

Since the sum of the length of the blue parts is equal to the length of the red one (see Fig. 7.4.2.1) we can estimate that (as a conservative assumption) the remaining load level is redistributing on the blue linear elastic fixed supports. To assume the maximum deflection of the slab first of all we need to calculate the deflection according to $q_1=3.75 \text{ kN/m}^2$ (linear elastic behaviour):

The approximated bending stiffness of the slab:

$$ D_{II} = \frac{E t^3}{12(1-\nu^2)} = \frac{33 \cdot 10^6 \cdot 0.2^3}{12 (1-0.2^2)} = 22917 \text{ kN m}^2/\text{m} $$

Based on this value the deflection (as a cantilever):

$$ e_1 \approx \frac{q_1 L_x^4}{8 D_{II}} = \frac{3.75 \cdot 4^4}{8 \cdot 22917} = 5.236 \text{ mm} $$
Since the remaining part of the load is redistributing to the blue (linear elastic) parts of the support we can assume that the remaining middle part of the distributed total load acts on the outer part. Therefore the remaining distributed load which has affect on the linear elastic parts of the line support:

\[ q_2^* = 2q_2 = 2 \cdot 6.25 = 12.5 \text{ kN/m}^2 \]

Because the length of the red part equals to blue ones.

According to this load the deflection of the slab after the linear elastic behaviour can be assumed as:

\[ e^2_2 \approx \frac{q_2^* L_4^4}{8 D_{11}} = \frac{12.5 \cdot 4^4}{8 \cdot 22917} = 17.454 \text{ mm} \]

Thus the approximation of the total deflection with elasto-plastic calculation is:

\[ e_{pl, tot} = e_1 + e_2 = 5.236 + 17.454 = 22.69 \text{ mm} \]

This is a very conservative result.

We modelled this case in FEM-Design. Let's see the results based on the finite element analysis. Fig. 7.4.2.4 shows the model with the adjusted parameters.

After the elasto-plastic analysis the last converged load level based on the program is 100% (see Fig. 7.4.2.5). Fig. 7.4.2.5 also shows the reactions. We can see that the reaction moment around the line support of the middle part is a constant \( 30.00 \text{ kNm/m} \) which is the expected value according to the adjusted limit moment value \( m_{pl} = m_{c'} = 30 \text{ kNm/m} \). The remaining part of the line support behaves linearly as expected. At the end of the linear behaviour (load level 37%)
the deflection is 5.74 mm (see Fig. 7.4.2.3). The difference between the FEM-Design and hand
calculation is less than 10% (5.74 mm vs. 5.236 mm). This difference comes from the very
conservative hand calculation formula and from the fact that FEM-Design considers the shear
deformations (Mindlin plate theory).

At load level 100% the deflection is 19.36 mm (see Fig. 7.4.2.5). The results based on the hand
calculation was 22.69 mm. The difference is around 15%. This difference comes from the very
conservative hand calculation formula and from the fact that FEM-Design considers the shear
deformations (Mindlin plate theory).

Thus we can say that the program gives an accurate elasto-plastic solution.
7.4.3 Elasto-plastic surface support with detach in an embedded plate

In this example the structure is an embedded rectangle plate \((a=6\text{ m}, b=4\text{ m})\). The plate is assumed to be infinitely rigid. There are three different load positions of compression point load \((N=3000 \text{ kN})\), see Fig. 7.4.3.1. The behaviour of the bedding of the plate will be considered in three different ways (see Fig. 7.4.3.2).

The table below shows the nine analyzed cases:

<table>
<thead>
<tr>
<th>No.</th>
<th>Eccentricity of the compression point load ((N = 3000 \text{ kN}))</th>
<th>Behaviour of the bedding</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>in y direction, (e_y=1 \text{ m})</td>
<td>linear elastic</td>
</tr>
<tr>
<td>2</td>
<td>in x direction, (e_x=2 \text{ m})</td>
<td>linear elastic</td>
</tr>
<tr>
<td>3</td>
<td>in y direction, (e_y=1 \text{ m}) and in x direction, (e_x=2 \text{ m})</td>
<td>linear elastic</td>
</tr>
<tr>
<td>4</td>
<td>in y direction, (e_y=1 \text{ m})</td>
<td>linear elastic, non-tension</td>
</tr>
<tr>
<td>5</td>
<td>in x direction, (e_x=2 \text{ m})</td>
<td>linear elastic, non-tension</td>
</tr>
<tr>
<td>6</td>
<td>in y direction, (e_y=1 \text{ m}) and in x direction, (e_x=2 \text{ m})</td>
<td>linear elastic, non-tension</td>
</tr>
<tr>
<td>7</td>
<td>in y direction, (e_y=1 \text{ m})</td>
<td>elasto-plastic, non-tension</td>
</tr>
<tr>
<td>8</td>
<td>in x direction, (e_x=2 \text{ m})</td>
<td>elasto-plastic, non-tension</td>
</tr>
<tr>
<td>9</td>
<td>in y direction, (e_y=1 \text{ m}) and in x direction, (e_x=2 \text{ m})</td>
<td>elasto-plastic, non-tension</td>
</tr>
</tbody>
</table>

7.4.3.1 – Dimensions of the embedded plate and the positions of the compression point load on it

7.4.3.2 – The material behaviour of the bedding of the plate \((K= 10^5 \text{ kN/m/m}^2, f_{\text{lim,comp}}=200 \text{ kPa})\)
Case 1):
In this case the behaviour of the bedding is linear elastic therefore the reaction forces can be calculated according to the superposition of pure compression and uniaxial bending:

\[
\sigma_{\text{max}}^{(1)} = -\frac{N}{A} + \frac{M_y}{I_y} y_{\text{max}} = -\frac{3000}{24} + \frac{3000}{32} 2 = +62.5 \text{ kPa}
\]

\[
\sigma_{\text{min}}^{(1)} = -\frac{N}{A} - \frac{M_y}{I_y} y_{\text{min}} = -\frac{3000}{24} - \frac{3000}{32} 2 = -312.5 \text{ kPa}
\]

Case 2):
In this case the behaviour of the bedding is linear elastic therefore the reaction forces can be calculated according to the superposition of pure compression and uniaxial bending:

\[
\sigma_{\text{max}}^{(2)} = -\frac{N}{A} + \frac{M_y}{I_y} x_{\text{max}} = -\frac{3000}{24} + \frac{6000}{72} 3 = +125 \text{ kPa}
\]

\[
\sigma_{\text{min}}^{(2)} = -\frac{N}{A} - \frac{M_y}{I_y} x_{\text{min}} = -\frac{3000}{24} - \frac{6000}{72} 3 = -375 \text{ kPa}
\]

Case 3):
In this case the behaviour of the bedding is linear elastic therefore the reaction forces can be calculated according to the superposition of pure compression and biaxial bending:

\[
\sigma_{\text{max}}^{(3)} = -\frac{N}{A} + \frac{M_x}{I_x} y_{\text{max}} + \frac{M_y}{I_y} x_{\text{max}} = -\frac{3000}{24} + \frac{3000}{32} 2 + \frac{6000}{72} 3 = +312.5 \text{ kPa}
\]

\[
\sigma_{\text{min}}^{(3)} = -\frac{N}{A} + \frac{M_x}{I_x} y_{\text{min}} + \frac{M_y}{I_y} x_{\text{min}} = -\frac{3000}{24} + \frac{3000}{32} 2 - \frac{6000}{72} 3 = -562.5 \text{ kPa}
\]

The specific results of the hand calculation can be seen in Fig. 7.4.3.3.
Verification Examples

Case 4):
In this case the behaviour of the bedding is linear elastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the linear stress volume must be equal to the acting point load:

\[
N = -\frac{6 \cdot 3}{2} \sigma_{\text{min}}^{(1)}
\]

\[
\sigma_{\text{min}}^{(1)} = -\frac{2N}{6\cdot3} = -\frac{2 \cdot 3000}{6 \cdot 3} = -333.3 \text{kPa}
\]

Case 5):
In this case the behaviour of the bedding is linear elastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the linear stress volume must be equal to the acting point load:

\[
N = -\frac{4 \cdot 3}{2} \sigma_{\text{min}}^{(2)}
\]

\[
\sigma_{\text{min}}^{(2)} = -\frac{2N}{4\cdot3} = -\frac{2 \cdot 3000}{4 \cdot 3} = -500 \text{kPa}
\]

Case 6):
In this case the behaviour of the bedding is linear elastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the linear stress volume must be equal to the acting point load:

\[
N = -\frac{[2 \cdot \sqrt{2} \cdot \sqrt{2} ]^2}{2} \cdot \frac{1}{3} \sigma_{\text{min}}^{(3)}
\]

\[
\sigma_{\text{min}}^{(3)} = \frac{6N}{4^2} = \frac{6 \cdot 3000}{16} = -1125 \text{kPa}
\]

The specific results of the hand calculation can be seen in Fig. 7.4.3.4.

7.4.3.4 – The distribution of the reaction forces [kPa] of the linear elastic, non-tension cases
In Case 7-9 the behaviours are elasto-plastic, non-tension thus it means that according to the given $f_{lim,comp}=200 \text{ kPa}$ limit force (see Fig. 7.4.3.2) there are different values of the load-bearing capacity due to the positions of the point load.

Case 7):
In this case the behaviour of the bedding is elasto-plastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the constant stress volume (with $f_{lim,comp}=200 \text{ kPa}$) must be equal to the acting point load:

\[
N_{limit}^{(1)} = 6 \cdot 2 \cdot f_{lim,comp} = 6 \cdot 2 \cdot 200 = 2400 \text{ kN}
\]

\[
\eta^{(1)} = \frac{N_{limit}^{(1)}}{N} = \frac{2400}{3000} = 80 \%
\]

Case 8):
In this case the behaviour of the bedding is elasto-plastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the constant stress volume (with $f_{lim,comp}=200 \text{ kPa}$) must be equal to the acting point load:

\[
N_{limit}^{(2)} = 4 \cdot 2 \cdot f_{lim,comp} = 4 \cdot 2 \cdot 200 = 1600 \text{ kN}
\]

\[
\eta^{(2)} = \frac{N_{limit}^{(2)}}{N} = \frac{1600}{3000} = 53.3 \%
\]

Case 9):
In this case the behaviour of the bedding is elasto-plastic, non-tension therefore the reaction forces can be calculated according to the theory of the stress volume. The amount of the resultant and the centroid of the constant stress volume (with $f_{lim,comp}=200 \text{ kPa}$) must be equal to the acting point load:

\[
N_{limit}^{(3)} = \left[ \frac{\sqrt{2} \left( \sqrt{2} + \frac{\sqrt{2}}{2} \right)}{2} \right]^2 \cdot f_{lim,comp} = \frac{3}{2} \cdot 200 = 900 \text{ kN}
\]

\[
\eta^{(3)} = \frac{N_{limit}^{(3)}}{N} = \frac{900}{3000} = 30 \%
\]

The specific results of the hand calculation can be seen in Fig. 7.4.3.5.
In the second part of this chapter we make a FEM-Design model for the problem and compare the results with the hand calculation. Fig. 7.4.3.6 shows the main input properties of the model (concrete slab with 2 m thickness). In FEM-Design the reaction result of a surface support element is the average value. For more precise results the average element (mesh) size was 0.2 m.

By the calculation of the surface support reactions we extrapolated the element average FEM-Design numeric values to the extreme fibre (edge of the plate) to get more comparable results. You can find below one by one the 9 different cases and their results based on FEM-Design.

Case 1) FEM (see Fig. 7.4.3.7):

\[
\sigma_{\text{max}}^{\text{FEM} (1)} = +64.7 \text{kPa}
\]

\[
\sigma_{\text{min}}^{\text{FEM} (1)} = -317.5 \text{kPa}
\]
Case 2) FEM (see Fig. 7.4.3.8):

\[ \sigma_{\text{max}}^{\text{FEM}(2)} = +123.4 \text{ kPa} \]
\[ \sigma_{\text{min}}^{\text{FEM}(2)} = -379 \text{ kPa} \]

CASE 3) FEM (see Fig. 7.4.3.9):

\[ \sigma_{\text{max}}^{\text{FEM}(3)} = +295.5 \text{ kPa} \]
\[ \sigma_{\text{min}}^{\text{FEM}(3)} = -561 \text{ kPa} \]
Case 4) FEM (see Fig. 7.4.3.10):

\[ \sigma_{\text{min}}^{\text{FEM}(1)} = -338 \text{ kPa} \]

7.4.3.10 – The reaction results in the surface support with the specific numerical values (element average) for Case 4) [kPa]

Case 5) FEM (see Fig. 7.4.3.11):

\[ \sigma_{\text{min}}^{\text{FEM}(2)} = -501.5 \text{ kPa} \]

7.4.3.11 – The reaction results in the surface support with the specific numerical values (element average) for Case 5) [kPa]
Case 6) FEM (see Fig. 7.4.3.12):

\[ \sigma_{\text{min}}^{\text{FEM}[3]} = -1095 \text{kPa} \]

Case 7) FEM (see Fig. 7.4.3.13):

Last converged load level and the bearing capacity.

\[ \eta_{\text{FEM}[1]} = 80\% \]
\[ N_{\text{lim}}^{\text{FEM}[1]} = 0.80 \times 3000 = 2400 \text{kN} \]
Case 8) FEM (see Fig. 7.4.3.14):

Last converged load level and the bearing capacity.

\[
\eta_{FEM}^{(2)} = 52\% \quad N_{limit}^{FEM(2)} = 0.52 \times 3000 = 1560 \text{kN}
\]

Case 9) FEM (see Fig. 7.4.3.15):

Last converged load level and the bearing capacity.

\[
\eta_{FEM}^{(3)} = 30\% \quad N_{limit}^{FEM(3)} = 0.30 \times 3000 = 900 \text{kN}
\]

The differences between the hand calculations and the FEM-Design results are less than 5%.
Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.3 Elasto-plastic surface supports with detach in an embedded plate.str
7.4.4 Elasto-plastic trusses in a multispans continuous beam

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam length</td>
<td>( L_b = 8 \text{ m} )</td>
</tr>
<tr>
<td>Truss length</td>
<td>( L_c = 2 \text{ m} )</td>
</tr>
<tr>
<td>Total distributed load</td>
<td>( q = 20 \text{ kN/m} )</td>
</tr>
<tr>
<td>Structural steel (trusses with plastic limit force)</td>
<td>S235, ( f_y = 235 \text{ MPa} )</td>
</tr>
<tr>
<td>Young's modulus of steel</td>
<td>( E_s = 210 \text{ GPa} )</td>
</tr>
<tr>
<td>Cross-sectional area (CHS 20-4.0)</td>
<td>( A_1 = 201 \text{ mm}^2 )</td>
</tr>
<tr>
<td>Cross-sectional area (CHS 20-2.0)</td>
<td>( A_2 = 113 \text{ mm}^2 )</td>
</tr>
<tr>
<td>Plastic limit force (CHS 20-4.0)</td>
<td>( F_{pl,lim1} = f_y A_1 = 47.2 \text{ kN} )</td>
</tr>
<tr>
<td>Plastic limit force (CHS 20-2.0)</td>
<td>( F_{pl,lim2} = f_y A_2 = 26.6 \text{ kN} )</td>
</tr>
<tr>
<td>Concrete (beam, linear elastic material model)</td>
<td>( C \text{ 25/30} )</td>
</tr>
<tr>
<td>Young's modulus of concrete</td>
<td>( E_c = 31 \text{ GPa} )</td>
</tr>
<tr>
<td>Cross-sectional area (rectangle)</td>
<td>( b = 200 \text{ mm}; h = 300 \text{ mm} )</td>
</tr>
<tr>
<td>Inertia</td>
<td>( I_b = 0.00045 \text{ m}^4 )</td>
</tr>
</tbody>
</table>

The problem is a concrete beam with five trusses as supports (see Fig. 7.4.4.1). The beam is linear elastic. The trusses are linear elastic, perfectly plastic (with limit force, see the input table above). Truss number 1, 2, 4 and 5 have \( F_{pl,lim1} = 47.2 \text{ kN} \) plastic limit force. Truss number 3 has \( F_{pl,lim2} = 26.6 \text{ kN} \) plastic limit force. The external load is a total distributed load \( q = 20 \text{ kN/m} \) (see also Fig. 7.4.4.1 for the geometry).

At this problem (loads and supports) the degree of static indeterminacy is three. We will solve the problem with force method. The optimal solution with the force method is when the primary structure is essentially a series of simply supported beams (see Fig. 7.4.4.2). Apply the unit redundant forces \( \bar{X}_i \) (\( i = 1, 2, 3 \)) in the lines of the removed constraints, pairs of opposite moments at the end cross-sections of beams connected to hinges created above the intermediate supports (see the primary structure Fig. 7.4.4.2).
We apply vertical springs to represent the trusses as supports. The flexibility of the trusses (in the elastic region):

\[ k_1 = \frac{L_c}{E_c A_1} = \frac{2}{210 \cdot 10^6 \cdot 201 \cdot 10^{-6}} = 4.738 \cdot 10^{-5} \text{ m/kN} \]

for truss number 1, 2, 4 and 5.

\[ k_2 = \frac{L_c}{E_c A_2} = \frac{2}{210 \cdot 10^6 \cdot 113 \cdot 10^{-6}} = 8.428 \cdot 10^{-5} \text{ m/kN} \]

for truss number 3.

The bending stiffness of the beam:

\[ E_c I_b = 31 \cdot 10^6 \cdot 0.00045 = 13950 \text{ kNm}^2 \]
Fig. 7.4.4.2 shows the statical system of the analyzed problem with the primary structure.

By force method first of all we need to calculate the flexibility matrix. Flexible coefficients (see Fig. 7.4.4.2 for the values which provide these coefficients according to the virtual force method):

\[
\begin{align*}
    a_{11} &= a_{33} = 2 \frac{L_i}{3 E_c I_b} + k_1 \frac{1}{2} \frac{1}{2} + k_1 \cdot 1 + k_2 \frac{1}{2} \frac{1}{2} = 1.7587 \cdot 10^{-4} \text{ rad} \frac{\text{kNm}}{} \\
    a_{22} &= 2 \frac{L_i}{3 E_c I_b} + 2 k_1 \frac{1}{2} \frac{1}{2} + k_2 \cdot 1 = 2.0355 \cdot 10^{-4} \text{ rad} \frac{\text{kNm}}{} \\
    a_{13} &= a_{31} = k_2 \frac{1}{2} \frac{1}{2} = 2.107 \cdot 10^{-5} \text{ rad} \frac{\text{kNm}}{} \\
    a_{12} &= a_{23} = a_{32} = \frac{L_i}{6 E_c I_b} - k_1 \frac{1}{2} - k_2 \frac{1}{2} = -4.1935 \cdot 10^{-5} \text{ rad} \frac{\text{kNm}}{}
\end{align*}
\]

These coefficients are independent from the external loads. The second step is the calculation of the load constants based on the total load intensity \( q \).

Load constants:

\[
\begin{align*}
    a_{10} &= a_{30} = 2 \frac{q L_i^3}{24 E_c I_b} + k_1 \frac{q L_i}{2} \frac{1}{2} - k_1 \frac{2 q L_i}{2} + k_2 \frac{2 q L_i}{2} \frac{1}{2} = 1.220 \cdot 10^{-3} \text{ rad} \\
    a_{20} &= 2 \frac{q L_i^3}{24 E_c I_b} + 2 k_1 \frac{q L_i}{2} \frac{1}{2} - k_2 \frac{2 q L_i}{2} \frac{1}{2} = -5.202 \cdot 10^{-4} \text{ rad}
\end{align*}
\]

The equation system of the force method:

\[
A \mathbf{x} + \mathbf{a}_0 = 0 \quad \text{where:}
\]

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix} = \begin{bmatrix}
    1.7587 \cdot 10^{-4} & -4.1935 \cdot 10^{-5} & 2.107 \cdot 10^{-5} \\
    -4.1935 \cdot 10^{-5} & 2.0355 \cdot 10^{-4} & -4.1935 \cdot 10^{-5} \\
    2.107 \cdot 10^{-5} & -4.1935 \cdot 10^{-5} & 1.7587 \cdot 10^{-4}
\end{bmatrix} \text{ rad} \frac{\text{kNm}}{} \quad \text{and}
\]

\[
\mathbf{a}_0 = \begin{bmatrix}
    1.220 \cdot 10^{-3} \\
    -5.202 \cdot 10^{-4} \\
    1.220 \cdot 10^{-3}
\end{bmatrix} \text{ rad}
\]
The inverse matrix for the solution:

\[
A^{-1} = \begin{bmatrix}
6013 & 1147 & -447.0 \\
1147 & 5385 & 1147 \\
-447.0 & 1147 & 6013
\end{bmatrix} \text{kNm} \\
\text{rad}
\]

The solution of the equation system:

\[
X = \begin{bmatrix}
-6.194 \\
0.003467 \\
-6.194
\end{bmatrix} \text{kNm}
\]

The calculation of the truss forces based on the force method (see also Fig. 7.4.4.2):

\[
S_k = S_{k0} + \sum_{i=1}^{3} S_{ki} X_i
\]

\[
S_1' = S_{10} + S_{11} X_1 + S_{12} X_2 + S_{13} X_3 = \frac{q L_1}{2} + 0.5 X_1 + 0.5 X_2 + 0.5 X_3 = 16.90 \text{kN}
\]

\[
S_2' = S_{20} + S_{21} X_1 + S_{22} X_2 + S_{23} X_3 = 2 \frac{q L_1}{2} - 1.0 X_1 + 1.0 X_2 + 0.0 X_3 = 46.20 \text{kN}
\]

\[
S_3' = S_{30} + S_{31} X_1 + S_{32} X_2 + S_{33} X_3 = 2 \frac{q L_1}{2} + 1.0 X_1 - 1.0 X_2 + 1.0 X_3 = 33.80 \text{kN}
\]

Since the plastic limit force for truss number 3 is \( F_{p,\text{lim2}} = 26.6 \text{ kN} \) the elastic behaviour with the described statical system ended at load level \( q_1 \):

\[
\frac{q_1}{q} \frac{F_{p,\text{lim2}}}{S_3'} = \frac{26.6}{33.80} = 0.7870
\]

\[
q_1 = 0.7870 q = 15.74 \text{ kN/m}
\]

At this load level the normal force in truss number 3:

\[
S_3 = \frac{q_1}{q} S_3' = 26.60 \text{ kN}
\]

After this load level the statical system is changing (see Fig. 7.4.4.3). For this system the degree of static indeterminancy is 2. We will solve the problem also with force method. The optimal solution with the force method is when the primary structure is essentially a series of simply
supported beams (see Fig. 7.4.4.3). Apply the unit redundant forces \( \overline{X}_i \) \((i = 1, 2) \) in the lines of the removed constraints, pairs of opposite moments at the end cross-sections of beams connected to hinges created above the intermediate supports (see the primary structure Fig. 7.4.4.3).

Flexible coefficients (see Fig. 7.4.4.3 for the values which provide these coefficients according to the virtual force method):

\[
\begin{align*}
    a_{11} &= a_{22} = \frac{L_1}{3E_c I_b} + \frac{L_2}{3E_c I_b} + k_1 \frac{1}{2} \frac{1}{2} + k_1 \frac{1}{4} \frac{1}{4} + k_1 \left( \frac{1}{2} + \frac{1}{4} \right)^2 = 1.8483 \times 10^{-4} \text{ rad kNm} \\
    a_{12} &= a_{21} = -2k_1 \frac{1}{4} \left( \frac{1}{2} + \frac{1}{4} \right) = 3.0022 \times 10^{-5} \text{ rad kNm}
\end{align*}
\]

The calculation of the load constants based on the total load intensity \( q \).
Verification Examples

Load constants:

\[
a_{10} = a_{20} = \frac{q L_1^3}{24 E_i I_b} + \frac{q L_2^3}{24 E_i I_b} + k_i \frac{q L_1}{2} \left( \frac{1}{2} + \frac{1}{4} \right) + k_i \frac{q L_2}{2} \left( \frac{1}{2} + \frac{1}{4} \right) = 3.353 \cdot 10^{-3} \text{ rad}
\]

The equation system is now:

\[
\mathbf{A} \mathbf{X} + \mathbf{a}_0 = 0
\]

where:

\[
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1.8483 \cdot 10^{-4} & 3.0022 \cdot 10^{-5} \\ 3.0022 \cdot 10^{-5} & 1.8483 \cdot 10^{-4} \end{bmatrix} \text{ rad/kNm}
\]

\[
\mathbf{a}_0 = \begin{bmatrix} 3.353 \cdot 10^{-5} \\ 3.353 \cdot 10^{-5} \end{bmatrix} \text{ rad}
\]

The inverse matrix for the solution:

\[
\mathbf{A}^{-1} = \begin{bmatrix} 5557 & -902.6 \\ -902.6 & 5557 \end{bmatrix} \text{ kNm/ rad}
\]

The solution of the equation system:

\[
\mathbf{X} = \begin{bmatrix} -15.61 \\ -15.61 \end{bmatrix} \text{ kNm}
\]

The calculation of the truss forces:

\[
S_1'' = S_3'' = S_{10} + S_{11} X_1 + S_{12} X_2 = \frac{q L_1}{2} + \frac{1}{2} X_1 + 0 \cdot X_2 = 12.20 \text{ kN}
\]

\[
S_2'' = S_4'' = S_{20} + S_{21} X_1 + S_{22} X_2 = \left( \frac{q L_1}{2} + \frac{q L_2}{2} \right) - \frac{1}{2} X_1 + \frac{1}{4} X_2 = 67.81 \text{ kN}
\]

Since the plastic limit force for truss number 2 and 4 is \( F_{pl,lim1} = 47.2 \text{ kN} \) this elastic behaviour for this second load step with the described statical system ended at load level \( q_2 \) (see also Fig. 7.4.4.4):

\[
q_2 = \frac{F_{pl,lim1}}{S_{2''}} = \frac{47.2}{67.81} = 0.70 \text{ kN/m}
\]

\[
q_2 = 0.70 \text{ kN/m}
\]
On this load level the normal force in truss number 2 and 4:

\[ S_2 = S_4 = \frac{q_1}{q} S_2' + \frac{q_2}{q} S_2'' = 47.20 \text{ kN} \]

The remaining load is:

\[ q_3 = q - q_1 - q_2 = 1.062 \text{ kN/m} \]

This load is applied now on a new statical system too (see Fig. 7.4.4.4).

\[ S_f' = \frac{q L_b}{2} = 80 \text{ kN} \]

Thus the final normal force in truss number 1:

\[ S_1 = \frac{q_1}{q} S_1' + \frac{q_2}{q} S_1'' + \frac{q_3}{q} S_1''' = 19.50 \text{ kN} \]

Figure 7.4.4.5 shows the truss normal forces and the internal forces for the different load levels with the different statical system. Fig. 7.4.4.5 also shows the final results of the problem (truss normal forces and internal forces).

After the hand calculation we provide the FEM-Design calculation applying these plastic limit forces for the trusses. Fig. 7.4.4.6 shows the FEM-Design model with the geometry and with the main modeling issues.
7.4.4.5 – The superposition of the different load level results and the final truss forces and internal forces for the problem.
After the nonlinear plastic calculation Fig. 7.4.4.7 shows the normal forces in the trusses. Fig. 7.4.4.8 shows the final moment diagram. The last converged equilibrium load level is at 100% of the total load, thus the equilibrium exists for the total load level.

We can say that the hand calculation and the FEM calculation results are identical!
Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.4 Elasto-plastic trusses in a multispan continuous beam.str
7.4.5 Elasto-plastic point-point connection between cantilevers

Inputs

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column height</td>
<td>H = 3 m</td>
</tr>
<tr>
<td>Point load</td>
<td>F₁ = 500 kN or F₂ = 600 kN</td>
</tr>
<tr>
<td>Structural steel</td>
<td>S 235</td>
</tr>
<tr>
<td>Young's and shear moduli of steel</td>
<td>Eₛ = 210 GPa, G = 80.77 GPa</td>
</tr>
<tr>
<td>Cross-section (both vertical columns)</td>
<td>HEA 600</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>Aₑ = 22646 mm²</td>
</tr>
<tr>
<td>Relevant inertia</td>
<td>Iₑ,1 = 0.001412 m⁴</td>
</tr>
<tr>
<td>Relevant shear correction factor</td>
<td>ρₑ,2 = 0.3314</td>
</tr>
<tr>
<td>Point connection with plastic compression limit</td>
<td>Nₑₚ = 200 kN</td>
</tr>
</tbody>
</table>

We have two vertical columns (HEA 600, S235) with fixed bottom and hinges at the top ends (see Fig. 7.4.5.1). We applied a horizontal point load (with different intensity, see Fig. 7.4.5.1) at the top of the left column. The tops of the columns are connected to each other with a point-point connection with a plastic compression limit Nₑₚ = 200 kN.

The bending moment distributions in the three different cases (see Fig. 7.4.5.1) according to the same bending stiffnesses and the plastic compression limit in the point-point connection are trivial.

7.4.5.1 – The three different problems and the different bending moment distributions in the columns.
If there is no limit compression force in the point-point connection the distribution of the point load is equal on both top ends of the columns:

\[ M_1^{(1)} = F_1/2 \cdot H = 500/2 \cdot 3 = 750 \text{ kNm} \]
\[ M_2^{(1)} = F_1/2 \cdot H = 500/2 \cdot 3 = 750 \text{ kNm} \]
\[ M_1^{(2)} = (F_1 - N_{pl}) H = (500 - 200)3 = 900 \text{ kNm} \]
\[ M_2^{(2)} = N_{pl} H = 200 \cdot 3 = 600 \text{ kNm} \]
\[ M_1^{(3)} = (F_2 - N_{pl}) H = (600 - 200)3 = 1200 \text{ kNm} \]
\[ M_2^{(3)} = N_{pl} H = 200 \cdot 3 = 600 \text{ kNm} \]

The displacements of the top points of the columns:

\[ e_1^{(1)} = \frac{(F_1/2) \cdot H^3}{3 E_s I_c} + \frac{F_1/2 \cdot H}{\rho_{c,z} G A_c} = \frac{(500/2) \cdot 3^3}{3 \cdot 210 \cdot 10^5 \cdot 0.001412} + \frac{500/2 \cdot 3}{0.3314 \cdot 80.77 \cdot 10^6 \cdot 2.2646 \cdot 10^{-2}} = 8.83 \text{ mm} \]
\[ e_2^{(1)} = \frac{(F_1/2) \cdot H^3}{3 E_s I_c} + \frac{F_1/2 \cdot H}{\rho_{c,z} G A_c} = \frac{(500/2) \cdot 3^3}{3 \cdot 210 \cdot 10^5 \cdot 0.001412} + \frac{500/2 \cdot 3}{0.3314 \cdot 80.77 \cdot 10^6 \cdot 2.2646 \cdot 10^{-2}} = 8.83 \text{ mm} \]
\[ e_1^{(2)} = \frac{(F_1 - N_{pl}) H^3}{3 E_s I_c} + \frac{(F_1 - N_{pl}) H}{\rho_{c,z} G A_c} = \frac{(500 - 200) \cdot 3^3}{3 \cdot 210 \cdot 10^5 \cdot 0.001412} + \frac{(500 - 200) \cdot 3}{0.3314 \cdot 80.77 \cdot 10^6 \cdot 2.2646 \cdot 10^{-2}} = 10.59 \text{ mm} \]
\[ e_2^{(2)} = \frac{N_{pl} H^3}{3 E_s I_c} + \frac{N_{pl} H}{\rho_{c,z} G A_c} = \frac{200 \cdot 3^3}{3 \cdot 210 \cdot 10^5 \cdot 0.001412} + \frac{200 \cdot 3}{0.3314 \cdot 80.77 \cdot 10^6 \cdot 2.2646 \cdot 10^{-2}} = 7.06 \text{ mm} \]
\[ e_1^{(3)} = \frac{(F_2 - N_{pl}) H^3}{3 E_s I_c} + \frac{(F_2 - N_{pl}) H}{\rho_{c,z} G A_c} = \frac{(600 - 200) \cdot 3^3}{3 \cdot 210 \cdot 10^5 \cdot 0.001412} + \frac{(600 - 200) \cdot 3}{0.3314 \cdot 80.77 \cdot 10^6 \cdot 2.2646 \cdot 10^{-2}} = 14.12 \text{ mm} \]
\[ e_2^{(3)} = \frac{N_{pl} H^3}{3 E_s I_c} + \frac{N_{pl} H}{\rho_{c,z} G A_c} = \frac{200 \cdot 3^3}{3 \cdot 210 \cdot 10^5 \cdot 0.001412} + \frac{200 \cdot 3}{0.3314 \cdot 80.77 \cdot 10^6 \cdot 2.2646 \cdot 10^{-2}} = 7.06 \text{ mm} \]

Fig. 7.4.5.2 shows the FEM-Design model and their input parameters. The results of the FE calculations according to the three different cases are in Fig. 7.4.5.3 and 7.4.5.4 (see also Fig. 7.4.5.1).

The bending moment diagrams and the displacement values of the top of the columns are identical with the hand calculation.
7.4.5.2 – The FEM-design model of the problem with the inputs

7.4.5.3 – The bending moment diagrams of the FEM solution for the three different cases
bending moments [kNm], connection force [kN]
7.4.5.4 – The displacement diagrams of the FEM solution for the three different cases [mm]

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.5 Elasto-plastic point-point connection between cantilevers.str
7.4.6 Elasto-plastic point-point connection with uplift in a multispan continuous beam

Inputs:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam length</td>
<td>$L_1 = 4 \text{ m}; L_2 = L_3 = 2 \text{ m}$</td>
</tr>
<tr>
<td>Partial distributed load</td>
<td>$q = 10 \text{ kN/m}$</td>
</tr>
<tr>
<td>Point load</td>
<td>$F = 5 \text{ kN}$</td>
</tr>
<tr>
<td>Concrete (beam, linear elastic material model)</td>
<td>C 25/30</td>
</tr>
<tr>
<td>Young's modulus of concrete</td>
<td>$E_c = 31 \text{ GPa}$</td>
</tr>
<tr>
<td>Cross-sectional area (rectangle)</td>
<td>$b = 120 \text{ mm}; h = 150 \text{ mm}$</td>
</tr>
<tr>
<td>Inertia</td>
<td>$I_b = 0.00003375 \text{ m}^4$</td>
</tr>
<tr>
<td>Plastic hinge above support $B$ (point-point connection)</td>
<td>$M_{pl,Rd} = 5 \text{ kNm}$</td>
</tr>
<tr>
<td>C support only bears compression</td>
<td>$\text{uplift}$</td>
</tr>
</tbody>
</table>

A multispan continuous beam, a plastic hinge with limit moment capacity (as a point-point connection) above support B and only compression resistance at support C with different types of loads are given in Fig. 7.4.6.1.

First of all we make a hand calculation and then make a FEM-Design plastic calculation. At the end of this chapter these two results will be compared.

According to the uplift and plastic behaviour we need to analyze and calculate the results in load steps. The first load step $q_1$, $F_1$ are at a certain load level where the plastic hinge reaches its limit moment value. On this load level there will be zero reaction force in support C, because this support only can bear compression.

The 50% of the total load level: $q_1 = 5 \text{ kN/m}$, $F_1 = 2.5 \text{ kN}$.

At this load level the specific bending moment values (see also Fig. 7.4.6.2):

\[
M_{A}^{B} = F_1 L_2 = 2.5 \cdot 2 = 5 \text{ kNm}
\]

this is the plastic limit moment capacity above support B.

\[
M_{1 \text{mid}}^{AB} = -\frac{F_1 L_2}{2} + \frac{q_1 L_1^2}{8} = -\frac{2.5 \cdot 2}{2} + \frac{5 \cdot 4^2}{8} = 7.50 \text{ kNm}
\]

\[
M_{1 \text{mid}}^{BC} = 0 \text{ kNm}
\]
Verification Examples

At this load level the specific reaction values (see also Fig. 7.4.6.2):

\[
A_j = \frac{q_j \frac{L_j^2}{2} - F_j L_j}{L_j} = \frac{5 \frac{4^2}{2} - 2.5 \cdot 2}{4} = 8.75 \text{kN}
\]

\[
B_j = \frac{q_j \frac{L_j^2}{2} + F_j \left(L_j + L_2\right)}{L_j} = 13.75 \text{kN}
\]

\[
C_j = 0 \text{kN}
\]

At this load level the specific deflection values according to virtual force method (see also Fig. 7.4.6.2):

\[
e_{ABmid}^i = \frac{1}{E_c I_b} \left[ -\frac{L_1 L_1^2}{4} M_j^B + 2 \left( q_j \frac{L_j^2}{2} M_j^B \right) \right] = 11.15 \text{mm}
\]

\[
e_{BCmid}^i = \frac{1}{E_c I_b} \left[ \frac{L_2 L_2}{2} M_j^B + \frac{M_j^B}{2} L_j \frac{L_2}{2} - \frac{q_j L_j^2}{8} L_j \frac{L_2}{3} \right] = -6.372 \text{mm}
\]

\[
e_{C}^i = \frac{1}{E_c I_b} \left[ M_j^B \left( \frac{L_2 + L_3}{2} + \frac{L_2 + L_3}{2} \right) + \frac{(L_2 + L_3)^2}{2} M_j^B - \frac{q_j L_j^2}{8} L_j \frac{L_2 + L_3}{3} \right] = -9.558 \text{mm}
\]

After this load step the remaining loads (q₂, F₂) are acting on a different statical system (see Fig. 7.4.6.3). The second 50% part of the total load level: q₂ = 5 kN/m, F₂ = 2.5 kN are acting on separate simply supported beams.

At this load step the specific bending moment values (see also Fig. 7.4.6.3):
At this load step the specific reaction values (see also Fig. 7.4.6.3):

\[ A_2 = q_2 \frac{L_1}{2} = 5 \cdot \frac{4}{2} = 10.0 \text{kN} \]

\[ B_2 = q_2 \frac{L_1}{2} + F_2 = 5 \cdot \frac{4}{2} + \frac{2.5}{2} = 11.25 \text{kN} \]

\[ C_2 = \frac{F_2}{2} = \frac{2.5}{2} = 1.25 \text{kN} \]

At this load step the specific deflection values according to virtual force method (see also Fig. 7.4.6.3):

\[ e_2^{\text{ABmid}} = \frac{5 q_2 L_1^4}{384 E_c I_b} = 15.93 \text{mm} ; \quad e_2^{\text{BCmid}} = \frac{F_2 (L_2 + L_3)^3}{48 E_c I_b} = 3.186 \text{mm} \]

The final results come from the superposition of the two former calculated cases.

The final specific bending moment values (see also Fig. 7.4.6.4):

\[ M_2^{\text{ABmid}} = \frac{q_2 L_1^2}{8} = \frac{5 \cdot 4^2}{8} = 10.0 \text{kNm} \]

\[ M_2^{\text{BCmid}} = F_2 \frac{L_2 + L_3}{4} = 2.5 \frac{2 + 2}{4} = 2.5 \text{kNm} \]

\[ M_2^{\text{ABmid}} = M_1^{\text{ABmid}} + M_2^{\text{ABmid}} = 7.50 + 10.0 = 17.5 \text{kNm} \]
\[ M_{BC\text{mid}} = M_{1,BC\text{mid}} + M_{2,BC\text{mid}} = 0 + 2.5 = 2.5 \text{kNm} \]

The final specific reaction values (see also Fig. 7.4.6.4):

\[ A = A_1 + A_2 = 8.75 + 10.0 = 18.75 \text{kN} \]
\[ B = B_1 + B_2 = 13.75 + 11.25 = 25.0 \text{kN} \]
\[ C = C_1 + C_2 = 0 + 1.25 = 1.25 \text{kN} \]

The final specific deflection values (see also Fig. 7.4.6.4):

\[ e^{AB\text{mid}} = e_1^{AB\text{mid}} + e_2^{AB\text{mid}} = 11.15 + 15.93 = 27.08 \text{mm} \]

At the final deflection calculation of the middle point of BC span we need to consider the fact that above support B a plastic hinge occurred and hence in support C compression is arising in the second load step.

\[ e^{BC\text{mid}} = e_1^{BC\text{mid}} - \frac{e_2^{C} + e_2^{BC\text{mid}}}{2} = -6.372 - \frac{-9.558}{2} + 3.186 = 1.593 \text{mm} \]
After the hand calculation we modelled this problem in FEM-Design. We adjusted a point-point connection with plastic limit moment value above support B (see Fig. 7.4.6.5) and uplift at support C (see Fig. 7.4.6.5).

Fig. 7.4.6.6 shows the reactions, bending moments and deflection values at load level 50%. The differences between hand and FEM calculations are less than 1%.
Fig. 7.4.6.7 shows the reactions, bending moments and deflection values at load level 100%. The differences between hand and FEM calculations are less than 1%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.6 Elasto-plastic point-point connection with uplift in a multispam continuous beam.str
7.4.7 Elasto-plastic edge connections in a building braced by shear walls

Inputs:

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Fig. 7.4.7.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal loads (on one storey, on one wall plane)</td>
<td>V = 10 kN</td>
</tr>
<tr>
<td>Concrete (walls and slabs)</td>
<td>C 25/30</td>
</tr>
<tr>
<td>Young’s modulus of concrete</td>
<td>E_c = 31 GPa</td>
</tr>
<tr>
<td>Thickness of the walls/slabs</td>
<td>t = 20 cm</td>
</tr>
<tr>
<td>Plastic edge connection shear force limit (above Wall4)</td>
<td>v_{pl} = 2.5 kN/m</td>
</tr>
<tr>
<td>Edge connection behaviour</td>
<td>Without detach</td>
</tr>
</tbody>
</table>

The shear wall problem is given in Fig. 7.4.7.1. The given loads are only two concentrated horizontal loads on the two storeys (see Fig. 7.4.7.1). The top of Wall4 only can bear \( v_{pl} = 2.5 \) kN/m plastic limit force.

We defined the problem and the model in FEM-Design (7.4.7.2).

If all of the elements behaviour are linear elastic then the solution of this problem based on the finite element method (according to the mechanical model of the elements) is “exact”.

Fig. 7.4.7.2 shows the finite element surface mesh of the problem.
Fig. 7.4.7.3 shows the resultants of the reactions and the edge connection forces/moments according to the linear elastic calculation.

We can see that the horizontal shear force in the edge connections (above Wall4, see Fig. 7.4.7.1 and 7.4.7.3) is:

\[ V_{el} = 15.6 \text{ kN} \]

The ultimate plastic resultant shear force bearing capacity of the edge connection above Wall4 is:

\[ V_{ult} = v_{pl} \cdot L = 2.5 \cdot 4 = 10.0 \text{ kN} \]

Fig. 7.4.7.4 shows the resultants of the reactions and the edge connection forces/moments according to the elasto-plastic calculation. The edge connection resultant shear force based on the FEM-Design calculation (Fig. 7.4.7.4):

\[ V_{ult,FEM} = 10.0 \text{ kN} \]
It is obvious that according to the elasto-plastic calculation due to the plastic deformations on the edge connections the final displacements need to be greater than the displacements of the linear elastic calculation.

7.4.7.3 – The resultants of the reactions and the edge connections according to linear elastic calculation (forces [kN], moments [kNm])

7.4.7.4 – The resultants of the reactions and the edge connections according to elasto-plastic calculation (forces [kN], moments [kNm])
Fig. 7.4.7.5 shows the displacements of the linear elastic (above) and the elasto-plastic (below) calculations.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.7 Elasto-plastic edge connections in a building braced by shear walls.str
7.4.8 Elasto-plastic edge connections with detach in a shear wall

Inputs:

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry</td>
<td>Fig. 7.4.8.1</td>
</tr>
<tr>
<td>Vertical distributed load</td>
<td>( q_V = 50 \text{ kN/m (N = 200 kN)} )</td>
</tr>
<tr>
<td>Horizontal distributed load intermediate levels</td>
<td>( q_{H2} = 15 \text{ kN/m (V = 60 kN)} )</td>
</tr>
<tr>
<td>Horizontal distributed load top level</td>
<td>( q_{H1} = 7.5 \text{ kN/m (V/2 = 30 kN)} )</td>
</tr>
<tr>
<td>Concrete (wall)</td>
<td>C 25/30</td>
</tr>
<tr>
<td>Young’s modulus of concrete</td>
<td>( E_c = 31 \text{ GPa} )</td>
</tr>
<tr>
<td>Thickness of the wall</td>
<td>( t = 20 \text{ cm} )</td>
</tr>
<tr>
<td>Plastic edge connection shear force limit</td>
<td>( v_{pl,1} = 20 \text{ kN/m} )</td>
</tr>
<tr>
<td>Plastic support shear force limit</td>
<td>( v_{pl,2} = 50 \text{ kN/m} )</td>
</tr>
<tr>
<td>Edge connection / Fixed support behaviour</td>
<td>Detach in vertical direction</td>
</tr>
</tbody>
</table>

Fig. 7.4.8.1 shows the shear wall problem with the geometry, external loads and the behaviour of the edge connections and the support.

\[
e_1 = \frac{M_1}{N_1} = \frac{90}{200} = 0.45 \text{ m} ; \quad e_2 = \frac{M_2}{N_2} = \frac{360}{400} = 0.90 \text{ m} ; \quad e_3 = \frac{M_3}{N_3} = \frac{810}{600} = 1.35 \text{ m}
\]
The eccentric normal force at the top level (see Fig. 7.4.8.2) is inside the Culmann's kernel therefore the specific normal force distribution can be calculated with the following equations and approximations:

\[
\begin{align*}
n_{z1}^+ &= \frac{N_1}{A} t + \frac{M_1}{I} x_{max} t = -\frac{200}{0.2 \cdot 4} 0.2 + \frac{90}{0.2 \cdot 4^3/12} 2 \cdot 0.2 = -16.25 \text{kN/m} \\
n_{z1}^- &= \frac{N_1}{A} t - \frac{M_1}{I} x_{max} t = -\frac{200}{0.2 \cdot 4} 0.2 - \frac{90}{0.2 \cdot 4^3/12} 2 \cdot 0.2 = -83.75 \text{kN/m}
\end{align*}
\]

The eccentric normal forces at the intermediate levels (see Fig. 7.4.8.2) are outside the Culmann's kernel therefore the specific normal force distribution can be calculated with the following equations and approximations (based on the resultant of the stress volume):

\[
\begin{align*}
n_{z2}^- &= \frac{N_2}{3 \left( \frac{L}{2} - e_2 \right)^{1/2}} = \frac{-400}{3 \left( \frac{4}{2} - 0.9 \right)^{1/2}} = -242.4 \text{kN/m} \\
n_{z3}^- &= \frac{N_3}{3 \left( \frac{L}{2} - e_3 \right)^{1/2}} = \frac{-600}{3 \left( \frac{4}{2} - 1.35 \right)^{1/2}} = -615.4 \text{kN/m}
\end{align*}
\]
The average shear forces at the different levels:

\[ n_{xz1,\text{average}} = \frac{V_1}{A} t = \frac{30}{0.2 \cdot 4} = 7.5 \text{ kN/m} \]

\[ n_{xz2,\text{average}} = \frac{V_2}{A_{\text{active2}}} t = \frac{90}{0.2 \cdot \left( \frac{4}{2} - 0.9 \right)} = 27.27 \text{ kN/m} \]

\[ n_{xz3,\text{average}} = \frac{V_3}{A_{\text{active3}}} t = \frac{150}{0.2 \cdot \left( \frac{4}{2} - 1.35 \right)} = 76.92 \text{ kN/m} \]

You can see these calculated values in Fig. 7.4.8.3 also.

The plastic limit shear forces differ from each other at the edge connections and the fixed supports hence one-by-one the load-bearing capacities are (see Fig. 7.4.8.1 and 7.4.8.3):

\[ \eta_1 = \frac{\nu_{pl,1}}{n_{xz1,\text{average}}} = \frac{20}{7.5} = 266.7\% \]

\[ \eta_2 = \frac{\nu_{pl,1}}{n_{xz2,\text{average}}} = \frac{20}{27.27} = 73.34\% \]

\[ \eta_3 = \frac{\nu_{pl,2}}{n_{xz3,\text{average}}} = \frac{50}{76.92} = 65.00\% \]

Thus the significant value which gives us the load-bearing capacity of the structure is the last value: **65%** of the external load.
After the hand calculation we have made a finite element calculation with FEM-Design. The model can be seen with the adjusted parameters in Fig. 7.4.8.4 (see also the input table for the data).

![FEM-Design model with fixed support and the vertical and horizontal forces](image)

*Figure 7.4.8.4 – FEM-Design model with fixed support and the vertical and horizontal forces*

The first FEM-Design results can be seen in Fig. 7.4.8.5. In this case the results are based on the uplift (detach) calculation (without plastic calculation).

The results (Fig. 7.4.8.5) are in good agreement with the hand calculation. Do not forget that by the hand calculation we assumed prismatic beam behaviour by the specific normal force calculation but by the FEM calculation the behaviour is more compound.
In the second calculation we considered plastic analysis with detach (uplift) in FEM-Design. According to the plastic behaviour the last converged (equilibrium) load level was:

$$\eta^{FEM} = 62\%$$

Fig. 7.4.8.6 shows the edge connection forces and the reactions for the last converged solution. The difference between the hand and FEM calculation is 4.6%.

$$\Delta = \frac{\eta_3^{HAND} - \eta_3^{FEM}}{\eta_3^{HAND}} = \frac{65\% - 62\%}{65\%} = 4.6\%$$
Figure 7.4.8.6 – Distribution of the reactions and connection forces with plastic detach calculation
specific normal forces [red and blue, kN/m] and shear forces [green lines, kN/m]
Last converged (equilibrium) load level: 62 %

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/7.4.8 Elasto-plastic edge
connections with detach in a shear wall.str
7.4.9 Elasto-plastic line-line connections in a square plate

Inputs:

<table>
<thead>
<tr>
<th>Item</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square slab</td>
<td>( L = L_x = L_y = 8 \text{ m} )</td>
</tr>
<tr>
<td>Slab thickness</td>
<td>( t = 20 \text{ cm} )</td>
</tr>
<tr>
<td>Applied distributed load</td>
<td>( p = 15 \text{ kN/m}^2 )</td>
</tr>
<tr>
<td>Concrete</td>
<td>( \text{C 25/30} )</td>
</tr>
<tr>
<td>Young's modulus of concrete</td>
<td>( E_c = 31 \text{ GPa} )</td>
</tr>
<tr>
<td>Isotropic positive plastic moment capacity</td>
<td>( m^x = m^y = 30 \text{ kNm/m} )</td>
</tr>
<tr>
<td>Line-line connection, with plastic limit specific moment (see the adjusted local coordinate system in Fig. 7.4.9.3)</td>
<td>( m^{x,y}_{\text{pl.limit}} = 30 \text{ kNm/m} )</td>
</tr>
</tbody>
</table>

In this example line-line connections with plastic limit force/moment will be used to calculate the plastic load-bearing capacity of a simply supported square plate with isotropic positive plastic moment capacity. If the slab has constant isotropic positive plastic moment capacity \( (m^x = m^y = 30 \text{ kNm/m}) \) the yield-line layout is known in this case [11] (see Fig. 7.4.9.1 and 7.4.9.2).

If the correct (real) yield-line layout is known the plastic load-bearing capacity can be calculated according to a kinematically admissible virtual displacement field which belong to this layout. The maximum total distributed load which causes the plastic failure of this square slab comes from the equality of the external and the internal virtual work.

\[
W_{\text{int}} = W_{\text{ext}}
\]
Verification Examples

The external work done by loads:

\[ W_{ext} = p_{\text{HAND}}^\text{max} \delta V = p_{\text{HAND}}^\text{max} L_x L_y \frac{1}{3} L = p_{\text{HAND}}^\text{max} L \frac{1}{3} L \]

The internal work done by resisting moments:

\[ W_{int} = m_x^+ \delta \Theta_y L_y + m_y^+ \delta \Theta_x L_x = m_x^+ \frac{4}{L_x} L_y + m_y^+ \frac{4}{L_y} L_x = 8 m^+ \]

Hence the plastic load-bearing capacity:

\[ p_{\text{HAND}}^\text{max} = \frac{24 m^+}{L^2} = \frac{24 \cdot 30}{8^2} = 11.25 \text{ kN/m}^2 \]

In FEM-Design along the yield lines line-line connections will be applied with adjusted plastic limit capacity \( m_x^{+\text{pl,limit}} = 30 \text{ kNm/m} \). Fig. 7.4.9.3 shows the local co-ordinate system of the line-line connections. We applied these systems because the resisting positive moments were assumed to be isotropic, hence the adjusted plastic limit capacity of the line-line connection was specified along the line (see Fig. 7.4.9.3).

Fig. 7.4.9.3 also shows the adjusted parameters, geometry and the applied \( p = 15 \text{ kN/m}^2 \) uniform distributed load.
With FEM-Design the last converged equilibrium load level was at 72%. It means that the load-bearing capacity due to the plastic calculation is:

\[ p_{\text{FEM}}^{\text{max}} = 0.72 \cdot p = 0.72 \cdot 15 = 10.8 \text{ kN/m}^2 \]

The difference between the hand calculation and FEM analysis is 4%.

\[ \Delta = \frac{p_{\text{HAND}}^{\text{max}} - p_{\text{FEM}}^{\text{max}}}{p_{\text{HAND}}^{\text{max}}} = \frac{11.25 - 10.80}{11.25} = 4\% \]

Fig. 7.4.9.4 shows the plastic connection forces in the adjusted line-line connections (which are the expected values according to the specific limit moment capacity).
Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/7.4.9 Elasto-plastic line-line connections in a square plate.str
7.5 Calculation with construction stages

7.5.1 A steel frame building with construction stages calculation

Inputs:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Strength of the steel</td>
<td>S235; E = 210 GPa</td>
</tr>
<tr>
<td>Columns</td>
<td>HEA 400</td>
</tr>
<tr>
<td>Beams</td>
<td>IPE 600</td>
</tr>
<tr>
<td>Line load on each floors</td>
<td>$p_{\text{Ed}} = 10 \text{kN/m}$</td>
</tr>
</tbody>
</table>

In this chapter we will show a short example about the construction stage calculation through a steel frame building with the “Incremental Tracking Method”. Fig. 7.5.1.1 shows the geometry of the frame building. In this figure we also indicated the loads and the three different stages which were considered in this analysis. In this example there were three different stages by the three different storeys. As a simplification by the three different stages there were only a 10 kN/m line load on the beams one-by-one on each stages (see also Fig. 7.5.1.1) for the better comparison.

By the incremental tracking method of the construction stage calculation FEM-Design handles the different stages as different structures. We apply the given loads on these different structures and apply a superposition by the internal forces and displacements respectively. Thus basically the construction stage calculation is a nonlinear calculation with nonlinear boundary conditions and statical systems through the whole analysis.

First of all we will show the different stage calculations one-by-one with the different boundary conditions and statical systems and superpose the results to verify the FEM-Design construction stage calculation method. In the end we will compare the results and show the differences between the regular and the construction stage calculation method as well.

7.5.1.1 – The geometry and the considered loads on a steel frame building
Fig. 7.5.1.2 shows the different stages with the different statical systems and loads. Next to the structural view you can see the translational displacements for each stages one-by-one.

With the construction stage calculation we accumulate these results therefore for the verification we need to superpose these values to get the final stage results. We calculated the displacements at specific points of the beams by the symmetry axis of the structure (see Fig. 7.5.1.2):

\[ e_{B1} = e_{ST1B1} + e_{ST2B1} + e_{ST3B1} = 5.95 + 2.56 + 1.67 = 10.18 \text{ mm} \]
\[ e_{B2} = e_{ST1B2} + e_{ST2B2} + e_{ST3B2} = 0 + 2.60 + 1.69 = 4.29 \text{ mm} \]
\[ e_{B3} = e_{ST1B3} + e_{ST2B3} + e_{ST3B3} = 0 + 0 + 1.74 = 1.74 \text{ mm} \]
Fig. 7.5.1.3 shows the different stages with the different statical systems and loads. Next to the structural view you can see the bending moment diagrams for each stages one-by-one.

With the construction stage calculation we accumulate these results therefore for the verification we need to superpose these values to get the final stage results. We calculated the bending moments at specific points of the beams by the symmetry axis of the structure (see Fig. 7.5.1.3):

\[ M_{B1} = M_{ST1B1} + M_{ST2B1} + M_{ST3B1} = 89.96 + 62.67 + 39.55 = 192.2 \text{ kNm} \]

\[ M_{B2} = M_{ST1B2} + M_{ST2B2} + M_{ST3B2} = 0 + 17.74 + 41.01 = 58.75 \text{ kNm} \]

\[ M_{B3} = M_{ST1B3} + M_{ST2B3} + M_{ST3B3} = 0 + 0 + (-2.18) = -2.18 \text{ kNm} \]
Fig. 7.5.1.4 shows the final accumulated displacement field at the end of the stage calculation in FEM-Design. Here you can see the results at specific points of the beams by the symmetry axis of the structure:

\[ e_{B1FEM-CS} = 10.18 \text{ mm} \]
\[ e_{B2FEM-CS} = 4.29 \text{ mm} \]
\[ e_{B3FEM-CS} = 1.74 \text{ mm} \]

Fig. 7.5.1.5 shows the final accumulated bending moments at the end of the stage calculation in FEM-Design. Here you can see the results at specific points of the beams by the symmetry axis of the structure:

\[ M_{B1FEM-CS} = 192.18 \text{ kNm} \]
\[ M_{B2FEM-CS} = 58.75 \text{ kNm} \]
\[ M_{B3FEM-CS} = -2.18 \text{ kNm} \]
The verification results are identical with the FEM-Design calculation.

At the end of this chapter let's compare the final construction stage results with the results of a calculation without construction stage calculation method (regular calculation on the frame with one statical system and load distribution). Fig. 7.5.1.6-7 show the displacements and the bending moments after a regular calculation.

The results based on a regular calculation:

\[
\begin{align*}
e_{B1}^{\text{FEM}} - withoutC &= 4.92 \text{ mm} \\
e_{B2}^{\text{FEM}} - withoutC &= 4.93 \text{ mm} \\
e_{B3}^{\text{FEM}} - withoutC &= 4.94 \text{ mm}
\end{align*}
\]

\[
\begin{align*}
M_{B1}^{\text{FEM}} - withoutC &= 81.78 \text{ kNm} \\
M_{B2}^{\text{FEM}} - withoutC &= 83.29 \text{ kNm} \\
M_{B3}^{\text{FEM}} - withoutC &= 65.52 \text{ kNm}
\end{align*}
\]

In this verification example the maximum displacement from the construction stage calculation is 2.07 times greater at the first floor (compare Fig. 7.5.1.4 with Fig. 7.5.1.6). The bending moment value is 2.35 times greater at the first floor by this verification example (compare Fig. 7.5.1.5 with Fig. 7.5.1.7).

7.5.1.6 – The translational displacements [mm] according to the regular calculation method

7.5.1.7 – The bending moment diagram [kNm] according to the regular calculation method
By a real frame structure the difference in the final results are not that much if we consider and add the live loads to the final stage results. Here in this example to show and emphasize the calculation method only a self-weight-like load was considered.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/7.5.1 A steel frame building with construction stages calculation.str
8 Dynamic response analysis

8.1 Footfall analysis of a concrete footbridge

Example taken from Ref. [13]. Let's take the following footbridge statical system from Fig. 8.1.1.

Inputs for the self excitation footfall analysis:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic elastic modulus of concrete</td>
<td>E = 38 GPa</td>
</tr>
<tr>
<td>The distributed load (load-mass conversion)</td>
<td>p = 18.13 kN/m</td>
</tr>
<tr>
<td>Number of considered mode shapes</td>
<td>N = 3</td>
</tr>
<tr>
<td>Inertia of the cross-section</td>
<td>I = 0.056 m⁴</td>
</tr>
<tr>
<td>Area of the section</td>
<td>A = 0.77 m²</td>
</tr>
<tr>
<td>Number of footsteps (conservative estimation)</td>
<td>N_{footstep} = 100 pcs</td>
</tr>
<tr>
<td>Mass of the walker</td>
<td>m = 71.36 kg</td>
</tr>
<tr>
<td>Frequency weighting curve</td>
<td>W_g</td>
</tr>
<tr>
<td>The excitation frequency interval</td>
<td>f_{p,\text{min}} = 1 Hz, f_{p,\text{max}} = 2.8 Hz</td>
</tr>
<tr>
<td>Frequency steps</td>
<td>steps = 100 pcs</td>
</tr>
<tr>
<td>The cut-off eigenfrequency</td>
<td>f_{cut} = 15 Hz</td>
</tr>
<tr>
<td>Damping</td>
<td>ζ = 1.5 %</td>
</tr>
<tr>
<td>Fourier coefficients</td>
<td>The Concrete Centre Table 4.3</td>
</tr>
</tbody>
</table>

The model is divided into 16 finite bar elements. The given distributed load is converted to mass with 1.0 factor (1848 kg/m) for the eigenfrequency calculation. The statical system is a beam with the given stiffness parameters and with 3 supports (see Fig. 8.1.1). All of the necessary parameters for the footfall analysis is given in the inputs. In FEM-Design the used excitation method was the self excitation method. For the self excitation method the adjusted region contained the full beam structure.

The first three mode shapes are visible in Fig. 8.1.2 based on the FEM-Design calculation. Table 8 contains the theoretical solutions about the eigenfrequencies of the first three modes according to Ref. [13] and FEM-Design results are also indicated. There are good agreements between the two results.
This footbridge is relatively soft, therefore the steady-state acceleration will be greater than the transient. As a simple hand calculation the RMS acceleration for walking at 2.102 Hz is the following:

The amplitude of the excitation force by the second harmonic:

\[ F_2 = \frac{71.36}{1000} \cdot 9.81 (0.069 + 0.0056 \cdot 2 \cdot 2.102) = 0.06478 \text{ kN} \]

In this case the second harmonic of the excitation frequency causes resonance.

The dynamic magnification factor for the accelerations by the 1\textsuperscript{st} mode shape and 2\textsuperscript{nd} harmonic:

\[
D_{1,2} = \frac{2^2 \left( \frac{f_p}{f_1} \right)^2}{\sqrt{1 - 2^2 \left( \frac{f_p}{f_1} \right)^2 + 4 \zeta^2 2^2 \left( \frac{f_p}{f_1} \right)^2}} = \frac{2^2 \left( \frac{2.102}{4.203} \right)^2}{\sqrt{1 - 2^2 \left( \frac{2.102}{4.203} \right)^2 + 4 \cdot 0.015^2 \cdot 2^2 \left( \frac{2.102}{4.203} \right)^2}} = \frac{1}{\sqrt{0 + 4 \cdot 0.015^2 \cdot 1}} = \frac{1}{2 \cdot 0.015} = 33.33
\]
Based on these values the RMS acceleration at mid-span (see Fig. 8.1.2 also):

\[ a_{w,midspan,RMS[steady\ state]} = \frac{1}{\sqrt{2}} \mu_{midspan,1}^2 \frac{F_2}{M_1} D_{I,2} W_2 = \frac{1}{\sqrt{2}} \left( \frac{1}{33.33} \right) \cdot \frac{0.06478}{1} \cdot 0.1645^2 = 0.04131 \text{ m/s}^2 \]

In Ref. [13] the peak acceleration value is \( a_{peak} = 0.06 \text{ m/s}^2 \), therefore the comparable RMS value is:

\[ a_{RMS} = a_{peak} \frac{\sqrt{2}}{\sqrt{2}} = 0.06 \text{ m/s}^2 = 0.04243 \text{ m/s}^2 \]

and the response factor based on Ref. [13]: \( R = 8.5 \)

Based on the FEM-Design calculation these two values are (see Fig. 8.1.3 as well):

\[ a_{RMS,FEM} = 0.0443 \text{ m/s}^2 \] and \( R = 8.86 \).

There are good agreements between the results. The difference comes from the fact that not only the first mode shape has effect on the accelerations however in Ref. [13] and the hand calculation here considered only the first mode.

Another interesting result could be the frequency curve. Fig. 8.1.4 shows the accelerations at the midspan point in function of the excitation frequencies. The red line is the steady-state response and the green one is the transient. Based on Fig. 8.1.4 we can say that in this example the transient response is really negligible compared to the steady-state response. The frequency curve clearly shows the resonance excitation frequencies where the peak RMS accelerations arise.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/8.1 Footfall analysis of a concrete footbridge.str
Figure 8.1.4 – Accelerations in function of excitation frequencies in FEM-Design

Red: steady-state response

Green: transient response
8.2 Footfall analysis of a composite floor

Example taken from Ref. [14]. Let's take a 130 mm deep normal weight concrete slab on top of 1.2 mm thick re-entrant deck. Slabs supported by 6.0 m span secondary beams at 2.48 m cross-centres which, in turn, are supported by 7.45 m span castellated primary beams in orthogonal direction, see Fig. 8.2.1. The input data and the geometry are available in Ref. [14].

Inputs for the self excitation footfall analysis:

<table>
<thead>
<tr>
<th>Input Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excitation region (see Fig. 8.2.1)</td>
<td>The whole floor slab</td>
</tr>
<tr>
<td>The distributed load (load-mass conversion)</td>
<td>p = 4.48 kN/m²</td>
</tr>
<tr>
<td>Number of footsteps (conservative estimation)</td>
<td>N_{footstep} = 100 pcs</td>
</tr>
<tr>
<td>Mass of the walker</td>
<td>m = 76 kg</td>
</tr>
<tr>
<td>Frequency weighting curve</td>
<td>W_g</td>
</tr>
<tr>
<td>The excitation frequency interval</td>
<td>f_{p,min} = 1.8 Hz, f_{p,max} = 2.2 Hz</td>
</tr>
<tr>
<td>Frequency steps</td>
<td>steps = 100 pcs</td>
</tr>
<tr>
<td>The cut-off eigenfrequency</td>
<td>f_{cut} = 15 Hz</td>
</tr>
<tr>
<td>Damping</td>
<td>ζ = 4.68 %</td>
</tr>
<tr>
<td>Fourier coefficients</td>
<td>SCI P354 Table 3.1</td>
</tr>
</tbody>
</table>

In Ref. [14] with the finite element calculation the first fundamental natural frequency was:

\[ f_1 = 10.80 \text{Hz} \]

In Ref. [14] with the finite element calculation the response factor was:

\[ R = 3.18 \]

With the given parameters above and considering the geometry and the material properties based on Ref. [14] FEM-Design calculation gives the following results (see Fig. 8.2.1 also):

\[ f_{FEM} = 10.82 \text{Hz} \quad \text{and} \quad R = 3.82 \]

We can say that there are good agreements between the results. However, it should be noted that in Ref. [14] the results of the calculation is given, but the details of the finite element model and calculation method is unclear, therefore there may be differences in the modeling methods. By this example it is very hard to say that the result in Ref. [14] is relevant because the hand calculation is quite different than the FEM calculation what was published in Ref. [14]. Based on our opinion the indicated FEM result in Ref. [14] belongs to the transient response as well as the result in FEM-Design.
Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/8.2 Footfall analysis of a composite floor.str

*Figure 8.2.1 – The model, the first mode shape [-] and the response factor [-] results in FEM-Design*
8.3 Footfall analysis of a lightweight floor

Example taken from Ref. [14]. Let’s take a chipboard flooring on lightweight steel beams, see Fig. 8.3.1. The input data and the geometry are available in Ref. [14].

Inputs for the full excitation footfall analysis:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excitation point (see Fig. 8.3.1)</td>
<td>In the middle of the floor</td>
</tr>
<tr>
<td>The distributed load (load-mass conversion)</td>
<td>p = 0.69 kN/m²</td>
</tr>
<tr>
<td>Number of footsteps (conservative estimation)</td>
<td>N_{\text{footstep}} = 100 pcs</td>
</tr>
<tr>
<td>Mass of the walker</td>
<td>m = 76 kg</td>
</tr>
<tr>
<td>Frequency weighting curve</td>
<td>W_g</td>
</tr>
<tr>
<td>The excitation frequency interval</td>
<td>f_{p,\text{min}} = 1.8 Hz, f_{p,\text{max}} = 2.2 Hz</td>
</tr>
<tr>
<td>Frequency steps</td>
<td>steps = 100 pcs</td>
</tr>
<tr>
<td>The cut-off eigenfrequency</td>
<td>f_{\text{cut}} = 15 Hz</td>
</tr>
<tr>
<td>Damping</td>
<td>\zeta = 5.0 %</td>
</tr>
<tr>
<td>Fourier coefficients</td>
<td>SCI P354 Table 3.1</td>
</tr>
</tbody>
</table>

In Ref. [14] with the finite element calculation the first fundamental natural frequency was:

\[ f_1 = 16.31 \text{ Hz} \]

In Ref. [14] with the finite element calculation the response factor was:

\[ R = 53.9 \]

In FEM-Design the average finite element size was 0.40 m. With the given parameters above and considering the geometry and the material properties based on Ref. [14] FEM-Design calculation gives the following results (see Fig. 8.3.1 also):

\[ f_{\text{FEM}} = 16.13 \text{ Hz} \quad \text{and} \quad R = 53.87 \]

Fig. 8.3.2 shows the response factors in function of the given interval of the excitation force based on FEM-Design calculation.

We can say that there are good agreements between the results. However, it should be noted that in Ref. [14] the results of the calculation is given, but the details of the finite element model and calculation method is unclear, therefore there may be differences in the modeling methods.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/8.3 Footfall analysis of a lightweight floor.str
Verification Examples

Figure 8.3.1 – The model, the first mode shape [-] and the response factor [-] results in FEM-Design

Figure 8.3.2 – Response factor in function of excitation frequencies in FEM-Design
Red: steady-state response
Green: transient response
8.4 Footfall analysis of a small stage with rhythmic crowd load

This calculation will be presented according to Danish Annex (Ref. [15]). The floor is a simply supported concrete slab. The half of the slab is a stage where the rhythmic crowd activity will be considered (see Fig. 8.4.1).

Inputs for the rhythmic crowd load footfall analysis:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus of concrete</td>
<td>( E = 31 \text{ GPa}, \nu = 0.2 )</td>
</tr>
<tr>
<td>Thickness of the concrete slab</td>
<td>( t = 250 \text{ mm} )</td>
</tr>
<tr>
<td>Self-weight plus the considered imposed load</td>
<td>( p = 6.75 \text{ kN/m}^2 )</td>
</tr>
<tr>
<td>Mean static crowd load (on half of the slab, Fig. 8.4.1)</td>
<td>( F_p = 1.0 \text{ kN/m}^2 )</td>
</tr>
<tr>
<td>The excitation frequency</td>
<td>( f_p = 3 \text{ Hz} )</td>
</tr>
<tr>
<td>The cut-off eigenfrequency</td>
<td>( f_{\text{cut}} = 30 \text{ Hz} )</td>
</tr>
<tr>
<td>Damping</td>
<td>( \zeta = 1.9 % )</td>
</tr>
<tr>
<td>Effective number of people</td>
<td>( n_e = 20 )</td>
</tr>
<tr>
<td>Fourier coefficients</td>
<td>According to Danish Annex</td>
</tr>
<tr>
<td></td>
<td>Reduced possibility to move about</td>
</tr>
</tbody>
</table>

The first eigenfrequency (based on finite element calculation):
\[
 f_1 = 12 \text{ Hz}
\]

In the Danish Annex the logarithmic decrement is given instead of critical damping ratio. The logarithmic decrement with the given critical damping ratio from the inputs:
\[
 (\delta_s + \delta_p) = 2 \pi \zeta = 2 \pi 0.019 = 0.12
\]
The frequency response factor in the Danish Annex is given with:

\[ H_j = \frac{1}{\sqrt{\left(1 - \left(j \cdot f_p \frac{f_1}{f_1}\right)^2\right)^2 + \left(\delta_s + \delta_p \frac{j \cdot f_p}{\pi f_1}\right)^2}} \]

Therefore:

\[ H_1 = \frac{1}{\sqrt{\left(1 - \left(1.3 \frac{12}{12}\right)^2\right)^2 + \left(0.12 \cdot 1.3 \frac{12}{\pi}\right)^2}} = 1.067 \]

\[ H_2 = \frac{1}{\sqrt{\left(1 - \left(2.3 \frac{12}{12}\right)^2\right)^2 + \left(0.12 \cdot 2.3 \frac{12}{\pi}\right)^2}} = 1.333 \]

\[ H_3 = \frac{1}{\sqrt{\left(1 - \left(3.3 \frac{12}{12}\right)^2\right)^2 + \left(0.12 \cdot 3.3 \frac{12}{\pi}\right)^2}} = 2.281 \]

The considered Fourier coefficients including the size reduction factor:

\[ \alpha_1 K_1 = 0.40 \]

\[ \alpha_2 K_2 = 0.25 \sqrt{0.1 + (1-0.1) \frac{1}{20}} = 0.0952 \]

\[ \alpha_3 K_3 = 0.05 \sqrt{0.01 + (1-0.01) \frac{1}{20}} = 0.0122 \]

The dynamic magnification factor for displacements (according to Danish Annex):

\[ k_F = \sqrt{\sum_{j=1}^{3} (\alpha_j K_j H_j)^2} = \sqrt{(0.4 \cdot 1.067)^2 + (0.0952 \cdot 1.333)^2 + (0.0122 \cdot 2.281)^2} \]

\[ k_F = 0.4461 \]

The acceleration response factor (according to Danish Annex):

\[ k_a = \sqrt{\frac{1}{2} \sum_{j=1}^{3} (j^2 \alpha_j K_j H_j)^2} = \frac{1}{\sqrt{2}} \sqrt{(1^2 \cdot 0.4 \cdot 1.067)^2 + (2^2 \cdot 0.0952 \cdot 1.333)^2 + (3^2 \cdot 0.0122 \cdot 2.281)^2} \]

\[ k_a = 0.5013 \]
The maximum deflection of the slab under the mean static crowd load on the half of the slab (based on a finite element calculation, see Fig. 8.4.2):

\[ u_p = 0.2132 \text{ mm} \]

The RMS acceleration of the structure induced by the vertical dynamic load (according to Danish Annex):

\[ a_a = k_a \left( \frac{2 \pi f_p}{u_p} \right)^2 = 0.5013 \cdot \left( \frac{2 \pi 3}{0.2132} \right)^2 \frac{0.2132}{1000} = 0.03797 \text{ m/s}^2 \]

The accelerations and the dynamic magnification factors for displacements based on the FEM-Design calculation (see Fig. 8.4.3):

\[ a_{FEM} = 0.03832 \text{ m/s}^2 \]
\[ k_{FEM} = 0.446 \]

The difference between the hand calculation and FEM-Design calculation is less than 1%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/8.4 Footfall analysis of a small stage with rhythmic crowd load.str
Figure 8.4.3 – The accelerations in [m/s^2] and the response factors [-] in FEM-Design
8.5 Dynamic response of a moving mass point on a simply supported beam

Example is taken from Ref. [18]. Let's take a simply supported beam with one moving mass point on it, see Fig. 8.5.1.

Inputs (as you can see in this case the moving object will be a “force” but its mass will be considered during the dynamic calculation):

<table>
<thead>
<tr>
<th>Analyzed position</th>
<th>Mid-span of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>The distributed mass of the beam</td>
<td>$m = 312 \text{ kg/m}$</td>
</tr>
<tr>
<td>Mass of the moving mass</td>
<td>$M = 1248 \text{ kg}$</td>
</tr>
<tr>
<td>Elastic modulus of the beam</td>
<td>$E = 206 \text{ GPa}$</td>
</tr>
<tr>
<td>Span length</td>
<td>$L = 20 \text{ m}$</td>
</tr>
<tr>
<td>Analyzed constant velocities of the moving mass</td>
<td>$v_1 = 18.12 \text{ km/h}$</td>
</tr>
<tr>
<td></td>
<td>$v_2 = 36.24 \text{ km/h}$</td>
</tr>
<tr>
<td></td>
<td>$v_3 = 72.48 \text{ km/h}$</td>
</tr>
<tr>
<td>Critical damping ratio</td>
<td>$\xi = 0 %$</td>
</tr>
<tr>
<td>Path length</td>
<td>$L_p = 40 \text{ m}$</td>
</tr>
<tr>
<td>Cross-section</td>
<td>$200 \text{ mm} \times 200 \text{ mm square}$</td>
</tr>
</tbody>
</table>

In FEM-Design by this example during the direct integration calculation the damping was neglected (thus in the Rayleigh damping matrix the $\alpha = 0$ and $\beta = 0$), therefore the system is undamped. In the FEM model we split the beam into 50 parts to consider a relevant continuous mass distribution, and 201 mass point positions, see Fig. 8.5.1. The results belong to those positions which are not on the structure will give us the free vibration response of the structure, see Fig. 8.5.4.

The first three eigenfrequencies of a simply supported beam with continuous distributed mass.

$$f_1 = \frac{\pi}{2L^2} \sqrt{\frac{EI}{m}} = \frac{\pi}{2 \cdot 20^2} \sqrt{\frac{206000000000 \cdot 0.2^4/12}{312}} = 1.165 \text{ Hz}$$

$$f_2 = \frac{\pi}{2(L/2)^2} \sqrt{\frac{EI}{m}} = \frac{\pi}{2 \left(\frac{20}{2}\right)^2} \sqrt{\frac{206000000000 \cdot 0.2^4/12}{312}} = 4.661 \text{ Hz}$$
Verification Examples

\[ f_3 = \frac{\pi}{2} \left( \frac{L}{3} \right)^2 \sqrt{\frac{206000000000 \cdot 0.2^4 / 12}{312}} = 10.49 \text{ Hz} \]

According to FEM-Design calculation the first three eigenfrequencies of the beam are as follows, see Fig. 8.5.2:

\[ f_{FEM1} = 1.165 \text{ Hz} \]
\[ f_{FEM2} = 4.658 \text{ Hz} \]
\[ f_{FEM3} = 10.47 \text{ Hz} \]

The critical velocity:

\[ v_{cr} = 2L f_1 = 2 \cdot 20 \cdot 1.165 = 46.6 \frac{\text{m}}{\text{s}} = 167.76 \frac{\text{km}}{\text{h}} \]

The analytical solutions based on Ref. [18] are the normalised dynamic factors of the beam mid-span under different velocities. These values are very close to the FEM-Design numerical calculations.

The maximum static deflection of the mid-span if the mass is at the mid-span:

\[ e_{max} = \frac{FL^3}{48EI} = \frac{MgL^3}{48EI} = \frac{1248 \cdot 9.81 \cdot 20^3}{48 \cdot 206000000000 \cdot 0.2^4 / 12} = 0.07429 \text{ m} \]
The maximum static deflection at mid-span based on FEM-Design calculation (see also Fig. 8.5.4):
\[ e_{FEM}^{max} = 0.07431 \text{ m} \]

The maximum normalized dynamic factor at the mid-span under \( v_1 \) velocity according to Ref. [18]:
\[ \left( \frac{Z_{\text{dynamic}}}{Z_{\text{static}}^{max}} \right)_{\text{max}} = 1.1 \]

This value based on FEM-Design, see Fig. 8.5.3-4:
\[ \left( \frac{Z_{\text{FEMdynamic}}}{Z_{\text{FEMstatic}}^{max}} \right)_{\text{max}} = 1.096 \]

Fig. 8.5.4 shows the static and dynamic displacements at the mid-span under the different load positions with \( v_1 \) velocity.

In Fig. 8.5.4 we can see the static and dynamic maximum deflection at the mid-span.
\[ e_{\text{dynamic max}}^{FEM v_1} = 0.08139 \text{ m} \]
Figure 8.5.4 – The dynamic and static displacements of the mid-span in function of mass positions with $v_1$ velocity with FEM-Design
The maximum normalized dynamic factor at the mid-span under $v_2$ velocity:

$$\left(\frac{Z_{\text{dynamic}}}{Z_{\text{static}}^{\max}}\right)^{\max} = 1.21$$

This value based on FEM-Design, see Fig. 8.5.5:

$$\left(\frac{Z_{\text{FEMdynamic}}^{\max}}{Z_{\text{FEMstatic}}^{\max}}\right)^{\max} = 1.21$$

Fig. 8.5.6 shows the static and dynamic displacements at the mid-span under the different load positions with $v_2$ velocity.

About Fig. 8.5.6 we can see the static and dynamic maximum deflection at the mid-span.

$$e^{\text{FEM } v_2}_{\text{dynamic max}} = 0.08991\, \text{m}$$
Figure 8.5.6 – The dynamic and static displacements of the mid-span in function of mass positions with $v^2$ velocity with FEM-Design
The maximum normalized dynamic factor at the mid-span under $v_3$ velocity:

$$\left(\frac{Z_{3\text{dynamic}}}{Z_{3\text{static}}}\right)_{\text{max}} = 1.85$$

This value based on FEM-Design, see Fig. 8.5.7:

$$\left(\frac{Z_{3\text{FEMdynamic}}}{Z_{3\text{FEMstatic}}}\right)_{\text{max}} = 1.76$$

Fig. 8.5.8 shows the static and dynamic displacements at the mid-span under the different load positions with $v_3$ velocity.
About Fig. 8.5.8 we can see the static and dynamic maximum deflection at the mid-span.

\[ e_{\text{FEM, v_3}}^{\text{dynamic, max}} = 0.1310 \text{ m} \]

We can say that there are good agreements between the results. On the FEM-Design dynamic deflection results we can see the free vibration behaviour of the structure as well.

Download link to the example file:
[http://download.strusoft.com/FEM-Design/inst190x/models/8.5 Dynamic response of a moving mass point on a simply supported beam.str](http://download.strusoft.com/FEM-Design/inst190x/models/8.5 Dynamic response of a moving mass point on a simply supported beam.str)
8.6 Dynamic response of a moving single force on a simply supported beam

![Diagram of a beam with a moving load](image)

Figure 8.6.1 – The geometry and the moving load positions on the path during the calculation

Inputs:

<table>
<thead>
<tr>
<th>Analyzed position</th>
<th>Mid-span of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analyzed position</td>
<td>Mid-span of the beam</td>
</tr>
<tr>
<td>The distributed mass of the beam</td>
<td>$\mu = 7950 \text{ kg/m}$</td>
</tr>
<tr>
<td>The concentrated moving force</td>
<td>$F = 480 \text{ kN}$</td>
</tr>
<tr>
<td>Bending stiffness</td>
<td>$EI = 4.26 \times 10^7 \text{ kNm}^2$</td>
</tr>
<tr>
<td>Span length</td>
<td>$L = 30 \text{ m}$</td>
</tr>
<tr>
<td>Analyzed constant velocities of the moving force</td>
<td>$v_1 = 50 \text{ m/s} = 180 \text{ km/h}$</td>
</tr>
<tr>
<td>Analyzed constant velocities of the moving force</td>
<td>$v_2 = 100 \text{ m/s} = 360 \text{ km/h}$</td>
</tr>
<tr>
<td>Damping</td>
<td>$\xi = 0 %$</td>
</tr>
<tr>
<td>Path length</td>
<td>$L_p = 90 \text{ m}$</td>
</tr>
</tbody>
</table>

In this calculation we neglect the mass effect of the moving force, because based on Ref. [19] we can get the analytical solution of the problem directly. Fig. 8.6.1 shows the problem with the considered moving force, positions and geometry.

The first eigen and angular natural frequency based on closed form with Bernoulli beam theory:

$$f_1 = \frac{\pi}{2 \cdot L^2} \sqrt{\frac{EI}{m}} = \frac{\pi}{2 \cdot 30^2} \sqrt{\frac{4.26 \times 10^7}{7.95}} = 4.040 \text{ Hz} \quad ; \quad \omega_{01} = 2 \pi f_1 = 25.38 \frac{\text{rad}}{\text{s}}$$

The first eigen and angular natural frequency based on FEM-Design calculation with Timoshenko beam theory:

$$f_{FEM1} = 3.997 \text{ Hz} \quad ; \quad \omega_{FEM\ 01} = 2 \pi f_{FEM1} = 25.11 \frac{\text{rad}}{\text{s}}$$

The analytical solution of the deflection of the beam in function of time and load position according to Ref. [19] if the moving force is on the structure can be calculated with the following formula:

$$e(x,t) = \sum_{r=1}^{n} v_r(x) \eta_r(t)$$
where:
\[ v_r(x) = \sqrt{\frac{2}{\mu L}} \sin \frac{r \pi x}{L} \quad \eta_r(t) = \frac{F}{\omega_{0r}} \sqrt{\frac{2}{\mu L}} \left( \frac{r \pi v}{L} \right)^2 \left( \frac{r \pi v}{L} \sin \omega_{0r} t - \omega_{0r} \sin \frac{r \pi v}{L} t \right) \]

In addition to the above mentioned parameters, \( r \) is the number of the eigenfrequencies.

At the midspan only the odd number of eigenfrequencies have effect on the final solution. In this case the dominant eigenfrequency is the first one, thus at the analytical solution we will use only the first eigenfrequency by the summation. In the complete analytical solution all of the eigenfrequencies have effect on the final solution, but in this case the rest of them have neglectable influence.

If \( 0 < t < L/v \) the approximate deflection at the mid-span in function of time:
\[ e\left( \frac{L}{2}, t \right) = \frac{2}{\mu L} \frac{F}{\omega_{01}} \left( \frac{1 \pi v}{L} \right)^2 \left( \frac{1 \pi v}{L} \sin \omega_{01} t - \omega_{01} \sin \frac{1 \pi v}{L} t \right) \]

The derivative of this function (velocity):
\[ \dot{e}\left( \frac{L}{2}, t \right) = \frac{2}{\mu L} \frac{F}{\omega_{01}} \left( \frac{1 \pi v}{L} \right)^2 \left( \cos \omega_{01} t - \cos \frac{1 \pi v}{L} t \right) \]

In that time value when the force is leaving the structure the initial values of the remaining free vibration part of the solution:

In case of \( v_i \):
\[ t_i = \frac{L}{v_i} = \frac{30}{50} = 0.6 \text{ s} \]
Displacement:
\[ e_{init} = e\left( \frac{L}{2}, 0.6 \right) = \frac{2}{7950 \cdot 30} \frac{480000}{25.38} \left( \frac{1 \pi 50}{30} \right)^2 \left( \frac{1 \pi 50}{30} \sin 25.38 \cdot 0.6 - 25.38 \sin \frac{1 \pi 50}{30} \cdot 0.6 \right) = -0.0006217 \text{ m} \]
Velocity:
\[ \dot{e}_{init} = \dot{e}\left( \frac{L}{2}, 0.6 \right) = \frac{2}{7950 \cdot 30} \frac{480000}{25.38} \left( \frac{1 \pi 50}{30} \right)^2 \left( \cos 25.38 \cdot 0.6 - \cos \frac{1 \pi 50}{30} \cdot 0.6 \right) = -0.003861 \text{ m/s} \]
In case of $v_2$:

$$t_2 = \frac{L}{v_2} = \frac{30}{100} = 0.3 \text{ s}$$

Displacement:

$$e_{init_2} = e\left(\frac{L}{2}, 0.3\right) = -0.0030183 \text{ m}$$

Velocity:

$$\dot{e}_{init_2} = \dot{e}\left(\frac{L}{2}, 0.3\right) = -0.097608 \text{ m/s}$$

After that the moving force has just reached the end of the beam the dynamic behaviour of the beam will be a free vibration (if no damping is considered). With the mentioned initial values based on that time value when the force will be at the end of the beam the function of the free vibration can be given:

If $L/v < t$ the approximate deflection at the mid-span in function of time:

$$e\left(\frac{L}{2}, t\right) = e_{init} \cos \left[\omega_{01} \left(t - \frac{L}{v}\right)\right] + \dot{e}_{init} \frac{\omega_{01}}{\omega_{01}} \sin \left[\omega_{01} \left(t - \frac{L}{v}\right)\right]$$

Fig. 8.6.2 shows the deflection function of the given analytical solution with the two different velocities.

Fig. 8.6.3-4 shows the dynamic and static deflection of the mid-span under the moving load based on FEM-Design calculation with the two different velocities.

**Figure 8.6.2 – The dynamic deflection of the mid-span with the two given velocities according to the analytical solution**
We can see in Fig. 8.6.2-4 that after the position when the load has just left the beam the dynamic response is free vibration.

According to the given solutions we can say that the FEM-Design result are identical with the analytical solution.

Download link to the example file: http://download.strusoft.com/FEM-Design/inst190x/models/8.6 Dynamic response of a moving single force on a simply supported beam.str

![Moving load dynamic analysis: Displacement dyn (180.00 km/h)](image)

*Figure 8.6.3 – The dynamic and static displacements of the mid-span in function of force positions with v, velocity with FEM-Design*
Figure 8.6.4 – The dynamic and static displacements of the mid-span in function of force positions with \( v_2 \) velocity with FEM-Design.
8.7 Dynamic response of a moving force group on a simply supported beam

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analyzed position</td>
<td>Mid-span of the beam</td>
</tr>
<tr>
<td>The distributed mass of the beam</td>
<td>m = 7950 kg/m</td>
</tr>
<tr>
<td>Bending stiffness</td>
<td>EI = 4.26*10^7 kNm^2</td>
</tr>
<tr>
<td>Span length</td>
<td>L = 30 m</td>
</tr>
<tr>
<td>Analyzed velocities of the moving force groups</td>
<td></td>
</tr>
<tr>
<td>Case a): v_1 = 50 m/s = 180 km/h</td>
<td></td>
</tr>
<tr>
<td>Case a): v_2 = 100 m/s = 360 km/h</td>
<td></td>
</tr>
<tr>
<td>Case b): v_1 = 100 m/s = 360 km/h</td>
<td></td>
</tr>
<tr>
<td>Case b): v_2 = 120 m/s = 432 km/h</td>
<td></td>
</tr>
<tr>
<td>Critical damping ratio</td>
<td>ξ = 1.5 % = 0.015</td>
</tr>
<tr>
<td>Path length</td>
<td>L_p = 500 m</td>
</tr>
</tbody>
</table>

In this calculation we neglect the mass effect of the moving force, because based on Ref. [19] we have benchmark results to compare our FEM-Design results. In this example the structure is the same as in the previous example, only the moving load differs from it.

In this example we considered two different load groups as the moving loads. Fig. 8.7.1 top shows case a) and bottom shows case b). In both cases we modeled 12 motor trains group as the moving load. Case a) includes 12 individual concentrated loads (12 x 544 kN) to consider the weights of the train cars. In case b) we considered that every train cars have 4 axes, thus the load group contains 12 x 4 x 136 kN. The resultants of the load groups are the same.

![Figure 8.7.1 – The considered load groups to the dynamic analysis](image)

Thus in these examples case a) and case b) load groups will be move on the path.

In this example we considered damping based on the equivalent Kelvin-Voigt damping.

Considering the given critical damping ratio and the first eigenfrequency of the given structure (based on the previous example) the Rayleigh damping parameters are as follows:

\[ \alpha = 0; \quad \beta = \frac{\xi}{\pi f_1} = \frac{0.015}{\pi 3.997} = 0.001195 \]
With the given input parameters Fig. 8.7.2 shows the dynamic response of the mid-span in function of the load position based on Ref. [19] in case a) load groups with \( v_1 = 50 \text{ m/s} = 180 \text{ km/h} \) and \( v_2 = 100 \text{ m/s} = 360 \text{ km/h} \).

![Figure 8.7.2 – The dynamic deflection of the mid-span based on Ref. [19] in case a) with the two velocities](image)

Fig. 8.7.3-4 shows the dynamic response of mid-span in function of the load positions based on the FEM-Design moving load dynamic calculation in case a) load group.

![Figure 8.7.3 – The dynamic and static displacements of the mid-span in function of force positions with \( v_1 \) velocity in FEM-Design in case a)](image)
We can say that the results are very close to each other.

Fig. 8.7.5 shows the dynamic response of the mid-span in function of the load position based on Ref. [19] in case b) load groups with $v_1 = 100 \text{ m/s} = 360 \text{ km/h}$ and $v_2 = 120 \text{ m/s} = 432 \text{ km/h}$.
Fig. 8.7.6-7 shows the dynamic response of mid-span in function of the load positions based on the FEM-Design moving load dynamic calculation in case b) load groups.

We can say that the results are very close to each other.

Download link to the example file:
Case a):
http://download.strusoft.com/FEM-Design/inst190x/models/8.7 Dynamic response of a moving force group on a simply supported beam case a.str

Case b):
http://download.strusoft.com/FEM-Design/inst190x/models/8.7 Dynamic response of a moving force group on a simply supported beam case b.str
Figure 8.7.7 – The dynamic and static displacements of the mid-span in function of force positions with $v_2$ velocity in FEM-Design in case b)
9 Design calculations

This chapter is unfinished.

According to Eurocode standard!

9.1 Foundation design

This chapter is unfinished.

9.1.1 Design of an isolated foundation

This chapter is unfinished.

9.1.2 Design of a wall foundation

This chapter is unfinished.

9.1.3 Design of a foundation slab

This chapter is unfinished.
9.2 Reinforced concrete design

In this chapter we will show some detailed verification calculations regarding to reinforced concrete design according to EN 1992-1-1.

9.2.1 Moment capacity calculation for beams under pure bending

In this chapter we calculate the moment capacity of reinforced concrete cross sections under pure bending (uniaxial bending).

The first example will be an under-reinforced cross section. The second one will be a normal-reinforced and the third one an over-reinforced section. After independent “hand” calculations we compare the results with FEM-Design values.

The following input parameters are common for the different cross sections regarding to this subchapter.

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete characteristic compressive strength</td>
<td>$f_{ck} = 30 \text{ N/mm}^2$ or $20 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>The end of the parabolic part (material model, see EC-2)</td>
<td>$\varepsilon_{c2} = 0.20 %$</td>
</tr>
<tr>
<td>Ultimate limit strain of concrete (see EC-2)</td>
<td>$\varepsilon_{cu2} = 0.35 %$</td>
</tr>
<tr>
<td>Partial factor of concrete</td>
<td>$\gamma_c = 1.50$</td>
</tr>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{yk} = 420 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Elastic modulus of reinforcing steel</td>
<td>$E_s = 200 \text{ GPa}$</td>
</tr>
<tr>
<td>Ultimate limit strain of reinforcing steel</td>
<td>$\varepsilon_{uk} = 2.5 %$</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
<tr>
<td>Behaviour of plastic part (see Fig. 9.2.1.1.4)</td>
<td>$k = 1.05$</td>
</tr>
<tr>
<td>Concrete cover (on stirrups)</td>
<td>$c = 20 \text{ mm}$</td>
</tr>
<tr>
<td>Stirrup diameter</td>
<td>$\phi_s = 8 \text{ mm}$</td>
</tr>
</tbody>
</table>

The external dimensions are the same for every cross section ($b = 300 \text{ mm}$, $h = 500 \text{ mm}$).

9.2.1.1 Under-reinforced cross section

In this example we put two longitudinal rebars with 16 mm diameter at the bottom left and right corner of the stirrups (see Fig. 9.2.1.1.1, the concrete is C30/37). We neglect the effect of the hangers. Two different “hand” calculation methods provided here. First of all with aim of the most simple material models for concrete and reinforcing steel and secondly with improved material models, which FEM-Design uses also.

First we calculate the moment capacity with the following material models (see Fig. 9.2.1.1.2).
Due to the under-reinforced section behaviour we assume that the rebars strains are at the design ultimate limit strain value.

Based on the sum of the forces we can get the height of the active compression concrete zone (see Fig. 9.2.1.3).

\[ b x_c f_{cd} - A_s f_{yd} = 0 \]

The compressed zone:

\[ x_c = \frac{A_s f_{yd}}{b f_{cd}} = \frac{2 \cdot 16^2 \cdot \pi \cdot 420}{4 \cdot 1.15 \cdot 300 \cdot 1.5} = 24.48 \text{ mm} \]
We need to check that the assumption of the rebar strains were proper or not.

Based on the assumption that the concrete reaches its ultimate compression strain limit then the strain in the rebars (see Fig. 9.2.1.1.3):

\[ \varepsilon_s = \frac{(d - 1.25 x_c) \varepsilon_{cu2}}{1.25 x_c} = \frac{(500 - 20 - 8 - 8 - 1.25 \cdot 24.48) \cdot 0.0035}{1.25 \cdot 24.48} = 4.957\% > \varepsilon_{ud} = 2.25\% \]

Thus the maximum concrete strain cannot be \( \varepsilon_{cu2} = 0.35\% \), due to this the rebars reach the ultimate strain, thus the assumption was correct, the failure mode is the rupture of the rebars (under-reinforced section). But it has no effect on the former equilibrium equation.

The moment capacity of the section with the simple material models:

\[ M_{Rd,1} = b x_c f_{cd} \left( d - \frac{x_c}{2} \right) = 300 \cdot 24.48 \cdot \frac{30}{1.5} \left( 464 - \frac{24.48}{2} \right) = 66.35 \text{kNm} \]

Secondly we calculate with improved material models, see the following equations.

Concrete (see Fig. 9.2.1.1.4 left side):

\[ \sigma_c(\varepsilon_c) = f_{cd} \left[ 1 - \left( 1 - \frac{\varepsilon_c}{\varepsilon_{c2}} \right)^2 \right] \text{ if } 0 \leq \varepsilon_c \leq \varepsilon_{c2} \]

\[ \sigma_c(\varepsilon_c) = f_{cd} \text{ if } \varepsilon_{c2} < \varepsilon_c \leq \varepsilon_{cu2} \]
Rebars (see Fig. 9.2.1.4 right side):

\[
\sigma_s(\varepsilon_s) = \varepsilon_s E_s \quad \text{if} \quad \varepsilon_s \leq \frac{f_{yd}}{E_s}
\]

\[
\sigma_s(\varepsilon_s) = f_{yd} + (k-1) f_{yd} \frac{\varepsilon_s - \frac{f_{yd}}{E_s}}{\varepsilon_{ud} - \frac{f_{yd}}{E_s}} \quad \text{if} \quad \frac{f_{yd}}{E_s} < \varepsilon_s \leq \varepsilon_{ud}
\]

![Figure 9.2.1.4 – The material models for the second calculation left: concrete (only compression), right: rebar (both tension and compression)](image)

The assumed stress and strain distributions (Fig. 9.2.1.5).

![Figure 9.2.1.5 – The assumed strains and the stresses](image)

The concrete strains depends on the curvature and based on the improved material models this led to a nonlinear equations.

\[
\varepsilon(x) = \kappa \cdot (x - x_n)
\]

The sum of the forces:

\[
N_c + N_s = 0
\]

Resultant force in concrete:

\[
N_c = b \int_0^h \sigma_c(\varepsilon_c) \, dx
\]
Resultant force in rebars:

\[ N_s = \sum_{i=1}^{2} A_{si} \sigma_{si}(\varepsilon_{si}) \]

We solved the equation system with independent numerical method as a “hand” calculation. The position of the neutral axis:

\[ x_n = 38.18 \text{ mm} \]

The stress and strain values which belong to the equilibrium state are shown in Fig. 9.2.1.1.6.

The resultant of the concrete stress volume according to the independent “hand” calculation:

\[ N_c = b \int_{0}^{h} \sigma_c(\varepsilon_c) \, dx = -153.4 \text{ kN} \]

The resultant force in the rebars:

\[ N_s = \sum_{i=1}^{2} A_{si} \sigma_{si}(\varepsilon_{si}) = 2 \cdot \frac{16^2 \cdot \pi}{4} \cdot \frac{1.05 \cdot 420}{1.15} = 154.2 \text{ kN} \]

The difference between the compression and tension forces is less than 1% therefore this is the correct position of the neutral axis and curvature according to the improved material models.

The centroid of the concrete stress volume measured from the top of the section:

\[ x_c = \frac{b \int_{0}^{h} x \sigma_c(\varepsilon_c) \, dx}{N_c} = 14.34 \text{ mm} \]
The moment around neutral axis provided by the concrete:

\[ M_c = N_c (x_n - x_c) = 153.4 \cdot (38.18 - 14.34) = 3.657 \text{ kNm} \]

Rebars moment around neutral axis:

\[ M_s = \sum_{i=1}^{2} A_i \sigma_i (d_i - x_n) = 2 \cdot 201.1 \cdot 383.5 (464 - 38.18) = 65.64 \text{ kNm} \]

The moment capacity with improved material models:

\[ M_{Rd,2} = M_c + M_s = 3.657 + 65.64 = 69.30 \text{ kNm} \]

Ratio between the two hand calculations with the different material models:

\[ \frac{M_{Rd,2}}{M_{Rd,1}} = \frac{69.30}{66.35} = 1.044 \]

The moment capacity of the same section with FEM-Design (Fig. 9.2.1.1.7):

\[ M_{Rd,FEM} = 68.98 \text{ kNm} \]

The difference between the two hand calculations is 4%.

The FEM-Design results of the stresses and strains are shown in Fig. 9.2.1.1.7. The strain values and neutral axis position value between the hand and FE calculations are under 3%, the moment capacity difference is less than 1%.
Download link to the example file:

http://download.strusoft.com/FEM-Design/inst180x/models/9.2.1 Moment capacity calculation for beams under pure bending.str
9.2.1.2 Normal-reinforced cross section

Here we put 6 longitudinal rebars with 16 mm diameter at the bottom of the cross section (see Fig. 9.2.1.1.8, the concrete is C30/37). We neglect the effect of the hangers. The following verification calculations will be performed with the improved material models (see Chapter 9.2.1.1).

The assumed stress and strain distributions in the section are shown in Fig. 9.2.1.1.8.

The concrete strain depends on the curvature and based on the improved material models this led to a nonlinear equation.

\[ \varepsilon(x) = \kappa \cdot (x - x_n) \]

The sum of the forces:

\[ N_c + N_s = 0 \]

Resultant force in concrete:

\[ N_c = b \int_0^h \sigma_c(\varepsilon_c) \, dx \]

Resultant force in rebars:

\[ N_s = \sum_{i=1}^{6} A_{si} \sigma_{si}(\varepsilon_{si}) \]

We solved the equation system with independent numerical method as a “hand” calculation.

The neutral axis position is:

\[ x_n = 93.06 \text{ mm} \]

The stress and strain values which belong to the equilibrium state are shown in Fig. 9.2.1.1.9.
The resultant of the concrete stress volume according to the independent “hand” calculation:

\[ N_c = b \int_0^h \sigma_c(\varepsilon_c) \, dx = -452.03 \text{kN} \]

The resultant force in the rebars:

\[ N_s = \sum_{i=1}^6 A_{si} \sigma_{si}(\varepsilon_{si}) = 6 \cdot 16^2 \cdot \frac{\pi}{4} 374.71 = 452.04 \text{kN} \]

The difference between the compression and tension forces is less than 1% therefore this is the correct position of the neutral axis and curvature according to the improved material models.

The centroid of the concrete stress volume measured from the top of the section:

\[ x_c = \frac{b \int_0^h x \sigma_c(\varepsilon_c) \, dx}{N_c} = 38.71 \text{mm} \]

The moment around neutral axis provided by the concrete:

\[ M_c = N_c(x_n - x_c) = 452.03(93.06 - 38.71) = 24.57 \text{kNm} \]

Rebars moment around neutral axis:

\[ M_s = \sum_{i=1}^6 A_{si} \sigma_{si}(\varepsilon_{si})(d_i - x_n) = 6 \cdot 201.1 \cdot 374.7(464 - 93.06) = 166.8 \text{kNm} \]

The moment capacity with improved material models:

\[ M_{Rd,2} = M_c + M_s = 24.57 + 166.8 = 191.4 \text{kNm} \]
The moment capacity with FEM-Design (Fig. 9.2.1.1.10):

\[ M_{\text{Ed,FEM}} = 191.0 \text{kNm} \]

The FEM-Design results of the stresses and strains are shown in Fig. 9.2.1.1.10. The strain values and neutral axis position value difference between the hand and FE calculation are less than 4%, the moment capacity difference is less than 1%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.2.1 Moment capacity calculation for beams under pure bending.str
9.2.1.3 Over-reinforced cross section

We put 12 longitudinal rebars with 20 mm diameter at the bottom (see Fig. 9.2.1.1.11, the concrete is C20/25). We neglect the effect of the hangers. The following verification calculations will be performed with the improved material models (see Chapter 9.2.1.1).

The assumed stress and strain distributions in the section are shown in Fig. 9.2.1.1.11.

The concrete strain depends on the curvature and based on the improved material models this led to a nonlinear equation.

\[ \varepsilon(x) = \kappa \cdot (x - x_n) \]

The sum of the forces:

\[ N_c + N_s = 0 \]

Resultant force in concrete:

\[ N_c = \int_{0}^{h} b \sigma_c(\varepsilon_c) \, dx \]

Resultant force in rebars:

\[ N_s = \sum_{i=1}^{12} A_{si} \sigma_{si}(\varepsilon_{si}) \]

We solved the equation system with independent numerical method as a “hand” calculation.

The neutral axis position is:

\[ x_n = 317.95 \text{ mm} \]

The stress and strain values which belong to the equilibrium state are shown in Fig. 9.2.1.1.12.
The resultant of the concrete stress volume according to the independent “hand” calculation:

\[ N_c = b \int_0^h \sigma_c(\varepsilon_c) \, dx = -1029.6 \text{kN} \]

The resultant force in the rebars:

\[ N_s = \sum_{i=1}^{12} A_{si} \sigma_{si}(\varepsilon_{si}) = 6 \cdot \frac{20^2 \cdot \pi}{4} \cdot 317.13 + 6 \cdot \frac{20^2 \cdot \pi}{4} \cdot 229.08 = 1029.06 \text{kN} \]

The difference between the compression and tension forces less than 1% therefore this is the correct position of the neutral axis and curvature according to the improved material models.

Centroid of the concrete stress volume measured from the top of the section:

\[ x_c = \frac{\int_0^h b x \sigma_c(\varepsilon_c) \, dx}{N_c} = 132.3 \text{ mm} \]

The moment around neutral axis provided by the concrete:

\[ M_c = N_c(x_n - x_c) = 1029.6 \cdot (317.95 - 132.3) = 191.1 \text{kNm} \]

Rebars moment around neutral axis:

\[ M_s = \sum_{i=1}^{12} A_{si} \sigma_{si}(\varepsilon_{si})(d_i - x_n) = 6 \cdot 314.1 \cdot \left[ 317.1 \cdot (462 - 317.95) + 229.1 \cdot (422 - 317.95) \right] = 131.0 \text{kNm} \]

The moment capacity with improved material models:

\[ M_{Rd,2} = M_c + M_s = 191.1 + 131.1 = 322.2 \text{kNm} \]
The moment capacity with FEM-Design (Fig. 9.2.1.1.13):

\[ M_{Ed,FEM} = 314.0 \text{kNm} \]

The FEM-Design results of the stresses and strains are shown in Fig. 9.2.1.1.13. The strain values and neutral axis position value difference between the hand and FE calculation are less than 2\%, the moment capacity difference is less than 3\%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.2.1 Moment capacity calculation for beams under pure bending.str
9.2.2 Required reinforcement calculation for a slab

In this example we calculate the required reinforcements of a slab due to elliptic and hyperbolic bending conditions. First of all the applied reinforcement is orthogonal and then the applied reinforcement is non-orthogonal. We calculate the required reinforcement with hand calculation and then compare the results with FEM-Design values.

Inputs:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The thickness</td>
<td>h = 200 mm</td>
</tr>
<tr>
<td>The elastic modulus of concrete</td>
<td>$E_{cm} = 33$ GPa, C30/37</td>
</tr>
<tr>
<td>The Poisson's ratio of concrete</td>
<td>$\nu = 0.2$</td>
</tr>
<tr>
<td>The design value of compressive strength</td>
<td>$f_{cd} = 20$ MPa</td>
</tr>
<tr>
<td>Elastic modulus of steel bars</td>
<td>$E_s = 200$ GPa</td>
</tr>
<tr>
<td>The design value of yield stress of steel bars</td>
<td>$f_{yd} = 434.8$ MPa</td>
</tr>
<tr>
<td>Diameter of the longitudinal reinforcement</td>
<td>$\phi_l = 10$ mm</td>
</tr>
<tr>
<td>Nominal concrete cover</td>
<td>$c_x = 20$ mm; $c_y = 30$ mm</td>
</tr>
<tr>
<td>Effective heights</td>
<td>$d_x = 175$ mm; $d_y = 165$ mm</td>
</tr>
</tbody>
</table>

9.2.2.1 Elliptic bending

In the first case the bending condition is an elliptic bending. In FEM-Design the model is a slab with statically determinant support system and specific moment loads at its edges for the pure internal force state (see Fig. 9.2.2.1).
Fig. 9.2.2.2 shows the constant internal forces in the slab due to the loads. Fig. 9.2.2.3 shows the principal moments and their directions based on the FEM-Design calculation. According to the pure stress state the principal moments and the directions are the same in each elements.

First of all the reinforcement is orthogonal and the hand calculation and the comparison are the following:

1. **Orthogonal reinforcement (φ=90°)**

The reinforcement is orthogonal and their directions coincide with the local system (x=ξ, y=9=η).

The moments in the slab (tensor of the applied moments):

\[
\begin{align*}
    m_x &= m_{\xi} = +16 \text{ kNm/m} \\
    m_y &= m_\eta = +8 \text{ kNm/m} \\
    m_{xy} &= m_{\xi \eta} = +6 \text{ kNm/m}
\end{align*}
\]
The first invariant of the tensor: \( m_x + m_y = +24 \text{kN/m} \)

The calculation of the principal moments and their directions:

\[
m_1 = \frac{m_x + m_y}{2} + \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^2} = \frac{16 + 8}{2} + \sqrt{\left(\frac{16 - 8}{2}\right)^2 + 6^2} = 19.21 \text{kN/m}
\]

\[
m_2 = \frac{m_x + m_y}{2} - \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^2} = \frac{16 + 8}{2} - \sqrt{\left(\frac{16 - 8}{2}\right)^2 + 6^2} = 4.79 \text{kN/m}
\]

\[
\alpha_0 = \arctan \frac{m_1 - m_x}{m_{xy}} = \arctan \frac{19.21 - 16}{6} = 28.15^\circ
\]

Compare these results with Fig. 9.2.2.3. The difference is 0%.

The design moments (according to [9][10]) if the reinforcement \((\xi, \eta)\) is orthogonal and their directions coincide with the local co-ordinate system \((x, y)\):

Case a)

\[
m_{ud, \xi} = m_\xi - m_\eta \cos \varphi \left(\frac{1 + 2 \cos \varphi}{1 + \cos \varphi}\right) = 16 - 8 \left(\frac{\cos 90^\circ}{1 + \cos 90^\circ}\right) = +22 \text{kN/m}
\]

\[
m_{ud, \eta} = m_\xi \frac{1}{1 + \cos \varphi} + m_\eta \frac{1}{\sin \varphi} = 8 \left(\frac{1}{1 + \cos 90^\circ}\right) + 6 \left(\frac{1}{\sin 90^\circ}\right) = +14 \text{kN/m}
\]

This is a valid solution! Because \( m_{ud, \xi} + m_{ud, \eta} = +36 \text{kN/m} > m_x + m_y = +24 \text{kN/m} \)

\[
m_{ud, \xi} = +22 \text{kN/m} \quad m_{ud, \eta} = +14 \text{kN/m}
\]

Case b)

\[
m_{ud, \xi} = m_\xi + m_\eta \cos \varphi \left(\frac{1 - 2 \cos \varphi}{1 + \cos \varphi}\right) = 16 + 8 \left(\frac{\cos 90^\circ}{1 + \cos 90^\circ}\right) - 6 \left(\frac{1 - 2 \cos 90^\circ}{\sin 90^\circ}\right) = +10 \text{kN/m}
\]

\[
m_{ud, \eta} = m_\xi \frac{1}{1 - \cos \varphi} - m_\eta \frac{1}{\sin \varphi} = 8 \left(\frac{1}{1 - \cos 90^\circ}\right) - 6 \left(\frac{1}{\sin 90^\circ}\right) = +2 \text{kN/m}
\]

Invalid solution! Because \( m_{ud, \xi} + m_{ud, \eta} = +12 \text{kN/m} < m_x + m_y = +24 \text{kN/m} \)

Case \(\xi\)

\[
m_{ud, \xi} = m_\xi - \frac{m_{xy}^2}{m_\xi} = 16 - \frac{6^2}{8} = +11.5 \text{kN/m}
\]

\[
m_{ud, \eta} = 0
\]
Invalid solution! Because $m_{ud\xi} + m_{ud\eta} = +11.5\text{ kNm/m} < m_x + m_y = +24\text{ kNm/m}$

Case $\eta)$

$m_{ud\xi} = 0$

$$m_{ud\eta} = \frac{m_\xi m_\eta - m_\xi^2}{m_\xi \sin^2 \varphi + m_\eta \cos^2 \varphi - m_\xi \sin 2 \varphi} = \frac{16 \cdot 8 - 6^2}{16 \cdot \sin^2 90^\circ + 8 \cdot \cos^2 90^\circ - 6 \cdot \sin (2 \cdot 90^\circ)} = +5.75\text{ kNm/m}$$

Invalid solution! Because $m_{ud\xi} + m_{ud\eta} = +5.75\text{ kNm/m} < m_x + m_y = +24\text{ kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.4 and 9.2.2.5. The difference between the hand and FE calculation is 0%.

Figure 9.2.2.4 – The $m_{ud\xi}$ design moment for elliptic bending with orthogonal reinforcement [kNm]
Calculation of the required reinforcement based on the valid design moments:

In x (ξ) direction:

Sum of the moments:

\[ m_{ud} = f_{cd} x_c \left( d - \frac{20}{2} \right) \]

\[ m_{ud} = 22000 = 20 x_c \left( 175 - \frac{x_c}{2} \right) \]

\[ x_c = 6.403 \text{ mm} \]

Sum of the forces:

\[ x_c f_{cd} = a_s \xi f_{yd} \]

\[ 6.403 \cdot 20 = a_s \cdot 434.8 \]

\[ a_s = 0.2945 \text{ mm}^2/\text{mm} = 294.5 \text{ mm}^2/\text{m} \]

Fig. 9.2.2.6 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.
In \( y (\eta) \) direction:

Sum of the moments:

\[
m_{ad \eta} = f_{cd} x_c \left( d_y - \frac{x_c}{2} \right) ; \quad 14000 = 20 x_c \left( 165 - \frac{x_c}{2} \right) ; \quad x_c = 4.298 \text{ mm}
\]

Sum of the forces:

\[
x_c f_{cd} = a_{s\eta} f_{yd} ; \quad 4.298 \cdot 20 = a_{s\eta} 434.8 ; \quad a_{s\eta} = 0.1977 \text{ mm}^2/\text{mm} = 197.7 \text{ mm}^2/\text{m}
\]

Fig. 9.2.2.7 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%. 

Secondly the reinforcement is non-orthogonal and the hand calculation and the comparison are the following:

2. Non-orthogonal reinforcement (\( \phi=75^\circ \) between \( \xi \) and \( \eta \))

The reinforcement is non-orthogonal and the \( \xi \) direction concides with the local x direction. Thus \( y=\theta \). The angle between the \( \xi \) directional reinforcement and \( \eta \) directional reinforcement is \( \phi=75^\circ \).

The moments in the slab (tensor of the applied moments):

\[
\begin{align*}
m_x &= m_\xi = +16 \text{ kNm/m} \\
m_y &= m_\eta = +8 \text{ kNm/m} \\
m_{xy} &= m_{\xi \eta} = +6 \text{ kNm/m}
\end{align*}
\]

The first invariant of the tensor: \( m_x + m_y = +24 \text{ kNm/m} \)
The design moments (according to [9][10]) if the reinforcement \((\xi, \eta)\) is non-orthogonal:

Case a)
\[
m_{ud,\xi} = \frac{m_s \cos \varphi}{1 + \cos \varphi} + \frac{1 - 2 \cos \varphi}{1 + \cos \varphi} = 16 - 8 \frac{\cos 75^\circ}{1 + \cos 75^\circ} + 6 \frac{1 - 2 \cos 75^\circ}{\sin 75^\circ} = +17.35 \text{ kNm/m}
\]
\[
m_{ud,\eta} = \frac{1}{1 + \cos \varphi} + \frac{m_s \sin \varphi}{\sin 75^\circ} = 8 \frac{1}{1 + \cos 75^\circ} + 6 \frac{1}{\sin 75^\circ} = +12.57 \text{ kNm/m}
\]

This is a valid solution! Because \(m_{ud,\xi} + m_{ud,\eta} = +29.92 \text{ kNm/m} > m_x + m_y = +24 \text{ kNm/m}\)
\[
m_{ud,\xi} = +17.35 \text{ kNm/m} \quad m_{ud,\eta} = +12.57 \text{ kNm/m}
\]

Case b)
\[
m_{ud,\xi} = \frac{m_s \cos \varphi}{1 - \cos \varphi} - \frac{1 + 2 \cos \varphi}{1 - \cos \varphi} = 16 + 8 \frac{\cos 75^\circ}{1 - \cos 75^\circ} - 6 \frac{1 + 2 \cos 75^\circ}{\sin 75^\circ} = +9.37 \text{ kNm/m}
\]
\[
m_{ud,\eta} = \frac{1}{1 - \cos \varphi} - \frac{m_s \sin \varphi}{\sin 75^\circ} = 8 \frac{1}{1 - \cos 75^\circ} - 6 \frac{1}{\sin 75^\circ} = +4.58 \text{ kNm/m}
\]

Invalid solution! Because \(m_{ud,\xi} + m_{ud,\eta} = +13.95 \text{ kNm/m} < m_x + m_y = +24 \text{ kNm/m}\)

Case \(\xi\)
\[
m_{ad,\xi} = m_s - m_s \frac{m_s^2}{m_s^2} = 16 - \frac{6^2}{8} = +11.5 \text{ kNm/m}
\]
\[
m_{ad,\eta} = 0
\]

Invalid solution! Because \(m_{ad,\xi} + m_{ad,\eta} = +11.5 \text{ kNm/m} < m_x + m_y = +24 \text{ kNm/m}\)

Case \(\eta\)
\[
m_{ad,\xi} = 0
\]
\[
m_{ad,\eta} = \frac{m_s m_s - m_s^2}{m_s \sin^2 \varphi + m_s \cos^2 \varphi - m_s \sin 2 \varphi} = \frac{16 \cdot 8 - 6^2}{16 \cdot \sin^2 75^\circ + 8 \cdot \cos^2 75^\circ - 6 \cdot \sin (2 \cdot 75^\circ)} = +7.38 \text{ kNm/m}
\]

Invalid solution! Because \(m_{ad,\xi} + m_{ad,\eta} = +7.38 \text{ kNm/m} < m_x + m_y = +24 \text{ kNm/m}\)

The results of the design moments based on FEM-Design are in Fig. 9.2.2.8 and 9.2.2.9. The difference between the hand and FE calculation is 0%. 
Calculation of the required reinforcement based on the valid design moments:

In x (\(\xi\)) direction:

Sum of the moments:

\[
m_{ud\xi} = f_{cd} x_c \left( d_x - \frac{x_c}{2} \right) ; \quad 17350 = 20 x_c \left( 175 - \frac{x_c}{2} \right) ; \quad x_c = 5.029 \text{ mm}
\]

Sum of the forces:

\[
x_c f_{cd} = a_{x\xi} f_{yd} ; \quad 5.029 \cdot 20 = a_{x\xi} \cdot 434.8 \quad ; \quad a_{x\xi} = 0.2313 \text{ mm}^2/\text{mm} = 231.3 \text{ mm}^2/\text{m}
\]

Fig. 9.2.2.10 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.
In \( \eta \) direction:

Sum of the moments:

\[
m_{\eta \eta} = f_{cd} x_c d_y \left( d_y - \frac{x_c}{2} \right) ; \quad 12570 = 20 x_c \left( 165 - \frac{x_c}{2} \right) ; \quad x_c = 3.854 \text{ mm}
\]

Sum of the forces:

\[
x_c f_{cd} = a_{sn} f_{yd} ; \quad 3.854 \cdot 20 = a_{sn} \cdot 434.8 ; \quad a_{sn} = 0.1773 \text{ mm}^2/\text{mm} = 177.3 \text{ mm}^2/\text{m}
\]

Fig. 9.2.2.11 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.
Verification Examples

Download links to the example files:

Elliptic bending, orthogonal reinforcement:
http://download.strusoft.com/FEM-Design/inst170x/models/9.2.2.1 Required reinforcement calculation in a slab with elliptic bending and orthogonal reinforcement.str

Elliptic bending, non-orthogonal reinforcement:
http://download.strusoft.com/FEM-Design/inst170x/models/9.2.2.1 Required reinforcement calculation in a slab with elliptic bending and skew reinforcement.str
**9.2.2.2 Hyperbolic bending**

In the second case the bending condition is a hyperbolic bending. In FEM-Design the model is a slab with statically determinant support system and specific moment loads at its edges for the pure internal force state (see Fig. 9.2.2.12).

![Figure 9.2.2.12 – The slab with the edge loads for pure stress state](image)

Fig. 9.2.2.13 shows the constant internal forces in the slab due to the loads. Fig. 9.2.2.14 shows the principal moments and their directions based on the FEM-Design calculation. According to the pure stress state the principal moments and the directions are the same in each elements.

![Figure 9.2.2.13 – The $m_x$, $m_y$, and $m_{xy}$ internal forces in the slab [kNm]](image)
Firstly the reinforcement is orthogonal and the hand calculation and the comparison are the following:

1. **Orthogonal reinforcement**

The reinforcement is orthogonal and their directions coincide with the local system (x=ξ, y=ϑ=η).

The moments in the slab (tensor of the applied moments):

\[ m_x = m_\xi = +16 \text{ kNm/m} \]
\[ m_y = m_\eta = -8 \text{ kNm/m} \]
\[ m_{xy} = m_{\xi\eta} = +6 \text{ kNm/m} \]

The first invariant of the tensor: \( m_x + m_y = +8 \text{ kNm/m} \)

The calculation of the principal moments and their directions:

\[
m_1 = \frac{m_x + m_y}{2} + \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^2} = \frac{16 + (-8)}{2} + \sqrt{\left(\frac{16 - (-8)}{2}\right)^2 + 6^2} = 17.42 \text{ kNm/m}
\]

\[
m_2 = \frac{m_x + m_y}{2} - \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^2} = \frac{16 + (-8)}{2} - \sqrt{\left(\frac{16 - (-8)}{2}\right)^2 + 6^2} = -9.42 \text{ kNm/m}
\]

\[ \alpha_\theta = \arctan \frac{m_x - m_y}{m_{xy}} = \arctan \frac{17.42 - 16}{6} = 13.32^\circ \]

Compare these results with Fig. 9.2.2.14. The difference is 0%.
The design moments (according to [9][10]) if the reinforcement $(\xi, \eta)$ is orthogonal:

Case a)

$$m_{ud\bar{z}} = m_{\bar{z}} - m_\alpha \frac{\cos \varphi}{1 + \cos \varphi} + m_\bar{z} \frac{1 - 2 \cos \varphi}{\sin \varphi} = 16 + 8 \frac{\cos 90^\circ}{1 + \cos 90^\circ} + 6 \frac{1 - 2 \cos 90^\circ}{\sin 90^\circ} = +22 \text{ kNm/m}$$

$$m_{ud\eta} = m_\alpha \frac{1}{1 + \cos \varphi} + m_\bar{z} \frac{1}{\sin \varphi} = -8 \frac{1}{1 + \cos 90^\circ} + 6 \frac{1}{\sin 90^\circ} = -2 \text{ kNm/m}$$

Invalid solution! Because their have different signs.

Case b)

$$m_{ud\bar{z}} = m_{\bar{z}} + m_\alpha \frac{\cos \varphi}{1 - \cos \varphi} - m_\bar{z} \frac{1 + 2 \cos \varphi}{\sin \varphi} = 16 - 8 \frac{\cos 90^\circ}{1 - \cos 90^\circ} - 6 \frac{1 + 2 \cos 90^\circ}{\sin 90^\circ} = +10 \text{ kNm/m}$$

$$m_{ud\eta} = m_\alpha \frac{1}{1 - \cos \varphi} - m_\bar{z} \frac{1}{\sin \varphi} = -8 \frac{1}{1 - \cos 90^\circ} - 6 \frac{1}{\sin 90^\circ} = -14 \text{ kNm/m}$$

Invalid solution! Because their have different signs.

Case $\xi$)

$$m_{ud\bar{z}} = m_{\bar{z}} - \frac{m_\bar{z}^2}{m_\alpha} = 16 - \frac{6^2}{-8} = +20.5 \text{ kNm/m}$$

$$m_{ud\eta} = 0$$

This is a valid solution at the bottom!

$$m_{ud\bar{z}} = +20.5 \text{ kNm/m} \quad m_{ud\eta} = 0 \text{ kNm/m}$$

Case $\eta$)

$$m_{ud\bar{z}} = 0$$

$$m_{ud\eta} = \frac{m_\bar{z} m_\alpha - m_\bar{z}^2}{m_\bar{z} \sin^2 \varphi + m_\alpha \cos^2 \varphi - m_\bar{z} \sin 2 \varphi} = \frac{16 \cdot (-8) - 6^2}{16 \cdot \sin^2 90^\circ + (-8) \cdot \cos^2 90^\circ - 6 \cdot \sin (2 \cdot 90^\circ)} = -10.25 \text{ kNm/m}$$

This is a valid solution at the top!

$$m_{ud\bar{z}} = 0 \text{ kNm/m} \quad m_{ud\eta} = -10.25 \text{ kNm/m}$$
The results of the design moments based on FEM-Design are in Fig. 9.2.2.15 and 9.2.2.16. The difference between the hand and FE calculation is 0%.

Calculation of the required reinforcement based on the valid design moments:

In $x$ ($\xi$) direction at the bottom:

Sum of the moments:

$$m_{ud,\xi} = f_{cd} x_c \left( d_x - \frac{x}{2} \right) ; \quad 20500 = 20 x_c \left( 175 - \frac{x}{2} \right) ; \quad x_c = 5.959 \text{ mm}$$
Verification Examples

Sum of the forces:

\[ x_c f_{cd} = a_{s\bar{y}} f_{yd} \quad ; \quad 5.959 \cdot 20 = a_{s\bar{y}} 434.8 \quad ; \quad a_{s\bar{y}} = 0.2741 \text{ mm}^2/\text{mm} = 274.1 \text{ mm}^2/\text{m} \]

Fig. 9.2.2.17 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

In \( y (\eta) \) direction at the top:

Sum of the moments:

\[ m_{yd\eta} = f_{cd} x_c \left( d_y - \frac{x_c}{2} \right) \; ; \quad 10250 = 20 x_c \left( 165 - \frac{x_c}{2} \right) \; ; \quad x_c = 3.136 \text{ mm} \]

Sum of the forces:

\[ x_c f_{cd} = a_{\eta y} f_{yd} \; ; \quad 3.136 \cdot 20 = a_{\eta y} 434.8 \quad ; \quad a_{\eta y} = 0.1443 \text{ mm}^2/\text{mm} = 144.3 \text{ mm}^2/\text{m} \]

Fig. 9.2.2.18 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%. 
Secondly the reinforcement is non-orthogonal and the hand calculation and the comparison are the following:

2. **Non-orthogonal reinforcement** \((\varphi = 75^\circ \text{ between } \xi \text{ and } \eta)\)

The reinforcement is non-orthogonal and the \(\xi\) direction concides with the local x direction. Thus \(y = \vartheta\).

The moments in the slab (tensor of the applied moments):

\[
\begin{align*}
  m_x &= m_\bar{\xi} = +16 \text{ kNm/m} \\
  m_y &= m_\vartheta = -8 \text{ kNm/m} \\
  m_{xy} &= m_{\bar{\xi}\vartheta} = +6 \text{ kNm/m}
\end{align*}
\]

The first invariant of the tensor: \(m_x + m_y = +8 \text{ kNm/m}\)

The design moments (according to [9][10]) if the reinforcement \((\xi, \eta)\) is non-orthogonal:

Case a)

\[
\begin{align*}
  m_{\text{ud} \xi} &= m_\bar{\xi} - m_\vartheta \frac{\cos \varphi}{1 + \cos \varphi} + m_{\bar{\xi}\vartheta} \frac{1 - 2 \cos \varphi}{\sin \varphi} = 16 + 8 \frac{\cos 75^\circ}{1 + \cos 75^\circ} + 6 \frac{1 - 2 \cos 75^\circ}{\sin 75^\circ} = +20.64 \text{ kNm/m} \\
  m_{\text{ud} \eta} &= m_\vartheta \frac{1}{1 + \cos \varphi} + m_{\bar{\xi}\vartheta} \frac{1}{\sin \varphi} = -8 \frac{1}{1 + \cos 75^\circ} + 6 \frac{1}{\sin 75^\circ} = -0.144 \text{ kNm/m}
\end{align*}
\]

Invalid solution! Because their have different signs.
Case b)

\[
\begin{align*}
  m_{ud, \dot{\varphi}} &= m_{\varphi} + m_{\varphi} \frac{\cos \varphi}{1 - \cos \varphi} - m_{\varphi} \frac{1 + 2 \cos \varphi}{\sin \varphi} = 16 - 8 \frac{\cos 75^\circ}{1 - \cos 75^\circ} - 6 \frac{1 + 2 \cos 75^\circ}{\sin 75^\circ} = +3.78 \text{kNm/m} \\
  m_{ud, \eta} &= m_{\theta} \frac{1}{1 - \cos \varphi} - m_{\varphi} \frac{1}{\sin \varphi} = -8 \frac{1}{1 - \cos 75^\circ} - 6 \frac{1}{\sin 75^\circ} = -17.01 \text{kNm/m}
\end{align*}
\]

Invalid solution! Because they have different signs.

Case \(\xi\))

\[
\begin{align*}
  m_{ud, \dot{\xi}} &= m_{\xi} - m_{\varphi} \frac{\dot{m}_{\varphi}}{m_{\varphi}} = 16 - \frac{6^2}{-8} = +20.5 \text{kNm/m} \\
  m_{ud, \eta} &= 0
\end{align*}
\]

This is a valid solution at the bottom!

\[
\begin{align*}
  m_{ud, \dot{\xi}} &= +20.5 \text{kNm/m} & m_{ud, \eta} &= 0 \text{kNm/m}
\end{align*}
\]

Case \(\eta\))

\[
\begin{align*}
  m_{ud, \dot{\eta}} &= 0 \\
  m_{ud, \dot{\xi}} &= m_{\xi} m_{\varphi} - m_{\varphi} \frac{\dot{m}_{\varphi}}{m_{\varphi}} = \frac{16 \cdot (-8) - 6^2}{16 \cdot \sin^2 75^\circ + (-8) \cdot \cos^2 75^\circ - 6 \cdot \sin (2 \cdot 75^\circ)} = -14.4 \text{kNm/m}
\end{align*}
\]

This is a valid solution at the top!

\[
\begin{align*}
  m_{ud, \dot{\xi}} &= 0 \text{kNm/m} & m_{ud, \eta} &= -14.4 \text{kNm/m}
\end{align*}
\]

The results of the design moments based on FEM-Design are in Fig. 9.2.2.19 and 9.2.2.20. The difference between the hand and FE calculation is 0%.
Calculation of the required reinforcement based on the valid design moments:

In \( x (\xi) \) direction at the bottom:

Sum of the moments:

\[
m_{ud\xi} = f_{cd} x_c \left(d_x \frac{x_c}{2} \right) ;
20500 = 20 x_c \left(175 - \frac{x_c}{2}\right) ;
x_c = 5.959 \text{ mm}
\]

Sum of the forces:

\[
x_c f_{yd} = a_{s\xi} f_{yd} ;
5.959 \cdot 20 = a_{s\xi} 434.8 ;
a_{s\xi} = 0.2741 \text{ mm}^2/\text{mm} = 274.1 \text{ mm}^2/\text{m}
\]

Fig. 9.2.2.21 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.
In η direction at the top:

Sum of the moments:

\[ m_{\eta d} = f_{cd} x_c \left( d_y - \frac{x_c}{2} \right) \quad ; \quad 14400 = 20 x_c \left( 165 - \frac{x_c}{2} \right) \quad ; \quad x_c = 4.423 \text{ mm} \]

Sum of the forces:

\[ x_c f_{cd} = a_{\eta y} f_{yd} \quad ; \quad 4.423 \cdot 20 = a_{\eta y} 434.8 \quad ; \quad a_{\eta y} = 0.2034 \text{ mm}^2/\text{mm} = 203.4 \text{ mm}^2/\text{m} \]

Fig. 9.2.2.22 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.
Download links to the example files

Hyperbolic bending, orthogonal reinforcement:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.2.2 Required reinforcement calculation in a slab with hyperbolic bending and orthogonal reinforcement.str

Hyperbolic bending, non-orthogonal reinforcement:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.2.2 Required reinforcement calculation in a slab with hyperbolic bending and skew reinforcement.str
9.2.3 Shear capacity calculation

In this chapter we will show detailed calculations of beams and slabs regarding to shear force.

9.2.3.1 Shear capacity of a beam

In this example we check the shear capacity of a cantilever beam (see Fig. 9.2.3.1.1). The input parameters and details are in the following table.

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete characteristic compressive strength ( f_{ck} )</td>
<td>25 N/mm²</td>
</tr>
<tr>
<td>Beam height ( h )</td>
<td>350 mm</td>
</tr>
<tr>
<td>Beam width ( b_w )</td>
<td>250 mm</td>
</tr>
<tr>
<td>Partial factor of concrete ( \gamma_c )</td>
<td>1.50</td>
</tr>
<tr>
<td>Reinforcing steel characteristic yield strength ( f_{yk} )</td>
<td>500 N/mm²</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel ( \gamma_s )</td>
<td>1.15</td>
</tr>
<tr>
<td>Stirrup distance ( s )</td>
<td>200 mm</td>
</tr>
<tr>
<td>Longitudinal rebar diameter ( \phi )</td>
<td>20 mm</td>
</tr>
<tr>
<td>Stirrup diameter ( \phi_s )</td>
<td>10 mm</td>
</tr>
<tr>
<td>Concrete cover ( c )</td>
<td>20 mm</td>
</tr>
</tbody>
</table>

First of all we need to check that we need any designed shear reinforcement or not:

\[
V_{Rd,c} = \max \left\{ 0.18 \frac{f_{ck}}{\gamma_c} \left( \frac{k}{\gamma_{min}} \right)^\frac{1}{3} b_w d \right\} = \max \left\{ 0.18 \cdot 1.803 \cdot \left( \frac{100 \cdot 0.016 \cdot 25}{0.424} \right)^\frac{1}{3} \right\} 250 \cdot 310 = 57.3 \text{kN}
\]

Figure 9.2.3.1.1 – The cantilever, the cross section and the design value of the shear force
where:
\[ d = h - c - \phi_s - \frac{\phi}{2} = 350 - 20 - 10 - \frac{20}{2} = 310 \text{ mm} \]

\[ k = \min \left( 1 + \sqrt{\frac{200}{d^2}} \right) = \min \left( 1 + \sqrt{\frac{200}{310^2}} \right) = 1.803 ; \]

\[ \rho_l = \min \left( \frac{A_{sl}}{b_n d} \right) = \min \left( \frac{4 \cdot 20^2 \cdot \pi}{4 \cdot 250 \cdot 310 \cdot 0.02} \right) = 0.016 ; \]

\[ \nu_{min} = 0.035 k^\frac{3}{2} f_{ck}^\frac{1}{2} = 0.035 \cdot 1.803^\frac{3}{2} \cdot 25^2 = 0.424 . \]

It is necessary to use designed shear reinforcement because:
\[ V_{Rd,c} = 57.3 \text{ kN} < V_{Ed} = 120 \text{ kN} \]

Before the calculation of the designed shear reinforcement we need to check that the dimensions of the cross section is enough to bear the designed value of the shear force or not.

Thus the upper limit of the shear force bearing capacity:
\[ V_{Rd,max} = \alpha_{cw} b_w z \nu f_{cd} \frac{\cot (\theta) + \cot (\alpha)}{1 + \cot^2 (\theta)} = 1 \cdot 250 \cdot 279 \cdot 0.54 \cdot 16.67 \cdot \frac{1.3 + 0}{1 + 1.3^2} = 303.4 \text{ kN} \]

where:
\[ \alpha_{cw} = 1.0 \quad \text{the normal force is zero in the cantilever;} \]
\[ z = 0.9 d = 0.9 \cdot 310 = 279 \text{ mm} \]
\[ \nu = 0.6 \left( 1 - \frac{f_{ck}}{250} \right) = 0.6 \left( 1 - \frac{25}{250} \right) = 0.540 \]
\[ \alpha = 90^\circ \]

\[ \cot (\theta) = 1.3 \quad \text{this value is adjustable in the program.} \]

The compressed concrete strut can bear the acting shear force thus we can calculate the designed shear reinforcement.
\[ V_{Ed} = 120 \text{ kN} < V_{Rd,max} = 303.4 \text{ kN} \]
The next step is to calculate the shear capacity of the beam according to the defined shear reinforcement (see Fig. 9.2.3.1.1).

\[ V_{Rd,s} = \sum_{S} A_{sw} f_{yd} (\cot(\alpha) + \cot(\theta)) \sin \alpha = \frac{279 \cdot 2 \cdot 10^3 \cdot \pi}{200 \cdot 4} 435 (0 + 1.3) 1 = 123.85 \text{kN} \]

\[ \frac{V_{Ed}}{V_{Rd,s}} = \frac{120 \text{kN}}{123.8 \text{kN}} = 97\% \]

The numerical results based on FEM-Design are shown in Fig. 9.2.3.1.2. The difference between the two calculations is 0%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.2.3.1 Shear capacity of a beam.str
9.2.3.2 Shear capacity of a slab

In this chapter we will calculate the shear capacity of a slab (see Fig. 9.2.3.2.1). In FEM-Design based on the internal forces we calculate the required shear capacity and based on the parameters of the slab and the applied longitudinal reinforcement in it the actual shear capacity is also computable.

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete characteristic compressive strength</td>
<td>$f_{ck} = 20$ N/mm$^2$</td>
</tr>
<tr>
<td>Partial factor of concrete</td>
<td>$\gamma_c = 1.50$</td>
</tr>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{yk} = 500$ N/mm$^2$</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
<tr>
<td>Longitudinal bar diameter in x direction</td>
<td>$\phi_{lx} = 14$ mm</td>
</tr>
<tr>
<td>Distance in x direction between the bars</td>
<td>$s_x = 150$ mm</td>
</tr>
<tr>
<td>Longitudinal bar diameter in y direction</td>
<td>$\phi_{ly} = 10$ mm</td>
</tr>
<tr>
<td>Distance in y direction between the bars</td>
<td>$s_y = 150$ mm</td>
</tr>
<tr>
<td>Concrete cover (on longitudinal bars in x direction)</td>
<td>$c = 20$ mm</td>
</tr>
<tr>
<td>Slab thickness</td>
<td>$t = 200$ mm</td>
</tr>
<tr>
<td>The effective depth of the x reinforcements</td>
<td>$d_x = 173$ mm</td>
</tr>
<tr>
<td>The effective depth of the y reinforcements</td>
<td>$d_y = 161$ mm</td>
</tr>
<tr>
<td>Normal force in x direction</td>
<td>$n_x = 20$ kN/m</td>
</tr>
<tr>
<td>Normal force in y direction</td>
<td>$n_y = 20$ kN/m</td>
</tr>
<tr>
<td>Vertical surface total load on the slab</td>
<td>$q_{z,Ed} = 10$ kN/m$^2$</td>
</tr>
</tbody>
</table>

Fig. 9.2.3.2.1 shows the slab. Two opposite edges of the slab are simply supported and the other two are free. The longitudinal reinforcements are also indicated in the figure.

![Figure 9.2.3.2.1 – The slab, the reinforcements and the design values of external forces](image-url)
First of all we need to calculate the required shear capacity of the slab which depends on the internal shear forces in the two perpendicular directions of the slab local coordinate system which were also indicated in Fig. 9.2.3.2.1.

The internal forces of the slab at a corner point (based on finite element analysis with 0.5 m average element size (see Fig. 9.2.3.2-3):

Shear forces:
\[ v_{xz} = 69.25 \text{ kN/m} \quad ; \quad v_{yz} = 17.31 \text{ kN/m} \]

Normal forces:
\[ n_x = -20 \text{ kN/m} \quad ; \quad n_y = -10 \text{ kN/m} \quad ; \quad n_{xy} = 0 \text{ kN/m} \]

The maximum shear force in the slab according to the two shear forces in the two directions:
\[ v_{max} = \sqrt{v_{xz}^2 + v_{yz}^2} = \sqrt{(69.25)^2 + (17.31)^2} = 71.38 \text{ kN/m} \]

The direction of the maximum shear force (right-handed coordinate system):
\[ \alpha = \arctan \left( \frac{v_{yz}}{v_{xz}} \right) = \arctan \left( \frac{17.31}{69.25} \right) = +14.03^\circ \]

The maximum shear force value could be different in every nodes and therefore the angle of it also. It means that the shear force capacity must be computed in every nodes in different directions (see the relevant formulas and equations below). Here is the calculation method for the mentioned corner point:

The shear capacity with the applied parameters in the calculated main direction:
\[ v_{Rd,c} = \max \left\{ 0.18 \gamma_c k \left( 100 \rho_x f_{ck} \frac{1}{\sigma_{cp}} \right)^{\frac{1}{3}} + k_1 \sigma_{cp} d_{eff} \right\} = \max \left\{ 0.18 \cdot 1.5 \cdot 2 \cdot \left( 100 \cdot 0.005967 \cdot 20 \right)^{\frac{1}{3}} + 0.15 \cdot 0.09705 \cdot 0.4427 + 0.15 \cdot 0.09705 \cdot 167 \right\} = 94.02 \text{ kN/m} \]

The longitudinal reinforcement in the main directions:
\[ a_{sx} = \frac{\phi_x \pi}{4} \frac{1000}{s_x} = \frac{14^2 \pi}{4} \frac{1000}{150} = 1026 \text{ mm}^2 / \text{m} \]
\[ a_{sy} = \frac{\phi_y \pi}{4} \frac{1000}{s_y} = \frac{10^2 \pi}{4} \frac{1000}{150} = 523.6 \text{ mm}^2 / \text{m} \]
The effective reinforcement area in the former calculated main direction:
\[ a_\alpha = a_\alpha \cos^2 (\alpha - 0^\circ) + a_{\bar{\alpha}} \cos^2 (\alpha - 90^\circ) = 1026 \cos^2 (14.03^\circ) + 523.6 \cos^2 (14.03^\circ - 90^\circ) = 996.5 \, \text{mm}^2 \]

Effective height:
\[ d_{\text{eff}} = \frac{d_x + d_y}{2} = \frac{173 + 161}{2} = 167 \, \text{mm} \]

Reinforcement ratio in the main direction:
\[ \rho_\alpha = \min \left\{ \frac{a_\alpha}{d_{\text{eff}}} = \frac{996.5}{167 \cdot 1000}, \frac{a_{\bar{\alpha}}}{d_{\text{eff}}} = \frac{0.02}{167} \right\} = 0.005967 \]

The effective normal force in the direction of the maximum shear force:
\[ n_\alpha = n_x \cos^2 \alpha + n_y \sin^2 \alpha + 2 n_{\bar{\alpha}} \cos \alpha \sin \alpha = \]
\[ = (-20) \cos^2 14.03^\circ + (-10) \sin^2 14.03^\circ + 2 \cdot 0 \cos 14.03^\circ \sin 14.03^\circ = -19.41 \, \text{kN} \, \text{m} \] (compression)

Normal stress in the maximum shear force direction (the transformed normal force is compression):
\[ \sigma_\alpha = \frac{n_\alpha}{t} = \frac{-19.41}{200} = 0.09705 \, \text{N/mm}^2 \]
\[ \sigma_{c_{\text{cp}}} = \min \left\{ \frac{\sigma_\alpha}{0.2 f_{cd}} = \frac{0.09705}{2.66} \right\} = 0.09705 \, \text{N/mm}^2 \]

Modifying factors:
\[ k = \min \left\{ 1 + \sqrt[2]{\frac{200}{d_{\text{eff}} \cdot 2.0}}, 1 + \sqrt[2]{\frac{200}{167 \cdot 2.0}} \right\} = 2 \]
\[ \nu_{\text{min}} = 0.035 k^{\frac{3}{2}} f_{ck}^{\frac{1}{2}} = 0.035 \cdot 0.15^{\frac{3}{2}} \cdot 20^{\frac{1}{2}} = 0.4427 \]
The results of the hand calculation are equal to the FEM-Design results (see Fig. 9.2.3.2.2-4).

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.2.3.2 Shear capacity of a slab.str
9.2.4 Crack width calculation of a beam

In this example we calculate the crack width of a simple supported normal-reinforced concrete beam under the given external load.

Inputs:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete characteristic compressive strength</td>
<td>$f_{ck} = 20$ N/mm$^2$</td>
</tr>
<tr>
<td>Concrete effective tension strength</td>
<td>$f_{ct,eff} = 2.2$ N/mm$^2$</td>
</tr>
<tr>
<td>Concrete's Young modulus</td>
<td>$E_{cm} = 30$ GPa</td>
</tr>
<tr>
<td>Beam height</td>
<td>$h = 400$ mm</td>
</tr>
<tr>
<td>Beam width</td>
<td>$b = 200$ mm</td>
</tr>
<tr>
<td>Partial factor of concrete</td>
<td>$\gamma_c = 1.50$</td>
</tr>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{yk} = 500$ N/mm$^2$</td>
</tr>
<tr>
<td>Young's modulus of rebars</td>
<td>$E_s = 200$ GPa</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
<tr>
<td>Longitudinal rebar diameter</td>
<td>$\phi = 20$ mm</td>
</tr>
<tr>
<td>Stirrup diameter</td>
<td>$\phi_s = 10$ mm</td>
</tr>
<tr>
<td>Longitudinal bars effective height</td>
<td>$d = 360$ mm</td>
</tr>
<tr>
<td>Cover (on stirrups)</td>
<td>$c = 20$ mm</td>
</tr>
<tr>
<td>Partial factor of rebars</td>
<td>$\gamma_s = 1.15$</td>
</tr>
</tbody>
</table>

First we calculate the crack width with \textit{linear-elastic} non-tension concrete material model. The crack width is calculated with the aim of the distance of the cracks and the difference between concrete and rebar strains.

\[ w_k = s_{r,max} \Delta e = 149.48 \cdot 0.001428 = 0.2135 \text{ mm} \]
In order to calculate strains we need to calculate the cracked reinforced cross-sectional data (Stadium II.: concrete and steel are linear-elastic and the cross-section is cracked).

The position of the neutral axis according to the cracked section (Stadium II.):

\[
\alpha_e = \frac{E_s}{E_{cm}} = \frac{200}{30} = 6.667 \quad ; \quad bx_{II} = \frac{x_{II}}{2} = \alpha_e A_s (d - x_{II}) \quad \text{thus:} \quad x_{II} = 136.75 \text{ mm}
\]

The moment of inertia:

\[
I_{II} = \frac{bx_{II}^3}{3} + A_s \alpha_e (d - x_{II})^2 = \frac{200 \cdot 136.75^3}{3} + \frac{4 \cdot 20^2 \cdot \pi}{4} \cdot 6.667 \cdot (360 - 136.75)^2
\]

\[
I_{II} = 5.878 \cdot 10^8 \text{ mm}^4
\]

The concrete and rebar strain difference:

\[
\Delta \varepsilon = \varepsilon_{\text{sm}} - \varepsilon_{\text{cm}} = \max \left( \frac{\sigma_s - k_t f_{p, \text{eff}} (1 + \alpha_e \rho_{p, \text{eff}})}{E_s} \right) = \max \left( \frac{303.8 - 0.4 \cdot 0.0716 \cdot (1 + 6.667 \cdot 0.0716)}{200000} \right)
\]

\[
\Delta \varepsilon = 0.001428
\]

Rebar stress:

\[
\sigma_s = \frac{M_{qp} (d - x_{II})}{I_{II} \alpha_e} = \frac{120 \cdot 10^6 \cdot (360 - 136.75)}{5.878 \cdot 10^8} \cdot 6.667 \quad ; \quad \sigma_s = 303.8 \frac{\text{N}}{\text{mm}^2}
\]

\(k_t\) depends the durability of the load, the given load is long-term loading:

\(k_t = 0.4\)

Effective tensile rebar ratio:

\[
\rho_{p, \text{eff}} = \frac{A_s}{A_{c, \text{eff}}} = \frac{4 \phi^2 \pi}{4} = \frac{4 \cdot 0.2^2 \pi}{4} \cdot \frac{2.5(h - d)}{h - x_{II}} \cdot \frac{3}{2} = 0.0716
\]

\(b = \min \left( \frac{2.5\cdot(400 - 360)}{400 - 136.75} \right) = 200 \cdot \min \left( \frac{3}{400} \right) = 0.0716\)
First we need to check if the longitudinal bars are close to each other or not:

\[ t_{\text{limit}} = 5 \left( (c + \phi_s) + \frac{\phi}{2} \right) = 5 \left( (20 + 10) + \frac{20}{2} \right) = 200 \text{ mm} \quad ; \quad t_{\text{actual}} = 40 \text{ mm} \]

\[ t_{\text{limit}} > t_{\text{actual}} \] therefore the rebars are close to each other, so the distance between cracks calculated as follows:

\[ s_{r,\text{max}} = 3.4 (c + \phi_s) + 0.425 k_1 k_2 \frac{\phi}{P_{\text{eff}}} = 3.4 \cdot (20 + 10) + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{20}{0.0716} = 149.48 \text{ mm} \]

\( k_1 \) depends the cohesion between the rebars and concrete, the reinforcements are ribbed:

\( k_1 = 0.8 \)

\( k_2 \) depends the strains in the cross-section, in this case we have pure bending:

\( k_2 = 0.5 \)

Numerical results:

\[ w_{k, \text{FEM}} = 0.22 \text{ mm} \]

The difference between the numerical and hand calculation is less than 3%.
Now we calculate the crack width with non-linear non-tension concrete material model:

The following figure shows (Fig. 9.2.4.3) the concrete material model (dashed curve) but FEM-Design modifies it a bit due to numerical stability (continuous curve).

First we need to calculate the neutral axis position and the beam curvature where the maximum moment occurs (in this case the middle cross-section). The $x$ is measured from the top of the cross-section and its positive direction meant to the bottom of the cross-section.

The equilibrium equation respect to the forces:

$$N_c - N_s = 0$$

The force in the rebars according to that the rebars are in the linear elastic region:

$$N_s = A_s E_s \varepsilon_s = A_s E_s \kappa (d - x_n)$$

The force in the concrete according to the nonlinear compression material model:

$$N_c = b \int_0^h \sigma_c(\varepsilon) \, dx = b \int_0^h f_{ck} \left(1 - \left(1 - \frac{\varepsilon}{\varepsilon_{c2}}\right)^2\right) \, dx = b \int_0^h \frac{f_{ck}}{\varepsilon_{c2}} \left(1 - \left(1 - \frac{\kappa \left(x_n - x\right)}{\varepsilon_{c2}}\right)^2\right) \, dx$$

With numerical calculation the neutral axis and the curvature are:

$$x_n = 180.62 \text{ mm} \quad ; \quad \kappa = 9.05797 \times 10^{-6} \frac{1}{\text{mm}}$$

The compression and the tension forces:

$$N_c = b \int_0^h \frac{f_{ck}}{\varepsilon_{c2}} \left(1 - \left(1 - \frac{\kappa \left(x_n - x\right)}{\varepsilon_{c2}}\right)^2\right) \, dx = 200 \int_0^{400} \left(1 - \frac{9.05797 \times 10^{-6} (180.62 - x)}{0.002}\right)^2 \, dx$$

$$N_c = 408.4 \text{ kN}$$

$$N_s = 1257 \times 200000 \times 9.05797 \times 10^{-6} \times (360 - 180.62) = 408.4 \text{ kN}$$
The crack width with non-linear concrete material model:

\[ w_k = s_{r, \text{max}} \Delta \varepsilon = 141.57 \cdot 0.001544 = 0.2186 \text{ mm} \]

The concrete and rebar strain difference:

\[
\Delta \varepsilon = \varepsilon_{\text{sm}} - \varepsilon_{\text{cm}} = \max \left\{ \frac{\sigma_s - k_t \frac{f_{\text{c,t}}}{P_{\text{p,eff}}} (1 + \alpha_e \rho_{\text{p,eff}})}{E_s} \right\} = \max \left\{ \frac{324.96 - 0.4 \cdot \frac{2.2}{0.0859} (1 + 6.667 \cdot 0.0859)}{200000} \right\} = 0.001544
\]

Rebar stress:

\[
\sigma_s = E_s \varepsilon_s = E_s k (d - x_n) = 200000 \cdot 9.05797 \cdot 10^{-6} (360 - 180.62) = 324.96 \text{ N/mm}^2
\]

\( k_t \) depends on the durability of the load, this load is long-term loading:

\( k_t = 0.4 \)

Effective tensile rebar ratio:

\[
\rho_{\text{eff}} = \frac{A_s}{A_{v,\text{eff}}} = \frac{4 \phi^2 \pi}{4} = \frac{2.5 (h - d)}{h - x_{II}} = \frac{2.5 (400 - 360)}{400 - 180.62} = 0.0859
\]

First we need to check if the longitudinal bars are close to each other:

\[
t_{\text{limit}} = 5 \left( c + \phi_s + \frac{\phi}{2} \right) = 5 \left( 20 + 10 + \frac{20}{2} \right) = 200 \text{ mm} ; \ t_{\text{actual}} = 40 \text{ mm}
\]

\( t_{\text{limit}} > t_{\text{actual}} \) therefore the rebars are close to each other, so the distance between cracks calculated as follows:

\[
s_{r, \text{max}} = 3.4 \left( c + \phi_s \right) + 0.425 k_t k_2 \frac{\phi}{\rho_{\text{eff}}} = 3.4 (20 + 10) + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{20}{0.0859} = 141.57 \text{ mm}
\]
$k_1$ depends the cohesion between the rebars and concrete, the reinforcements are ribbed:

\[ k_1 = 0.8 \]

$k_2$ depends the strains in the cross-section, in this case we have pure bending:

\[ k_2 = 0.5 \]

The difference between the hand calculations are under 3% and the results of the second hand calculation coincide with the FEM-Design results.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.4 Crack width calculation of a beam.str
9.2.5 Crack width calculation of a slab

In this example we calculate the crack width of a slab due to *elliptic* and *hyperbolic* bending conditions. First of all the applied reinforcement is *orthogonal* and then the applied reinforcement is *non-orthogonal*. We calculate the crack width with hand calculation and then compare the results with FEM-Design values. The crack width calculation is relevant in SLS combination, thus the internal forces (moments) in the examples come from a quasi-permanent serviceability combination.

The calculation of the crack directions is based on [10] and [12] according to the tensor of the reserve forces due to an arbitrary internal force and reinforcement distribution in the slab.

Fig. 9.2.5 shows the notation system of the applied angles in this chapter. The figure is valid for bottom and top reinforcement separately.

The $x$-$y$ system denotes the direction of the local co-ordinate system of the slab.

The $\xi$ and $\eta$ directions are the directions of the reinforcements.

The $\xi$ and $\eta$ angles are the angles between the reinforcement directions and axis $x$.

The $\varphi$ angle is the angle between the two directional reinforcements.

The $\alpha$ angle is the angle between axis $x$ and the direction of the crack.

The $\alpha_0$ angle is the angle perpendicular to the direction of the crack.

![Figure 9.2.5 – The notation of the angles](image)

The concrete covers in the different directions are valid on top and bottom equally even if there is no need for reinforcement in one direction.
Level arm of internal forces

9.2.5.1 Elliptic bending

The SLS moments in the slab in shell local system due to elliptic bending:

\[ m_x = +16 \text{kNm/m} \quad \text{the resultant of the } x \text{ direction normal stresses.} \]

\[ m_y = +8 \text{kNm/m} \quad \text{the resultant of the } y \text{ direction normal stresses.} \]

\[ m_{xy} = +6 \text{kNm/m} \quad \text{the resultant of the } x-y \text{ direction shear stresses.} \]

Orthogonal reinforcement (\( \phi = 90^\circ \) between \( \xi \) and \( \eta \))

The reinforcement is orthogonal and their directions coincide with the local system (\( x=\xi, y=\eta \)). Fig. 9.2.5.1.1 shows the applied reinforcements and the concrete covers. Thus \( \xi = 0^\circ; \eta = 90^\circ \).
The applied reinforcement in the slab:

Only bottom reinforcement is necessary (see Chapter 9.2.2 for further information).

\[
a_s = \frac{\phi_s \pi}{4} \frac{1000}{s} = \frac{10^2 \pi}{4} \frac{1000}{120} = 654.5 \text{ mm}^2/m
\]

\[
a = \frac{\phi_s \pi}{4} \frac{1000}{s} = \frac{10^2 \pi}{4} \frac{1000}{180} = 436.3 \text{ mm}^2/m
\]

Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:

\[
E_{ex} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} E_{sx} & E_{sy} \\ E_{xy} & E_{sy} \end{bmatrix} = \begin{bmatrix} 114.3 & 42.86 \\ 42.86 & 57.14 \end{bmatrix} \text{kN/m}, \text{ where}
\]

\[
E_s = \frac{m_s}{z} = \frac{16}{0.14} = 114.3 \text{ kN/m}
\]

\[
E_y = \frac{m_y}{z} = \frac{8}{0.14} = 57.14 \text{ kN/m}
\]

\[
E_{xy} = \frac{m_{xy}}{z} = \frac{6}{0.14} = 42.86 \text{ kN/m}
\]

The tensor of the resisting (yield) forces based on the resistance of the reinforcement:

\[
R_{xy} = \begin{bmatrix} R_x & 0 \\ 0 & R_y \end{bmatrix} = \begin{bmatrix} R_{sx} & R_{sy} \\ R_{xy} & R_{sy} \end{bmatrix} = \begin{bmatrix} 327.3 & 0 \\ 0 & 218.2 \end{bmatrix} \text{kN/m}, \text{ where}
\]

\[
A_{sx} = a_s f_{y} = 654.5 \cdot 500 = 327.3 \text{ kN/m}
\]

\[
A_{sy} = a_s f_{y} = 436.3 \cdot 500 = 218.2 \text{ kN/m}
\]

\[
R_s = A_{sx} \cos^2(\xi) + A_{sy} \cos^2(\eta) = 327.3 \cos^2(0°) + 218.2 \cos^2(90°) = 327.3 \text{ kN/m}
\]

\[
R_y = A_{sx} \sin^2(\xi) + A_{sy} \sin^2(\eta) = 327.3 \sin^2(0°) + 218.2 \sin^2(90°) = 218.2 \text{ kN/m}
\]

\[
R_{xy} = A_{sx} \cos(\xi) \sin(\xi) + A_{sy} \cos(\eta) \sin(\eta) = 327.3 \cos(0°) \sin(0°) + 218.2 \cos(90°) \sin(90°) = 0 \text{ kN/m}
\]
The tensor of the reserve forces:

\[
\mathbf{r}^* = \begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} \equiv \begin{bmatrix} r_x^* & r_y^* \\ r_{xy}^* & r_{yx}^* \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} - r \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} R_x - rE_x & R_{xy} - rE_{xy} \\ R_{xy} - rE_{xy} & R_y - rE_y \end{bmatrix},
\]

where \( r \) is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:

\( r_2^* = 0 \)

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

\[
r_2^* = \frac{(r_x^* + r_y^*)}{2} - \sqrt{\frac{(r_x^* - r_y^*)^2}{4} + r_{xy}^*} = 0
\]

\[
r_2^* = \frac{(R_x - rE_x + R_y - rE_y)}{2} - \sqrt{\frac{(R_x - rE_x - R_y + rE_y)^2}{4} + (R_{xy} - rE_{xy})^2} = 0
\]

This equation gives two solutions for the \( r \) scalar internal force multiplier. The smallest positive \( r \) value has physical meaning. Without further detailed calculation the relevant \( r \) scalar load multiplier is:

\( r = 2.120 \)

Based on this scalar value the reserve force tensor:

\[
\mathbf{r}^* = \begin{bmatrix} 182.1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 84.98 & -90.86 \\ -90.86 & 97.06 \end{bmatrix} \text{ kN/m}
\]

The principal direction of the first principal reserve force gives the direction of the crack:

\[
\alpha = \arctan \frac{r_x^* - r_y^*}{r_{xy}^*} = \arctan \frac{182.1 - 84.98}{-90.86} = -47.00^\circ
\]

Now we can calculate in this direction the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

\[
\alpha_0 = \alpha + 90^\circ = 43^\circ
\]
The effective reinforcement area in this direction:
\[ a_{\alpha_0} = a_{\xi} \cos^2(\alpha_0 - \xi) + a_{\eta} \cos^2(\alpha_0 - \eta) = 654.5 \cos^2(43^\circ - 0^\circ) + 436.3 \cos^2(43^\circ - 90^\circ) = 553.0 \text{ mm}^2 \]

The bending moment in the direction perpendicular to the crack direction:
\[ m_{\alpha_0} = m_x \cos^2 \alpha_0 + m_y \sin^2 \alpha_0 + 2 m_{xy} \cos \alpha_0 \sin \alpha_0 =
\]
\[ = (16) \cos^2 43^\circ + (8) \sin^2 43^\circ + 2 \cdot (6) \cos 43^\circ \sin 43^\circ = 18.26 \text{ kNm/m} \]

The position of the neutral axis according to the uncracked section (Stadium I.):
\[ \alpha_e = \frac{E_s}{E_{cm}} = 200 \quad x_I = \frac{h^2 + \alpha_e a_{\alpha_0} d}{h + \alpha_e a_{\alpha_0}} = 101.3 \text{ mm} \]

The moment of inertia (Stadium I.):
\[ I_I = \frac{x_I^3}{3} + \frac{(h-x_I)^3}{3} + \alpha_e a_{\alpha_0} (d-x_I)^2 = 6.844 \cdot 10^8 \text{ mm}^4 \]

Concrete tensile stress (Stadium I.) to check if the crack exist or not:
\[ \sigma_{c,a_0} = \frac{m_{\alpha_0} (h-x_I)}{I_I} = \frac{18.26 \cdot 10^3 \cdot (200 - 101.3)}{6.844 \cdot 10^5} = 2.63 \frac{\text{N}}{\text{mm}^2} > f_{cm} = 2.2 \text{ MPa} \quad \text{crack occurred.} \]

The position of the neutral axis according to the cracked section (Stadium II.):
\[ x_{II} = \frac{x_{II}}{2} = \alpha_e a_{\alpha_0} (d-x_{II}) \quad \text{thus:} \quad x_{II} = 31.92 \text{ mm} \]

The moment of inertia (Stadium II.):
\[ I_{II} = \frac{x_{II}^3}{3} + \alpha_e a_{\alpha_0} (d-x_{II})^2 = \frac{31.92^3}{3} + 6.667 \cdot 0.553 \cdot (170 - 31.92)^2 = 8.114 \cdot 10^7 \text{ mm}^4 \]

Rebar stress (Stadium II.):
\[ \sigma_{s,a_0} = \frac{m_{\alpha_0} (d-x_{II})}{I_{II}} = \frac{6.667 \cdot 18.26 \cdot 10^3 \cdot (170 - 31.92)}{8.114 \cdot 10^4} = 207.2 \frac{\text{N}}{\text{mm}^2} \]
Effective tensile rebar ratio:

\[
\rho_{p,\text{eff}} = \frac{a_{e_s}}{a_{c,\text{eff}}} = \frac{a_{e_s}}{h_{c,\text{eff}}} = \frac{0.553}{\min \left\{ \frac{2.5(h-d)}{h-x_{II}}, \frac{2.5(200-170)}{200-31.92}, \frac{3}{2} \right\}} = \frac{0.553}{\min \left\{ \frac{75}{100}, \frac{75}{56.03} \right\}} = 0.009870
\]

The concrete and rebar strain difference:

\[
\Delta \varepsilon = \varepsilon_{sm} - \varepsilon_{cm} = \max \left\{ \frac{\sigma_{s,\alpha_s} - k_s f_{p,\text{eff}}}{E_s} \left( 1 + \alpha_s \rho_{p,\text{eff}} \right), \frac{\sigma_{s,\alpha_s}}{0.6 E_s} \right\}
= \max \left\{ \frac{207.2 - 0.4 \cdot 2.2}{0.00987} \cdot \left( 1 + 6.667 \cdot 0.00987 \right), \frac{200000}{0.6 \cdot 207.2} \right\}
= \max \left\{ \frac{0.0005609}{0.0006216} = 0.0006216 \right\}
\]

The criterium of the spacing of the bonded bars that they are close to each other or not:

\[sp_{cr} = 5 \left( c + \phi_l / 2 \right) = 5 \left( 25 + 10 / 2 \right) = 150 \text{ mm}\]

The effective spacing of the bonded bars in the direction of the crack:

\[sp = \frac{\phi_l \pi / 4}{a_{e_s}} = \frac{10^2 \pi / 4}{0.553} = 142.0 \text{ mm}\]

Now the maximum crack spacing:

\[sp \leq sp_{cr} \text{ thus: } s_{r,\text{max}} = 3.4 c + 0.425 k_1 k_2 p_{p,\text{eff}} = 3.4 \cdot 25 + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{10}{0.00987} = 257.2 \text{ mm}\]

Thus the crack width:

\[w_k = s_{r,\text{max}} \Delta \varepsilon = 257.2 \cdot 0.0006216 = 0.1599 \text{ mm}\]

Numerical results:

\[w_{k,FEM} = 0.1559 \text{ mm}\]
Fig. 9.2.5.1.2 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is less than 3%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.2.5.1 Crack width calculation in a slab with elliptic bending and orthogonal reinforcement.str
Non-orthogonal reinforcement ($\varphi=75^\circ$ between $\xi$ and $\eta$)

The reinforcement is non-orthogonal and the $\xi$ direction coincides with the local $x$ direction. The angle between the $\xi$ directional reinforcement and $\eta$ directional reinforcement is $\varphi=75^\circ$. Fig. 9.2.5.1.3 shows the applied reinforcements and the concrete covers. Thus $\xi = 0^\circ$; $\eta = 75^\circ$.

![Diagram showing applied skew reinforcement for elliptic bending](image)

The applied reinforcement in the slab:

Only bottom reinforcement necessary (see Chapter 9.2.2 for further information).

$$
\begin{align*}
    a_{s,\xi} &= \frac{\varphi_{\xi}^2 \pi}{4} \frac{1000}{s_{\xi}} = \frac{10^2 \pi}{4} \frac{1000}{160} = 490.9 \text{ mm}^2 / \text{m} \\
    a_{s,\eta} &= \frac{\varphi_{\eta}^2 \pi}{4} \frac{1000}{s_{\eta}} = \frac{10^2 \pi}{4} \frac{1000}{200} = 392.7 \text{ mm}^2 / \text{m}
\end{align*}
$$

Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:

$$
E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} 114.3 & 42.86 \\ 42.86 & 57.14 \end{bmatrix} \text{ kN/m, where}
$$

$$
E_x = \frac{m_x}{z} = \frac{16}{0.14} = 114.3 \text{ kN/m},
$$

$$
E_y = \frac{m_y}{z} = \frac{8}{0.14} = 57.14 \text{ kN/m},
$$

$$
E_{xy} = \frac{m_{xy}}{z} = \frac{6}{0.14} = 42.86 \text{ kN/m}.
$$

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:

$$
R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} = \begin{bmatrix} 258.7 & 49.1 \\ 49.1 & 183.2 \end{bmatrix} \text{ kN/m, where}
$$
Verification Examples

\[ A_{\xi \xi} = a_{\xi \xi} f_{\xi \xi} = 490.9 \cdot 500 = 245.5 \text{ kN/m} , \]
\[ A_{\xi \eta} = a_{\xi \eta} f_{\xi \eta} = 392.7 \cdot 500 = 196.4 \text{ kN/m} , \]
\[ R_x = A_{\xi \xi} \cos^2(\xi) + A_{\xi \eta} \cos^2(\eta) = 245.5 \cos^2(0^\circ) + 196.4 \cos^2(75^\circ) = 258.7 \text{ kN/m} , \]
\[ R_y = A_{\xi \xi} \sin^2(\xi) + A_{\xi \eta} \sin^2(\eta) = 245.5 \sin^2(0^\circ) + 196.4 \sin^2(75^\circ) = 183.2 \text{ kN/m} , \]
\[ R_{xy} = A_{\xi \xi} \cos(\xi) \sin(\xi) + A_{\xi \eta} \cos(\eta) \sin(\eta) = 245.5 \cos(0^\circ) \sin(0^\circ) + 196.4 \cos(75^\circ) \sin(75^\circ) = 49.1 \text{ kN/m} . \]

The tensor of the reserve forces:
\[ r^* = \begin{bmatrix} r_{1}^* & 0 \\ 0 & r_{2}^* \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} - r \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} R_x - r E_x & R_{xy} - r E_{xy} \\ R_{xy} - r E_{xy} & R_y - r E_y \end{bmatrix} , \]

where \( r \) is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:
\[ r_{2}^* = 0 \]

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:
\[ r_{2}^* = \left( \frac{r_{x}^* + r_{y}^*}{2} \right) - \sqrt{\left( \frac{r_{x}^* - r_{y}^*}{2} \right)^2 + r_{xy}^*} = 0 \]
\[ r_{2}^* = \left( \frac{R_x - r E_x + R_y - r E_y}{2} \right) - \sqrt{\left( \frac{R_x - r E_x - R_y + r E_y}{2} \right)^2 + (R_{xy} - r E_{xy})^2} = 0 \]

This equation gives two solutions for the \( r \) scalar internal force multiplier. The smallest positive \( r \) value has physical meaning. Without further detailed calculation the relevant \( r \) scalar load multiplier is:
\[ r = 2.058 \]

Based on this scalar value the reserve force tensor:
\[ r^* = \begin{bmatrix} r_{1}^* & 0 \\ 0 & r_{2}^* \end{bmatrix} = \begin{bmatrix} 89.08 & 0 \\ 0 & 65.61 \end{bmatrix} = \begin{bmatrix} 23.47 & -39.11 \\ -39.11 & 65.61 \end{bmatrix} \text{ kN/m} . \]
The principal direction of the first principal reserve force gives the direction of the crack:
\[ \alpha = \arctan \frac{r^* \cdot r_x}{r_y} = \arctan \frac{89.08 - 23.47}{-39.11} = -59.20^\circ \]

Now we can calculate in this direction the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:
\[ \alpha_0 = \alpha + 90^\circ = 30.8^\circ \]

The effective reinforcement area in this direction:
\[ a_{\alpha_0} = a_s \cos^2 (\alpha_0 - \xi) + a_x \cos^2 (\alpha_0 - \eta) = 490.9 \cos^2 (30.8^\circ - 0^\circ) + 392.7 \cos^2 (30.8^\circ - 75^\circ) = 564.0 \text{ mm}^2 \]

The bending moment in the direction perpendicular to the crack direction:
\[ m_{\alpha_0} = m_x \cos^2 \alpha_0 + m_y \sin^2 \alpha_0 + 2 m_{xy} \cos \alpha_0 \sin \alpha_0 = 19.18 \text{ kNm} \]

The position of the neutral axis according to the uncracked section (Stadium I.):
\[ x_I = \frac{h^2}{2} + \frac{\alpha_e a_{\alpha_0} d}{h + \alpha_e a_{\alpha_0}} = 101.3 \text{ mm} \]

The moment of inertia (Stadium I.):
\[ I_I = \frac{x_I^3}{3} + \frac{(h-x_I)^3}{3} + \alpha_e a_{\alpha_0} (d-x_I)^2 = 6.848 \cdot 10^8 \text{ mm}^4 \]

Concrete tensile stress (Stadium I.) to check if the crack exist or not:
\[ \sigma_{c,\alpha_0} = \frac{m_{\alpha_0} (h-x_I)}{I_I} = \frac{19.18 \cdot 10^3 \cdot (200 - 101.3)}{6.848 \cdot 10^5} = 2.76 \text{ MPa} > f_{c,m} = 2.2 \text{ MPa} \quad \text{crack occurred.} \]
The position of the neutral axis according to the cracked section (Stadium II.):

\[ x_{II} = \frac{x_{II}}{2} + \alpha_e a_{\alpha_e} \left( d - x_{II} \right) \]  thus:  \[ x_{II} = 32.19 \text{ mm} \]

The moment of inertia (Stadium II.):

\[ I_{II} = \frac{x_{II}^3}{3} + \alpha_e a_{\alpha_e} \left( d - x_{II} \right)^2 = \frac{32.19^3}{3} + 6.667 \cdot 0.564 \cdot (170 - 32.19)^2 = 8.253 \cdot 10^7 \text{ mm}^4 \]

Rebar stress:

\[ \sigma_{s,\alpha_0} = \frac{m_{\alpha_e} \left( d - x_{II} \right)}{I_{II}} = 6.667 \cdot \frac{19.18 \cdot 10^3 \cdot (170 - 32.19)}{8.253 \cdot 10^4} = 213.5 \text{ N/mm}^2 \]

Effective tensile rebar ratio:

\[ \rho_{p,\text{eff}} = \frac{a_{\sigma_s}}{a_{c,\text{eff}}} = \frac{0.564}{\min \left\{ \frac{2.5 (h - d)}{h - x_{II}}, \frac{2.5 (h - d)}{h} \right\}} = \frac{0.564}{\min \left\{ \frac{2.5 (200 - 170)}{200 - 32.19}, \frac{2.5 (200 - 170)}{200} \right\}} = \frac{0.564}{0.01008} = 0.01008 \]

The concrete and rebar strain difference:

\[ \Delta \varepsilon = \varepsilon_{sm} - \varepsilon_{cm} = \max \left\{ \frac{\sigma_{s,\alpha_0} - k_t f_{ct,\text{eff}} \left( 1 + \alpha_e \rho_{p,\text{eff}} \right)}{E_s}, 0.6 \frac{\sigma_{s,\alpha_0}}{E_s} \right\} = \max \left\{ \frac{213.5 - 0.4 \cdot 2.2}{0.01008 \cdot 0.01008} \left[ 1 + 6.667 \cdot 0.01008 \right], 0.6 \cdot \frac{213.5}{200000} \right\} = \max \left\{ 0.0006017, 0.0006405 \right\} = 0.0006405 \]
The criterium of the spacing of the bonded bars that they are close to each other or not:

\[ sp_{cr} = 5\left(c + \frac{\phi_l}{2}\right) = 5\left(25 + \frac{10}{2}\right) = 150 \text{ mm} \]

The effective spacing of the bonded bars in the direction of the crack:

\[ sp = \frac{\phi_l^2 \pi / 4}{a_{\alpha_e}} = \frac{10^2 \pi / 4}{0.564} = 139.3 \text{ mm} \]

Now the maximum crack spacing:

\[ sp \leq sp_{cr} \quad \text{thus:} \]

\[ s_{r, max} = 3.4c + 0.425k_1k_2\frac{\phi_l}{p_{p, eff}} = 3.4 \cdot 25 + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{10}{0.01008} = 253.7 \text{ mm} \]

Thus the crack width:

\[ w_k = s_{r, max} \Delta \varepsilon = 253.7 \cdot 0.0006405 = 0.1625 \text{ mm} \]

Numerical results:

\[ w_{k, FEM} = 0.1579 \text{ mm} \]

**Fig. 9.2.5.1.4 – The crack width [mm] and the direction of the cracks at the bottom**

Fig. 9.2.5.1.4 shows the FEM-Design results.
The difference between the hand and FEM-Design calculations is less than 3%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.5.1 Crack width calculation in a slab with elliptic bending and skew reinforcement.str
9.2.5.2 Hyperbolic bending

The SLS moments in the slab in shell local system due to hyperbolic bending:

\[ m_x = +32 \text{kNm/m} \] the resultant of the \( x \) directional normal stresses.

\[ m_y = -16 \text{kNm/m} \] the resultant of the \( y \) directional normal stresses.

\[ m_{xy} = +12 \text{kNm/m} \] the resultant of the \( x-y \) directional shear stresses.

Orthogonal reinforcement (\( \phi = 90^\circ \) between \( \xi \) and \( \eta \))

The reinforcement is orthogonal and their directions coincide with the local system (\( x = \xi \), \( y = \eta \)). Fig. 9.2.5.2.1 shows the applied reinforcements and the concrete covers. Thus \( \xi = 0^\circ \); \( \eta = 90^\circ \).

The applied reinforcement in the slab:

Top and bottom reinforcement are also necessary but in this case only in one direction (see Chapter 9.2.2 for further information).

Bottom:

\[ a_{s_{\xi}}^{\text{bot}} = \frac{\phi_{s_{\xi}}^2 \pi}{4} \frac{1000}{s_{\xi}} = \frac{10^2 \pi}{4} \frac{1000}{60} = 1309 \text{mm}^2/\text{m} \]

\[ a_{s_{\eta}}^{\text{bot}} = 0 \]

Top:

\[ a_{s_{\xi}}^{\text{top}} = 0 \]

\[ a_{s_{\eta}}^{\text{top}} = \frac{\phi_{s_{\eta}}^2 \pi}{4} \frac{1000}{s_{\eta}} = \frac{10^2 \pi}{4} \frac{1000}{120} = 654.5 \text{mm}^2/\text{m} \]

Crack width on bottom:

Calculation of the direction of the crack based on the tensor of the reserve forces.
The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:
\[ E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \approx \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} 228.5 & 85.71 \\ 85.71 & -114.3 \end{bmatrix} \text{ kN} / \text{m} \], where
\[ E_x = \frac{m_x}{z} = \frac{32}{0.14} = 228.6 \text{ kN} / \text{m} \],
\[ E_y = \frac{m_y}{z} = \frac{-16}{0.14} = -114.3 \text{ kN} / \text{m} \],
\[ E_{xy} = \frac{m_{xy}}{z} = \frac{12}{0.14} = 85.71 \text{ kN} / \text{m} \].

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:
\[ R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \approx \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} = \begin{bmatrix} 654.5 & 0 \\ 0 & 62.5 \end{bmatrix} \text{ kN} / \text{m} \], where
\[ A_{\xi} = a_{\xi}^\text{bot} \cdot f_{yk} = 1309 \cdot 500 = 654.5 \text{ kN} / \text{m} \].

If there is no reinforcement on bottom in the other direction we need to consider somehow the tensile resistance of the concrete, because this may effect the relevant direction of the crack.
\[ A_{\eta} = c \cdot f_{cm} = 25 \cdot 2.2 = 62.5 \text{ kN} / \text{m} \],
\[ R_x = A_{\xi} \cos^2(\xi) + A_{\eta} \cos^2(\eta) = 654.5 \cos^2(0^\circ) + 62.5 \cos^2(90^\circ) = 654.5 \text{ kN} / \text{m} \],
\[ R_y = A_{\xi} \sin^2(\xi) + A_{\eta} \sin^2(\eta) = 654.5 \sin^2(0^\circ) + 62.5 \sin^2(90^\circ) = 62.5 \text{ kN} / \text{m} \],
\[ R_{xy} = A_{\xi} \cos(\xi) \sin(\xi) + A_{\eta} \cos(\eta) \sin(\eta) = 654.5 \cos(0^\circ) \sin(0^\circ) + 62.5 \cos(90^\circ) \sin(90^\circ) = 0 \text{ kN} / \text{m} \].

The tensor of the reserve forces:
\[ r^* = \begin{bmatrix} r^*_1 & 0 \\ 0 & r^*_2 \end{bmatrix} \approx \begin{bmatrix} r^*_x & r^*_{xy} \\ r^*_{xy} & r^*_y \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} - r \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} R_x - r E_x & R_{xy} - r E_{xy} \\ R_{xy} - r E_{xy} & R_y - r E_y \end{bmatrix} \],
where \( r \) is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:
\[ r^*_2 = 0 \]
It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

\[
    r_2^* = \left( \frac{r_x^* + r_y^*}{2} \right) \mp \left( \frac{(r_x^* - r_y^*)^2}{2} + r_{xy}^* \right)^{1/2} = 0
\]

\[
    r_2^* = \left( \frac{R_x - rE_x + rE_y - rE_y}{2} \right) \mp \left( \frac{(R_x - rE_x - rE_y + rE_y)^2}{2} + (r_{xy} - rE_{xy})^2 \right)^{1/2} = 0
\]

This equation gives two solutions for the \( r \) scalar internal force multiplier. The smallest positive \( r \) value has physical meaning. Without further detailed calculation the relevant \( r \) scalar load multiplier is:

\[ r = 2.337 \]

Based on this scalar value the reserve force tensor:

\[
    r^* = \begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} = \begin{bmatrix} 450.1 & 0 \\ 0 & 0 \end{bmatrix} \mp \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} 120.5 & -200.3 \\ -200.3 & 329.6 \end{bmatrix} \text{kN/m}
\]

The principal direction of the first principal reserve force gives the direction of the crack:

\[
    \alpha = \arctan \left( \frac{r_1^* - r_x^*}{r_{xy}^*} \right) = \arctan \left( \frac{450.1 - 120.5}{-200.3} \right) = -58.71^\circ
\]

Now we can calculate in this direction at the bottom the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

\[
    \alpha_0 = \alpha + 90^\circ = 31.29^\circ
\]

The effective reinforcement area in this direction:

\[
    a_{\alpha_0}^{\text{bot}} = a_{sx}^{\text{bot}} \cos^2(\alpha_0 - \xi) + a_{sy}^{\text{bot}} \cos^2(\alpha_0 - \eta) =
\]

\[
    = 1309 \cos^2(31.29^\circ - 0^\circ) + 0 \cos^2(31.29^\circ - 90^\circ) = 955.9 \text{mm}^2 / \text{m}
\]

\[
    a_{\alpha_0}^{\text{top}} = a_{sx}^{\text{top}} \cos^2(\alpha_0 - \xi) + a_{sy}^{\text{top}} \cos^2(\alpha_0 - \eta) =
\]

\[
    = 0 \cos^2(31.29^\circ - 0^\circ) + 654.5 \cos^2(31.29^\circ - 90^\circ) = 186.8 \text{mm}^2 / \text{m}
\]
The bending moment in the direction perpendicular to the crack direction:

\[ m_{\alpha} = m_x \cos^2 \alpha_0 + m_y \sin^2 \alpha_0 + 2m_{xy} \cos \alpha_0 \sin \alpha_0 = \]

\[ = (32 \cos^2 31.29^\circ - 16 \sin^2 31.29^\circ + 2 \cdot 12 \cos 31.29^\circ \sin 31.29^\circ) = 29.70 \text{ kNm} \]

The position of the neutral axis according to the uncracked section (Stadium I.):

\[ \alpha_e = \frac{E_s}{E_{cm}} \frac{200}{30} = 6.667 \quad \frac{h^2}{2} + \alpha_e a_{\alpha_s}^{bot} (d_\xi - x_I) + \alpha_e a_{\alpha_s}^{top} (d_\eta - x_I)^2 = 7.070 \cdot 10^5 \text{ mm}^4 \]

Concrete tensile stress (Stadium I.) to check the crack or not:

\[ \sigma_{e,\alpha} = \frac{m_{\alpha} (h - x_I)}{I_I} = \frac{29.70 \cdot 10^3 \cdot (200 - 101.9)}{7.070 \cdot 10^5} = 4.12 \text{ N/mm}^2 > f_{cm} = 2.2 \text{ MPa} \quad \text{crack occurred.} \]

The position of the neutral axis according to the cracked section (Stadium II.):

In case of this hyperbolic bending the applied reinforcements are only in one direction on one side, thus we considered the real effective height of the tensile and compressed bars instead of the average values which ones were introduced at the beginning of this chapter.

\[ x_{II} = \frac{x_{II}}{2} + \alpha_e a_{\alpha_s}^{bot} (x_{II} - d_\eta) = \alpha_e a_{\alpha_s}^{bot} (d_\xi - x_{II}) \quad \text{thus:} \quad x_{II} = 41.13 \text{ mm} \]

The moment of inertia (Stadium II.):

\[ I_{II} = \frac{x_{II}^3}{3} + \alpha_e a_{\alpha_s}^{bot} (d_\xi - x_{II})^2 + \alpha_e a_{\alpha_s}^{top} (x_{II} - d_\eta)^2 = \]

\[ = \frac{41.13^3}{3} + 6.667 \cdot 0.9559 \cdot (175 - 41.13)^2 + 6.667 \cdot 0.1868 \cdot (41.13 - 35)^2 = 1.3745 \cdot 10^8 \text{ mm}^4 \]

Rebar stress:

\[ \sigma_{s,\alpha} = \frac{m_{\alpha} (d_\xi - x_{II})}{I_{II}} = \alpha_e \frac{29.70 \cdot 10^3 \cdot (175 - 41.13)}{1.3745 \cdot 10^5} = 192.9 \text{ N/mm}^2 \]
Effective tensile rebar ratio:

\[
\rho_{p,\text{eff}} = \frac{a_{\alpha,\text{eff}}}{a_{\alpha}} = \frac{0.9559}{\text{min} \left( \frac{2.5(h-d_\xi)}{h-x_{II}}, \frac{2.5(200-175)}{200-41.13} \right) \min \left( \frac{3}{h}, \frac{2}{3} \right)} = \frac{0.9559}{\text{min} \left( \frac{62.5}{100}, 52.96 \right) = 0.01805}
\]

The concrete and rebar strain difference:

\[
\Delta \varepsilon = \varepsilon_{\alpha,\text{cm}} - \varepsilon_{\alpha,\text{sm}} = \max \left( \frac{\sigma_{s,\alpha} - k_{t} f_{ct,\text{eff}} (1 + \alpha_{e} \rho_{p,\text{eff}})}{0.6 \frac{\sigma_{s,\alpha}}{E_{s}}} \right) = \max \left( \frac{192.9 - 0.4 \cdot \frac{2.2}{0.01805} (1 + 6.667 \cdot 0.01805)}{200000} \frac{0.6 \cdot 192.9}{200000} \right) = \max \left( \frac{0.0006914}{0.0005787} = 0.0006914 \right)
\]

The criterium of the spacing of the bonded bars that they are close to each other or not:

\[
s_{p,\text{cr}} = 5(c_\xi + \phi_l/2) = 5(20 + 10/2) = 125 \text{mm}
\]

The effective spacing of the bonded bars in the direction of the crack:

\[
s_{p} = \frac{\phi_l^2 \pi / 4}{a_{\text{bot}}} = \frac{10^2 \pi / 4}{0.9559} = 82.16 \text{mm}
\]

Now the maximum crack spacing:

\[
s_{p} < s_{p,\text{cr}} \text{ thus:}
\]

\[
s_{r,\text{max}} = 3.4 c_\xi + 0.425 k_{t} k_{2} \frac{\phi_l}{\rho_{p,\text{eff}}} = 3.4 \cdot 20 + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{10}{0.01805} = 162.2 \text{mm}
\]

Thus the crack width:

\[
w_{k,\text{bot}} = s_{r,\text{max}} \Delta \varepsilon = 162.2 \cdot 0.0006914 = 0.1121 \text{mm}
\]

Numerical results:
Fig. 9.2.5.2.2 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is less than 1%.
Crack width on top:
Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:
\[
E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} -228.5 & -85.71 \\ -85.71 & 114.3 \end{bmatrix} \text{kN/m}, \text{ where}
\]
\[
E_x = \frac{-m_x}{z} = \frac{-32}{0.14} = -228.6 \text{kN/m},
\]
\[
E_y = \frac{-m_y}{z} = \frac{+16}{0.14} = +114.3 \text{kN/m},
\]
\[
E_{xy} = \frac{-m_{xy}}{z} = \frac{-12}{0.14} = -85.71 \text{kN/m}.
\]

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:
\[
R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} = \begin{bmatrix} 62.5 & 0 \\ 0 & 327.25 \end{bmatrix} \text{kN/m}, \text{ where}
\]
\[
A_{xy} = A_{xy} f_{yk} = 654.5 \cdot 500 = 327.25 \text{kN/m}.
\]

If there is no reinforcement on top in the other direction we need to consider somehow the tensile resistance of the concrete, because this may effect the relevant direction of the crack.
\[
A_{\bar{s}} = c \cdot f_{cm} = 25 \cdot 2.2 = 62.5 \text{kN/m},
\]
\[
R_x = A_{\bar{s}} \cos^2(\bar{\xi}) + A_{xy} \cos^2(\eta) = 62.5 \cos^2(0\degree) + 327.25 \cos^2(90\degree) = 62.5 \text{kN/m},
\]
\[
R_y = A_{\bar{s}} \sin^2(\bar{\xi}) + A_{xy} \sin^2(\eta) = 62.5 \sin^2(0\degree) + 327.25 \sin^2(90\degree) = 327.25 \text{kN/m},
\]
\[
R_{xy} = A_{\bar{s}} \cos(\bar{\xi}) \sin(\eta) + A_{xy} \cos(\eta) \sin(\eta) = 62.5 \cos(0\degree) \sin(0\degree) + 327.25 \cos(90\degree) \sin(90\degree) = 0 \text{kN/m}.
\]

The tensor of the reserve forces:
\[
r^* = \begin{bmatrix} r_{11}^* & 0 \\ 0 & r_{22}^* \end{bmatrix} = \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} - r \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} R_x - r E_x & R_{xy} - r E_{xy} \\ R_{xy} - r E_{xy} & R_y - r E_y \end{bmatrix},
\]
where \( r \) is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve
force tensor is equal to zero:
\[ r_2^* = 0 \]

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:
\[
r_2^* = \left( \frac{r_x^*+r_y^*}{2} \right) - \sqrt{\left( \frac{r_x^*-r_y^*}{2} \right)^2 + r_{xy}^*} = 0
\]
\[
r_2^* = \left( \frac{R_x - r E_x + R_y - r E_y}{2} \right) - \sqrt{\left( \frac{R_x - r E_x - R_y + r E_y}{2} \right)^2 + (R_{xy} - r E_{xy})^2} = 0
\]

This equation gives two solutions for the \( r \) scalar internal force multiplier. The smallest positive \( r \) value has physical meaning. Without further detailed calculation the relevant \( r \) scalar load multiplier is:
\[ r = 2.291 \]

Based on this scalar value the reserve force tensor:
\[
\begin{bmatrix}
  r_1^* & 0 \\
  0 & r_2^*
\end{bmatrix} =
\begin{bmatrix}
  651.4 & 0 \\
  0 & 0
\end{bmatrix} \pm
\begin{bmatrix}
  r_x^* & r_{xy}^* \\
  r_{xy}^* & r_y^*
\end{bmatrix} =
\begin{bmatrix}
  586.0 & 196.4 \\
  196.4 & 65.39
\end{bmatrix} \text{ kN}
\]

The principal direction of the first principal reserve force gives the direction of the crack:
\[
\alpha = \arctan \frac{r_2^* - r_x^*}{r_{xy}^*} = \arctan \frac{651.4 - 586.0}{196.4} = 18.42^\circ
\]

Now we can calculate in this direction at the bottom the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:
\[
\alpha_0 = \alpha + 90^\circ = 108.4^\circ
\]

The effective reinforcement area in this direction:
\[
a_{\alpha_0} = a_{ax}^\text{bot} \cos^2 (\alpha_0 - \xi) + a_{ax}^\text{bot} \cos^2 (\alpha_0 - \eta) =
\]
\[
= 1309 \cos^2 (108.4^\circ - 0^\circ) + 0 \cos^2 (108.4^\circ - 90^\circ) = 130.4 \text{ mm}^2 \text{ m}
\]
\[
a_{\alpha_0} = a_{ax}^\text{top} \cos^2 (\alpha_0 - \xi) + a_{ax}^\text{top} \cos^2 (\alpha_0 - \eta) =
\]
\[
= 0 \cos^2 (108.4^\circ - 0^\circ) + 654.5 \cos^2 (108.4^\circ - 90^\circ) = 589.3 \text{ mm}^2 \text{ m}
\]
The bending moment in the direction perpendicular to the crack direction:

\[ m_{\alpha_0} = m_x \cos^2 \alpha_0 + m_y \sin^2 \alpha_0 + 2 m_{xy} \cos \alpha_0 \sin \alpha_0 = \]

\[ = (32 \cos^2 108.4^\circ + (-16) \sin^2 108.4^\circ + 2 \cdot 12 \cos 108.4^\circ \sin 108.4^\circ) = -18.41 \text{kNm} / \text{m} \]

The position of the neutral axis according to the uncracked section (Stadium I.):

\[ \alpha = \frac{E_s}{E_{cm}} = \frac{200}{30} = 6.667; \quad x_f = \frac{h^2}{2} \frac{I_{\alpha}}{h + \alpha e a_{\alpha_{top}} + \alpha e a_{\alpha_{bot}}} = 100.9 \text{mm} \]

The moment of inertia (Stadium I.):

\[ I_I = \frac{x_I^3}{3} + \frac{(h-x_I)^3}{3} + \alpha e a_{\alpha_{top}} (d_{\eta} - x_I)^2 + \alpha e a_{\alpha_{bot}} (x_I - d_{\xi})^2 = 6.880 \cdot 10^8 \text{mm}^4 / \text{m} \]

Concrete tensile stress (Stadium I.) to check the crack exist or not:

\[ \sigma_{ct, \alpha_0} = -m_{\alpha_0} (h-x_I) \frac{I_I}{6.880 \cdot 10^8} = 2.65 \text{N/mm}^2 > f_{cm} = 2.2 \text{MPa} \quad \text{crack occurred.} \]

The position of the neutral axis according to the cracked section (Stadium II.):

In case of this hyperbolic bending the applied reinforcements are only in one direction on one side, thus we considered the real effective height of the tensile and compressed bars instead of the average values which ones were introduced at the beginning of this chapter.

\[ x_{II} = \frac{x_{II}}{2} + \alpha e a_{\alpha_{top}} (x_{II} - d_{\eta}) = \alpha e a_{\alpha_{top}} (d_{\eta} - x_{II}) \quad \text{thus:} \quad x_{II} = 32.12 \text{mm} \]

The moment of inertia (Stadium II.):

\[ I_{II} = \frac{x_{II}^3}{3} + \alpha e a_{\alpha_{top}} (d_{\eta} - x_{II})^2 + \alpha e a_{\alpha_{bot}} (x_{II} - d_{\xi})^2 = \]

\[ = \frac{32.12^3}{3} + 6.667 \cdot 0.5893 \cdot (165 - 32.12)^2 + 6.667 \cdot 0.1304 \cdot (32.12 - 25)^2 = 8.046 \cdot 10^7 \text{mm}^4 / \text{m} \]

Rebar stress:

\[ \sigma_{s, \alpha_{top}} = \frac{\alpha e a_{\alpha_{top}} (d_{\eta} - x_{II})}{I_{II}} \frac{18.41 \cdot 10^3 \cdot (165 - 32.12)}{8.046 \cdot 10^4} = 202.7 \text{N/mm}^2 \]
Effective tensile rebar ratio:

\[
\rho_{p,\text{eff}} = \frac{a_{\text{top}}}{a_{\text{eff}}} = \frac{a_{\text{top}}}{h_{\text{eff}}} = \frac{0.5893}{\min\left(\frac{2.5(h-d_\eta)}{h-x_{II}}, \frac{2.5(200-165)}{200-32.12}, \frac{3}{2}\right)} \approx 0.01053
\]

The concrete and rebar strain difference:

\[
\Delta \varepsilon = \varepsilon_{\text{sm}} - \varepsilon_{\text{cm}} = \max\left\{\frac{\sigma_{s,\varepsilon_0} - k_f f_{ct,\text{eff}} (1 + \alpha_s \rho_{p,\text{eff}})}{E_s}, \frac{\sigma_{s,\varepsilon_0}}{0.6 E_s} \right\} = \max\left\{\frac{202.7 - 0.4 \cdot \frac{2.2}{0.01053} (1 + 6.667 \cdot 0.01053)}{200000}, \frac{0.6 \cdot 202.7}{200000} \right\} = \max\left\{0.0005663, 0.0006081\right\} = 0.0006081
\]

The criterium of the spacing of the bonded bars that they are close to each other or not:

\[
s_{p,c} = 5(c_\eta + \phi_l/2) = 5(30 + 10/2) = 175 \text{ mm}
\]

The effective spacing of the bonded bars in the direction of the crack:

\[
s_{p} = \frac{\phi_l^2 \pi / 4}{a_{\text{eff}}} = \frac{10^2 \pi / 4}{0.5893} = 133.3 \text{ mm}
\]

Now the maximum crack spacing:

\[
s_{r,\text{max}} = 3.4 c_\eta + 0.425 k_f k_p \frac{\phi_l}{\rho_{p,\text{eff}}} = 3.4 \cdot 30 + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{10}{0.01053} = 263.4 \text{ mm}
\]

Thus the crack width:

\[
w_{k,\text{top}} = s_{r,\text{max}} \Delta \varepsilon = 263.4 \cdot 0.0006081 = 0.1602 \text{ mm}
\]

Numerical results:
$w_{k,FEM}^{top} = 0.1602 \text{ mm}$

Fig. 9.2.5.2.3 shows the FEM-Design results.

**The difference between the hand and FEM-Design calculations is 0%**.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.5.2 Crack width calculation in a slab with hyperbolic bending and orthogonal reinforcement.str
Non-orthogonal reinforcement ($\varphi=75^\circ$ between $\xi$ and $\eta$)

The reinforcement is non-orthogonal and the $\xi$ direction concides with the local $x$ direction. The angle between the $\xi$ directional reinforcement and $\eta$ directional reinforcement is $\varphi=75^\circ$. Fig. 9.2.5.2.4 shows the applied reinforcements and the concrete covers. Thus $\xi = 0^\circ$; $\eta = 75^\circ$.

![Diagram showing non-orthogonal reinforcement](image)

**Figure 9.2.5.2.4 – The applied skew reinforcement for elliptic bending**

The applied reinforcement in the slab:

Top and bottom reinforcement are also necessary but in this case only in one direction (see Chapter 9.2.2 for further information).

**Bottom:**

$$a_{s,\xi}^{bot} = \frac{\varphi^2 \pi}{4} \frac{1000}{s_{\xi}} = \frac{10^2 \pi}{4} \frac{1000}{60} = 1309 \text{ mm}^2 / \text{m}$$

$$a_{s,\eta}^{bot} = 0$$

**Top:**

$$a_{s,\xi}^{top} = \frac{\varphi^2 \pi}{4} \frac{1000}{s_{\eta}} = \frac{10^2 \pi}{4} \frac{1000}{90} = 872.7 \text{ mm}^2 / \text{m}$$

$$a_{s,\eta}^{top} = 0$$

**Crack width on bottom:**

Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:

$$E = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} = \begin{bmatrix} E_x & E_{xy} & E_y \\ E_{xy} & E_x & E_y \end{bmatrix} = \begin{bmatrix} 228.5 & 85.71 \\ 85.71 & -114.3 \end{bmatrix} \text{ kN/m}, \text{ where}$$

$$E_s = \frac{m_s}{z} = \frac{32}{0.14} = 228.6 \text{ kN/m},$$
\[ E_y = \frac{m_y}{z} = \frac{-16}{0.14} = -114.3 \text{ kN/m}, \]
\[ E_{xy} = \frac{m_{xy}}{z} = \frac{12}{0.14} = 85.71 \text{ kN/m}. \]

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:
\[ \mathbf{R} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} = \begin{bmatrix} 654.5 & 0 \\ 0 & 62.5 \end{bmatrix} \text{ kN/m}, \]
where
\[ A_{s\xi} = a_{s\xi} f_{yk} = 1309 \cdot 500 = 654.5 \text{ kN/m}. \]

If there is no reinforcement on top in the other direction we need to consider somehow the tensile resistance of the concrete perpendicular to the rebars, because this may effect the relevant direction of the crack.
\[ A_{s\xi+90^\circ} = c \cdot f_{cm} = 25 \cdot 2.2 = 62.5 \text{ kN/m}, \]
\[ R_x = A_{s\xi} \cos^2(\xi) + A_{s(\xi+90^\circ)} \cos^2(\xi + 90^\circ) = 654.5 \cos^2(0^\circ) + 62.5 \cos^2(90^\circ) = 654.5 \text{ kN/m}, \]
\[ R_y = A_{s\xi} \sin^2(\xi) + A_{s(\xi+90^\circ)} \sin^2(\xi + 90^\circ) = 654.5 \sin^2(0^\circ) + 62.5 \sin^2(90^\circ) = 654.5 \text{ kN/m}, \]
\[ R_{xy} = A_{s\xi} \cos(\xi) \sin(\xi) + A_{s(\xi+90^\circ)} \cos(\xi + 90^\circ) \sin(\xi + 90^\circ) = 654.5 \cos(0^\circ) \sin(0^\circ) + 62.5 \cos(90^\circ) \sin(90^\circ) = 0 \text{ kN/m}. \]

The tensor of the reserve forces:
\[ \mathbf{r}^* = \begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} = \begin{bmatrix} r_x^* & r_{xy}^* \\ r_{xy}^* & r_y^* \end{bmatrix} - r \begin{bmatrix} E_x & E_{xy} \\ E_{xy} & E_y \end{bmatrix} = \begin{bmatrix} R_x - r E_x & R_{xy} - r E_{xy} \\ R_{xy} - r E_{xy} & R_y - r E_y \end{bmatrix}, \]
where \( r \) is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:
\[ r_2^* = 0. \]

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:
\[ r_2^* = \left( \frac{r_x^* + r_y^*}{2} \right) - \sqrt{\left( \frac{r_x^* - r_y^*}{2} \right)^2 + r_{xy}^*} = 0. \]
\[ r_2^* = \left( \frac{R_x - r E_x + R_y - r E_y}{2} \right)^2 + \sqrt{\left( \frac{R_x - r E_x - R_y + r E_y}{2} \right)^2 + (R_{xy} - r E_{xy})} = 0 \]

This equation gives two solutions for the \( r \) scalar internal force multiplier. The smallest positive \( r \) value has physical meaning. Without further detailed calculation the relevant \( r \) scalar load multiplier is:

\[ r = 2.337 \]

Based on this scalar value the reserve force tensor:

\[ r^* = \begin{bmatrix} r_1^* & 0 \\ 0 & r_2^* \end{bmatrix} = \begin{bmatrix} 450.1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} r_x^* & r_y^* \\ r_{xy}^* & r_y^* \end{bmatrix} = \begin{bmatrix} 120.5 & -200.3 \\ -200.3 & 329.6 \end{bmatrix} \text{kN/m} \]

The principal direction of the first principal reserve force gives the direction of the crack:

\[ \alpha = \arctan \frac{r_1^* - r_x^*}{r_{xy}^*} = \arctan \frac{450.1 - 120.5}{-200.3} = -58.71^\circ \]

Now we can calculate in this direction at the bottom the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

\[ \alpha_0 = \alpha + 90^\circ = 31.29^\circ \]

The effective reinforcement area in this direction:

\[ a_{\alpha_0} = a_{\xi} \cos^2 (\alpha_0 - \xi) + a_{\eta} \cos^2 (\alpha_0 - \eta) = \]

\[ = 1309 \cos^2 (31.29^\circ - 0^\circ) + 0 \cos^2 (31.29^\circ - 75^\circ) = 955.9 \text{ mm}^2 / \text{m} \]

\[ a_{\alpha_0} = a_{\xi} \cos^2 (\alpha_0 - \xi) + a_{\eta} \cos^2 (\alpha_0 - \eta) = \]

\[ = 0 \cos^2 (31.29^\circ - 0^\circ) + 872.7 \cos^2 (31.29^\circ - 75^\circ) = 456.0 \text{ mm}^2 / \text{m} \]

The effective internal forces in the direction perpendicular to the crack direction:

\[ m_{\alpha_0} = m_x \cos^2 \alpha_0 + m_y \sin^2 \alpha_0 + 2 m_{xy} \cos \alpha_0 \sin \alpha_0 = \]

\[ = (32) \cos^2 31.29^\circ + (-16) \sin^2 31.29^\circ + 2 \cdot 12 \cos 31.29^\circ \sin 31.29^\circ = 29.70 \text{ kNm/m} \]
The position of the neutral axis according to the uncracked section (Stadium I.):

\[
\alpha_e = \frac{E_s}{E_{cm}} = 200/30 = 6.667; \quad x_I = \frac{h^2}{2} + \alpha_e a_{\alpha_e}^{bot} d_\frac{\alpha}{\eta} + \alpha_e a_{\alpha_e}^{top} d_\frac{\alpha}{\eta} = 101.3 \text{ mm}
\]

The moment of inertia (Stadium I.):

\[
I_I = \frac{x_I^3}{3} + \left(\frac{h-x_I}{3}\right)^2 + \alpha_e a_{\alpha_e}^{bot} (d_\frac{\alpha}{\eta} - x_I)^2 + \alpha_e a_{\alpha_e}^{top} (x_I - d_\frac{\alpha}{\eta})^2 = 7.150 \cdot 10^8 \text{ mm}^4/m
\]

Concrete tensile stress (Stadium I.) to check the crack exist or not:

\[
\sigma_{c,\alpha_0} = \frac{m_{\alpha_0} (h-x_I)}{I_I} = \frac{29.70 \cdot 10^3 \cdot (200 - 101.3)}{7.150 \cdot 10^5} = 4.10 \frac{\text{N}}{\text{mm}^2} > f_{cm} = 2.2 \text{ MPa} \quad \text{crack occurred.}
\]

The position of the neutral axis according to the cracked section (Stadium II.):

In case this of hyperbolic bending the applied reinforcements are only in one direction on one side, thus we considered the real effective height of the tensile and compressed bars instead of the average values which ones were introduced at the beginning of this chapter.

\[
x_{II} = \frac{x_{II}^3}{3} + \alpha_e a_{\alpha_e}^{bot} (d_\frac{\alpha}{\eta} - x_{II})^2 + \alpha_e a_{\alpha_e}^{top} (x_{II} - d_\frac{\alpha}{\eta})^2 =
\]

\[
= \frac{40.91^3}{3} + 6.667 \cdot 0.9559 \cdot (175 - 40.91)^2 + 6.667 \cdot 0.456 \cdot (40.91 - 35)^2 = 1.3752 \cdot 10^8 \text{ mm}^4/m
\]

Rebar stress:

\[
\sigma_{s,\alpha_e} = \frac{m_{s} (d_\frac{\alpha}{\eta} - x_{II})}{I_{II}} = 6.667 \cdot \frac{29.70 \cdot 10^3 \cdot (175 - 40.91)}{1.3752 \cdot 10^8} = 193.1 \frac{\text{N}}{\text{mm}^2}
\]
Effective tensile rebar ratio:
\[
\rho_{p, eff} = \frac{a_{\text{bot}}^{\text{tensile}}}{a_{\text{eff}}^{\text{tensile}}} = \frac{0.9559}{2.5 \left( h - d_z \right) / h} = \frac{0.9559}{200 - 40.91 / 2} \approx 0.9559 \min \left\{ \frac{2.5}{3}, \frac{200}{2} \right\} = 0.01803
\]

The concrete and rebar strain difference:
\[
\Delta \varepsilon = \varepsilon_{\text{sm}} - \varepsilon_{\text{cm}} = \max \left\{ \sigma_{s, \alpha_s} - k_e f_{\text{ct, eff}} (1 + \alpha_e \rho_{p, eff}) \right\} = \max \left\{ \frac{0.6 \sigma_{s, \alpha_s}}{E_s} \right\}
\]
\[
= \max \left\{ \frac{193.1 - 0.4 \cdot 2.2}{0.01803} \cdot \frac{1 + 6.667 \cdot 0.01803}{200000} \cdot \frac{0.0006921}{0.0005793} = 0.0006921 \right\}
\]

The criterium of the spacing of the bonded bars that they are close to each other or not:
\[
s_{p, cr} = 5 (c_z + \phi_l / 2) = 5 (20 + 10 / 2) = 125 \text{ mm}
\]

The effective spacing of the bonded bars in the direction of the crack:
\[
s_p = \frac{\phi_l \pi / 4}{a_{\text{bot}}^{\text{tensile}}} = \frac{10 \pi / 4}{0.9559} = 82.16 \text{ mm}
\]

Now the maximum crack spacing:
\[
s_p < s_{p, cr} \quad \text{thus:}
\]
\[
s_{r, \max} = 3.4 c_z + 0.425 k_e k_t \frac{\phi_l}{\rho_{p, eff}} = 3.4 \cdot 20 + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{10}{0.01805} = 162.2 \text{ mm}
\]

Thus the crack width:
\[
W_{k, \text{bot}} = s_{r, \max} \Delta \varepsilon = 162.2 \cdot 0.0006921 = 0.1123 \text{ mm}
\]

Numerical results:
Fig. 9.2.5.2.5 shows the FEM-Design results. The difference between the hand and FEM-Design calculations is less than 1%.
Verification Examples

Crack width on top:

Calculation of the direction of the crack based on the tensor of the reserve forces.

The tensor of the applied forces (effect) based on the internal forces in the quasi-permanent combination:

\[
\mathbf{E} = \begin{bmatrix}
E_1 & 0 \\
0 & E_2
\end{bmatrix} = \begin{bmatrix}
E_x & E_{xy} \\
E_{xy} & E_y
\end{bmatrix} = \begin{bmatrix}
-228.5 & -85.71 \\
-85.71 & 114.3
\end{bmatrix} \text{kN/m}, \text{ where}
\]

\[
E_x = \frac{-m_x}{z} = \frac{-32}{0.14} = -228.6 \text{kN/m},
\]

\[
E_y = \frac{-m_y}{z} = \frac{16}{0.14} = 114.3 \text{kN/m},
\]

\[
E_{xy} = \frac{-m_{xy}}{z} = \frac{-12}{0.14} = -85.71 \text{kN/m}.
\]

The tensor of the resisting (yield) forces based on the resistance on the reinforcement:

\[
\mathbf{R} = \begin{bmatrix}
R_1 & 0 \\
0 & R_2
\end{bmatrix} = \begin{bmatrix}
R_x & R_{xy} \\
R_{xy} & R_y
\end{bmatrix} = \begin{bmatrix}
87.55 & 93.48 \\
93.48 & 411.4
\end{bmatrix} \text{kN/m}, \text{ where}
\]

\[
A_{xy} = a^\text{top}_{xy} f_{yk} = 872.7 \cdot 500 = 436.4 \text{kN/m}.
\]

If there is no reinforcement in the other direction on top we need to consider somehow the tensile resistance of the concrete perpendicular to the rebars, because this may effect the relevant direction of the crack.

\[
A_{x(y+90^\circ)} = c \cdot f_{cm} = 25 \cdot 2.2 = 62.5 \text{kN/m},
\]

\[
R_x = A_{x(y+90^\circ)} \cos^2(\eta+90^\circ) + A_{xy} \cos^2(\eta) = 62.5 \cos^2(165^\circ) + 436.4 \cos^2(75^\circ) = 87.55 \text{kN/m},
\]

\[
R_y = A_{x(y+90^\circ)} \sin^2(\eta+90^\circ) + A_{xy} \sin^2(\eta) = 62.5 \sin^2(165^\circ) + 436.4 \sin^2(75^\circ) = 411.4 \text{kN/m},
\]

\[
R_{xy} = A_{x(y+90^\circ)} \cos(\eta+90^\circ) \sin(\eta+90^\circ) + A_{xy} \cos(\eta) \sin(\eta) =
\]

\[
= 62.5 \cos(165^\circ) \sin(165^\circ) + 436.4 \cos(75^\circ) \sin(75^\circ) =
\]

\[
= 93.48 \text{kN/m}.
\]

The tensor of the reserve forces:

\[
\mathbf{r}^* = \begin{bmatrix}
r_1^* & 0 \\
0 & r_2^*
\end{bmatrix} = \begin{bmatrix}
r_x^* & r_{xy}^* \\
r_{xy}^* & r_y^*
\end{bmatrix} - \mathbf{r} \begin{bmatrix}
E_x & E_{xy} \\
E_{xy} & E_y
\end{bmatrix} = \begin{bmatrix}
R_x - r E_x & R_{xy} - r E_{xy} \\
R_{xy} - r E_{xy} & R_y - r E_y
\end{bmatrix},
\]
where $r$ is a scalar multiplier of the internal forces.

Yielding occurs (described by Gvozdiev [12]), when the smaller principal value of the reserve force tensor is equal to zero:

$$r^*_2 = 0$$

It gives the following equation based on the well known calculation method of the smaller principal values of a tensor:

$$r^*_2 = \left( \frac{r^* + r^*}{2} \right) - \sqrt{\left( \frac{r^* - r^*}{2} \right)^2 + r^*_{xy}^2} = 0$$

$$r^*_2 = \left( \frac{R_x - rE_x + R_y - rE_y}{2} \right) - \sqrt{\left( \frac{R_x - rE_x - R_y + rE_y}{2} \right)^2 + (R_{xy} - rE_{xy})^2} = 0$$

This equation gives two solutions for the $r$ scalar internal force multiplier. The smallest positive $r$ value has physical meaning. Without further detailed calculation the relevant $r$ scalar load multiplier is:

$$r = 2.375$$

Based on this scalar value the reserve force tensor:

$$r^* = \begin{bmatrix} r^*_1 & 0 \\ 0 & r^*_2 \end{bmatrix} = \begin{bmatrix} 770.2 & 0 \\ 0 & 297 \end{bmatrix} \begin{bmatrix} 630.2 & 297.0 \\ 297.0 & 140.0 \end{bmatrix} \text{kN}$$

The principal direction of the first principal reserve force gives the direction of the crack:

$$\alpha = \arctan \frac{r^*_1 - r^*_2}{r^*_{xy}} = \arctan \frac{770.2 - 630.2}{297} = 25.24^\circ$$

Now we can calculate in this direction at the bottom the crack width based on the standard formulas. The internal forces and the reinforcement need to considered in the perpendicular direction of the crack:

$$\alpha_0 = \alpha + 90^\circ = 115.2^\circ$$

The effective reinforcement area in this direction:

$$a_{\alpha_0} = a_{\xi} \cos^2 \left( \alpha_0 - \xi \right) + a_{\eta} \cos^2 \left( \alpha_0 - \eta \right) =$$

$$= 1309 \cos^2 \left( 115.2^\circ - 0^\circ \right) + 0 \cos^2 \left( 115.2^\circ - 75^\circ \right) = 237.3 \text{ mm}^2$$

$$a_{\alpha_0} = a_{\xi} \cos^2 \left( \alpha_0 - \xi \right) + a_{\eta} \cos^2 \left( \alpha_0 - \eta \right) =$$
The effective internal forces in the direction perpendicular to the crack direction:
\[ m_{\vartheta} = m_x \cos^2 \alpha_0 + m_y \sin^2 \alpha_0 + 2 m_{xy} \cos \alpha_0 \sin \alpha_0 = \]
\[ = (32) \cos^2 115.2^\circ + (-16) \sin^2 115.2^\circ + 2 \cdot 12 \cos 115.2^\circ \sin 115.2^\circ = -16.54 \text{ kNm} \]

The moment of inertia (Stadium I.):
\[ I_I = \frac{x_I^3}{3} + \frac{(h-x_I)^3}{3} + \alpha_e a_{\vartheta_{\text{bot}}} (d_\eta - x_I)^2 + \alpha_e a_{\vartheta_{\text{top}}} (x_I - d_\xi)^2 = 6.899 \cdot 10^8 \text{ mm}^4 \]

Concrete tensile stress (Stadium I.) to check the crack exist or not:
\[ \sigma_{c, \alpha_0} = \frac{-m_{\vartheta} (h-x_I)}{I_I} = \frac{16.54 \cdot 10^3 \cdot (200-100.5)}{6.899 \cdot 10^8} = 2.39 \text{ N/mm}^2 > f_{\text{cm}} = 2.2 \text{ MPa} \quad \text{crack occurred.} \]

The position of the neutral axis according to the uncracked section (Stadium I.):
\[ \alpha_e = \frac{E_s}{E_{cm}} = \frac{200}{30} = 6.667 \quad ; \quad x_I = \frac{h^2}{2} + \alpha_e a_{\vartheta_{\text{bot}}} d_\xi + \alpha_e a_{\vartheta_{\text{top}}} d_\eta \]
\[ = \frac{100.5}{100.5} = 100.5 \text{ mm} \]

The moment of inertia (Stadium II.):
\[ I_{II} = \frac{x_{II}^3}{3} + \alpha_e a_{\vartheta_{\text{bot}}} (d_\eta - x_{II})^2 + \alpha_e a_{\vartheta_{\text{top}}} (x_{II} - d_\xi)^2 = \]
\[ = \frac{30.02^3}{3} + 6.667 \cdot 0.5091 \cdot (165 - 30.02)^2 + 6.667 \cdot 0.2373 \cdot (30.02 - 25)^2 = 7.090 \cdot 10^7 \text{ mm}^4 \]
Verification Examples

Rebar stress:

$$\sigma_{s,\alpha} = \alpha_e \frac{-m_{\alpha}(d_\eta - x_{II})}{l_{II}} = 6.667 \cdot \frac{16.54 \cdot 10^3 \cdot (165 - 30.02)}{7.090 \cdot 10^4} = 209.9 \text{ N/mm}^2;$$

Effective tensile rebar ratio:

$$\rho_{p,\text{eff}} = \frac{a_{\alpha_{\text{top}}}}{a_{c,\text{eff}}} = \frac{0.5091}{\min \left( \frac{2.5(h-d_\eta)}{h-x_{II}}, \frac{2.5(200-165)}{200-30.02} \right)} = \frac{0.5091}{\min \left( \frac{3}{h}, \frac{3}{200} \right)} = \frac{0.5091}{\min \left( \frac{3}{200}, \frac{3}{100} \right)} = \frac{0.5091}{\min \left( 87.5, 56.66 \right)} = 0.008985$$

The concrete and rebar strain difference:

$$\Delta \varepsilon = \varepsilon_{sm} - \varepsilon_{cm} = \max \left\{ \frac{\sigma_{s,\alpha} - k_{ct} f_{p,\text{eff}} (1 + \alpha_e \rho_{p,\text{eff}})}{E_s}, \frac{0.6 \sigma_{s,\alpha}}{E_s} \right\} = \max \left\{ \frac{209.9 - 0.4 \cdot 0.008985 \cdot (1 + 6.667 \cdot 0.008985)}{200000}, \frac{0.6 \cdot 209.9}{200000} \right\} = \max \left\{ 0.0005305, 0.0006297 \right\} = 0.0006297$$

The criterium of the spacing of the bonded bars that they are close to each other or not:

$$s_{p,ct} = 5(c_\eta + \phi_l/2) = 5(30 + 10/2) = 175 \text{ mm}$$

The effective spacing of the bonded bars in the direction of the crack:

$$s_{p} = \frac{\phi_l \pi / 4}{a_{\alpha_{\text{top}}}} = \frac{10 \pi / 4}{0.5091} = 154.3 \text{ mm}$$

Now the maximum crack spacing:

$$s_{r,\text{max}} = 3.4 c_\eta + 0.425 \frac{k_1 k_2 \phi_l}{\rho_{p,\text{eff}}} = 3.4 \cdot 30 + 0.425 \cdot 0.8 \cdot 0.5 \cdot \frac{10}{0.008985} = 291.2 \text{ mm}$$
Thus the crack width:

\[ w_{k,\text{top}} = s_{r,\text{max}} \Delta \varepsilon = 263.4 \cdot 0.0006297 = 0.1834 \text{ mm} \]

Numerical results:

\[ w_{k,FEM}^{\text{top}} = 0.1827 \text{ mm} \]

Fig. 9.2.5.2.6 shows the FEM-Design results.

The difference between the hand and FEM-Design calculations is less than 1%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.5.2 Crack width calculation in a slab with hyperbolic bending and skew reinforcement.str
### 9.2.6 Punching calculation of a slab

In this section we will check five different types of punching reinforcement: bended bar, circular stirrups, open stirrups, stud rail, PSB stud rail.

**Inputs:**

<table>
<thead>
<tr>
<th>Concrete</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete characteristic compressive strength</td>
<td>$f_{ck} = 25 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Plate height</td>
<td>h = 200 mm</td>
</tr>
<tr>
<td>Cover</td>
<td>c = 20 mm</td>
</tr>
<tr>
<td>Partial factor of concrete</td>
<td>$\gamma_c = 1.50$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reinforcement</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{yk} = 500 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Rebars Young modulus</td>
<td>$E_s = 200 \text{ GPa}$</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
<tr>
<td>Longitudinal rebar diameter</td>
<td>$\phi_x = 20 \text{ mm}$</td>
</tr>
<tr>
<td>Distance between longitudinal reinforcements</td>
<td>$s_x = 150 \text{ mm}$</td>
</tr>
<tr>
<td>Longitudinal bars effective distance</td>
<td>$d_{lx} = 170 \text{ mm}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Geometry</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Plate width in x direction</td>
<td>$L_x = 4.0 \text{ m}$</td>
</tr>
<tr>
<td>Plate width in y direction</td>
<td>$L_y = 4.0 \text{ m}$</td>
</tr>
<tr>
<td>Circle column diameter</td>
<td>$d_{\text{column}} = 25 \text{ cm}$</td>
</tr>
<tr>
<td>Specific normal force in x direction</td>
<td>$n_{x,Ed} = 40 \text{ kN/m}$</td>
</tr>
<tr>
<td>Specific normal force in y direction</td>
<td>$n_{y,Ed} = 40 \text{ kN/m}$</td>
</tr>
<tr>
<td>Vertical surface total load</td>
<td>$q_{z,Ed} = 25 \text{ kN/m}^2$</td>
</tr>
</tbody>
</table>
Fig. 9.2.6.1 shows the analyzed punching problem.

9.2.6.1 Bended bars

Inputs (see Fig. 9.2.6.1.1):

<table>
<thead>
<tr>
<th>Reinforcement</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>( f_{ywk} = 500 \text{ N/mm}^2 )</td>
</tr>
<tr>
<td>Bended bar diameter</td>
<td>( \phi_{bb} = 14 \text{ mm} )</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>( \gamma_s = 1.15 )</td>
</tr>
<tr>
<td>Bended bar distance</td>
<td>( s_r = 120 \text{ mm} )</td>
</tr>
</tbody>
</table>

\[ d_{\text{column}} = 25 \text{ cm} \]

\[ 0.4d \approx 60 \text{ mm} \]

\[ \alpha = 45^\circ \]

\[ \phi_{20/150} \]

Figure 9.2.6.1.1 – The applied punching reinforcement in x direction (the same valid in y direction)
The effective longitudinal reinforcement ratios:
\[
\rho_{l,x} = \frac{\phi_x \pi 1000}{d_{l,x} \cdot 100} = \frac{20 \pi 1000}{170 \cdot 100} = 0.01232 \quad ; \quad \rho_{l,y} = \frac{\phi_y \pi 1000}{d_{l,y} \cdot 100} = \frac{20 \pi 1000}{150 \cdot 100} = 0.01396
\]
\[
\rho_l = \min \left\{ \sqrt{\rho_{l,x} \cdot \rho_{l,y}}, 0.02 \right\} = \min \left\{ \sqrt{0.01232 \cdot 0.01396}, 0.02 \right\} = 0.01312
\]

The average effective height of the longitudinal reinforcements:
\[
d = \frac{d_{l,x} + d_{l,y}}{2} = \frac{170 + 150}{2} = 160 \text{ mm}
\]

**Concrete compression resistance**

The column reaction:
\[
V_{Ed} = L_x L_y q_{z,Ed} = 4 \cdot 1.25 = 500 \text{ kN}
\]

The perimeter of the column:
\[
u_0 = d_{\text{column}} \pi = 250 \cdot \pi = 785.40 \text{ mm}
\]

Because this column is an inner column the \( \beta \) is:
\[
\beta = 1.15
\]

The specific shear force at \( u_0 \):
\[
v_{Ed} = \beta \frac{V_{Ed}}{d u_0} = 1.15 \frac{400000}{160 \cdot 785.40} = 3.66 \text{ N/mm}^2
\]

The concrete compression resistance:
\[
v_{Rd,\text{max}} = 0.5 v f_{cd} = 0.5 \cdot 0.54 \cdot \frac{25}{1.5} = 4.50 \text{ N/mm}^2
\]

where:
\[
v = 0.6 \left( 1 - \frac{f_{ck}}{250} \right) = 0.6 \left( 1 - \frac{25}{250} \right) = 0.54
\]

Concrete compression utilization:
\[
\frac{v_{Ed}}{v_{Rd,\text{max}}} = \frac{3.66}{4.5} = 81.35\%
\]

Utilization is less than 100\% thus the slab can bear the acting shear force. Fig. 9.2.6.1.2 shows the relevant results with FEM-Design. The results are identical with the hand calculation.

**Shear reinforcement resistance**

Punching perimeter at \( 2d \) distance from the edge of the column:
\[
u_1 = 2 \left( \frac{d_{\text{column}}}{2} + 2d \right) \pi = 2 \left( \frac{250}{2} + 2 \cdot 160 \right) \pi = 2796 \text{ mm}
\]
The shear force at \( u_1 \) perimeter:
\[
v_{Ed} = \beta \frac{V_{Ed}}{d u_1} = 1.15 \frac{40000}{160 \cdot 2796} = 1.028 \frac{N}{mm^2}
\]

Modifying factors for concrete shear resistance:
\[
C_{Rd,c} = \frac{0.18}{V_c} = \frac{0.18}{1.5} = 0.12 \quad ; \quad k_1 = 0.1
\]

Average normal stress in the concrete:
\[
\sigma_{cp} = \min \left( \frac{n_{x,Ed}}{1000h} + \frac{n_{y,Ed}}{1000h} \right) = \min \left( \frac{10000 + 40000}{2} \right) = 0.2 \frac{N}{mm^2}
\]
\[
k = \min \left( 1 + \sqrt{\frac{200}{d}} \right) = \min \left( 1 + \sqrt{\frac{200}{160}} \right) = \min \left( 2.12 \right) = 2
\]
\[
\nu_{min} = 0.035 k^{1.5} f_{ck}^{0.5} = 0.035 \cdot 2^{1.5} \cdot 25^{0.5} = 0.495 \frac{N}{mm^2}
\]

The shear resistance of concrete:
\[
\nu_{Rd,c} = \max \left( C_{Rd,c} k (\frac{100 \rho_l f_{ck}}{V_{min}})^{\frac{3}{2}} \right) + k_1 \sigma_{cp} = \max \left( \frac{0.12 \cdot 2 \cdot (100 \cdot 0.01312 \cdot 25)^{3}}{434.78} \right) + 0.1 \cdot 0.2 = 0.788 \frac{N}{mm^2}
\]

We need to check if we need any punching reinforcement or not:
\[
\frac{v_{Ed}}{\nu_{Rd,c}} = \frac{1.028}{0.788} = 1.305 > 1.0
\]

Thus we need punching reinforcement.

Bended bars area in the \( u_1 \) perimeter:
\[
A_{sw} = 4 \frac{\phi_{bh}^2 \pi}{4} = 4 \frac{14^2 \pi}{4} = 615.75 \text{ mm}^2
\]

Effective tensile strength of the bended bars:
\[
f_{y,sw,eff} = \min \left( \frac{f_{ywd}}{250 + \frac{d}{4}} \right) = \min \left( \frac{f_{ywd}}{250 + \frac{160}{4}} \right) = \min \left( \frac{500}{1.15} \right) = 290 \frac{N}{mm^2}
\]
The punching reinforcement resistance:

\[ v_{Rd,sw} = 1.5 \frac{d}{s_r} A_{sw} f_y,sw,eff \frac{1}{u_i d} \sin \alpha = 1.5 \frac{160}{120} \frac{615.75 - 290}{2796 - 160} \sin 45^\circ = 0.564 \frac{N}{mm^2} \]

Punching shear resistance:

\[ v_{Rd,cs} = \min \left[ 0.75 \frac{v_{Rd,c} + v_{Rd,sw}}{k_{max} v_{Rd,c}} \right] = \min \left[ 0.75 \cdot 0.788 + 0.564 \right] = 1.155 \frac{N}{mm^2} \]

Punching shear resistance utilization:

\[ \frac{v_{Ed}}{v_{Rd,cs}} = \frac{1.028}{1.155} = 88.98 \% \]

Fig. 9.2.6.1.2 and the table below shows the relevant FEM-Design results.

<table>
<thead>
<tr>
<th></th>
<th>Concrete compression</th>
<th>Shear reinforcement</th>
<th>Concrete shear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hand calculation</td>
<td>4.5 N/mm²</td>
<td>1.155 N/mm²</td>
<td>0.788 N/mm²</td>
</tr>
<tr>
<td>FEM-Design</td>
<td>4.5 N/mm²</td>
<td>1.18 N/mm²</td>
<td>0.79 N/mm²</td>
</tr>
<tr>
<td>Difference</td>
<td>0.00%</td>
<td>2.10 %</td>
<td>0.20 %</td>
</tr>
</tbody>
</table>

The differences between the numerical and hand calculations are less than 3\%.


**Verification Examples**

**FEM-Design 19**

**Figure 9.2.6.1.2 – The punching detailed results in FEM-Design**

**Concrete compression resistance - Part 1.1: 6.4.3**

L.C.: "Load"

\[
\varepsilon_{\text{Ed}} = \frac{\beta \cdot \varepsilon_{\text{rel,ed}}}{u \cdot d} = \frac{1.15 \cdot 399.691.34}{784 \cdot 160} = 3.66 \text{ N/mm}^2 \quad (6.38)
\]

\[
\varepsilon_{\text{Rel,med}} = 0.5 \cdot \gamma \cdot f_{cd} = 0.5 \cdot 0.54 \cdot 16.67 = 4.50 \text{ N/mm}^2
\]

\[
\varepsilon_{\text{Ed}} = 3.66 \text{ N/mm}^2 \leq \varepsilon_{\text{Rel,med}} = 4.50 \text{ N/mm}^2 \quad (6.53) - \text{OK}
\]

**Shear reinforcement resistance - Part 1.1: 6.4.3**

\[
\varepsilon_{\text{Ed}} = \frac{\beta \cdot \varepsilon_{\text{rel,ed}}}{u \cdot d} \quad (6.38)
\]

\[
\varepsilon_{\text{Rel,ex}} = 1.5 \frac{d}{\delta} A_{\text{ex}} f_{\text{pm},\text{ex}} \frac{1}{u \cdot d} \sin (\alpha)
\]

\[
\varepsilon_{\text{Rel,ca}} = \min (0.75 \varepsilon_{\text{Rel,ca}} + \varepsilon_{\text{Rel,ex}} k_{\text{max}}, \varepsilon_{\text{Rel,ca}})
\]

<table>
<thead>
<tr>
<th>Perimeter index</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dist [mm]</td>
<td>320</td>
<td>340</td>
<td>360</td>
<td>380</td>
</tr>
<tr>
<td>$V_{\text{Ed}}$ [kN]</td>
<td>399.69</td>
<td>399.69</td>
<td>399.69</td>
<td>399.69</td>
</tr>
<tr>
<td>$\beta$ [-]</td>
<td>1.15</td>
<td>1.15</td>
<td>1.15</td>
<td>1.15</td>
</tr>
<tr>
<td>$d$ [mm]</td>
<td>160</td>
<td>160</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td>$u$ [m]</td>
<td>2.79</td>
<td>2.92</td>
<td>3.04</td>
<td>3.17</td>
</tr>
<tr>
<td>$v_{\text{Ed}}$ [N/mm²]</td>
<td>1.03</td>
<td>0.99</td>
<td>0.94</td>
<td>0.91</td>
</tr>
<tr>
<td>$v_{\text{Rel,ca}}$ [N/mm²]</td>
<td>0.79</td>
<td>0.79</td>
<td>0.79</td>
<td>0.79</td>
</tr>
<tr>
<td>$v_{\text{Rel,ca}}$ [N/mm²]</td>
<td>0.60</td>
<td>0.57</td>
<td>0.55</td>
<td>0.53</td>
</tr>
<tr>
<td>$v_{\text{Rel,ca}}$ [N/mm²]</td>
<td>1.16</td>
<td>1.16</td>
<td>1.14</td>
<td>1.12</td>
</tr>
<tr>
<td>Utilization [%]</td>
<td>87</td>
<td>85</td>
<td>83</td>
<td>81</td>
</tr>
</tbody>
</table>

**Concrete shear resistance - Part 1.1: 6.4.3**

L.C.: "Load"

\[
\varepsilon_{\text{Ed}} = \frac{\beta \cdot \varepsilon_{\text{rel,ed}}}{u \cdot d} = \frac{1.15 \cdot 399.691.34}{784 \cdot 160} = 0.77 \text{ N/mm}^2 \quad (6.38)
\]

\[
\varepsilon_{\text{Rel,ex}} = \max (v_{\text{Rel,ex}} \cdot k \cdot (100 - \gamma \cdot f_{cd}^{1/3}, \varepsilon_{\text{hi}}) + k \cdot \sigma_{\text{cr}} =
\]

\[
= \max (0.12 \cdot 2.00 \cdot (100 - 0.0131 \cdot 25.00)^{1/3}, 0.49) + 0.10 \cdot 0.20 = 0.79 \text{ N/mm}^2 \quad (6.47)
\]

\[
\varepsilon_{\text{Ed}} = 0.77 \text{ N/mm}^2 \leq v_{\text{Rel,ca}} = 0.79 \text{ N/mm}^2 \cdot \text{OK}
\]
Verification Examples

9.2.6.2 Circular stirrups

Inputs (see Fig. 9.2.6.2.1):

<table>
<thead>
<tr>
<th>Reinforcement</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{yw,k} = 500 , \text{N/mm}^2$</td>
</tr>
<tr>
<td>Circular stirrup diameter</td>
<td>$\phi_{stirrup} = 10 , \text{mm}$</td>
</tr>
<tr>
<td>Stirrup height</td>
<td>$h_s = 120 , \text{mm}$</td>
</tr>
<tr>
<td>Stirrup width</td>
<td>$w_s = s_r = 120 , \text{mm}$</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
</tbody>
</table>

Concrete compression resistance

The calculation and the results are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Shear reinforcement resistance

The results of the shear resistance of the concrete without punching reinforcement are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Thus it means that we need punching reinforcement.
Punching perimeter at \(2d\) distance from the edge of the column:
\[
u_1 = 2\left(\frac{d_{\text{column}}}{2} + 2d\right)\pi = 2\left(\frac{250}{2} + 2 \cdot 160\right)\pi = 2796 \text{ mm}
\]

The area of the 12 stirrups at the \(u_1\) perimeter (see Fig. 9.2.6.2.1):
\[
A_{sw} = 12 \frac{\phi_{\text{stirrup}}^2 \pi}{4} = 12 \cdot \frac{10^2 \pi}{4} = 942.48 \text{ mm}^2
\]

Effective tensile strength of the stirrups:
\[
f_{y,sw,\text{eff}} = \min \left(\frac{f_{ywd}}{250 + \frac{d}{4}}, \frac{f_{ywk}}{250 + \frac{160}{4}}\right) = \min \left(\frac{500}{250 + \frac{160}{4}}, \frac{290}{2796 - 160}\right) = 290 \text{ N/mm}^2
\]

The punching reinforcement resistance:
\[
v_{Rd,sw} = 1.5 \frac{d_s}{s_c} A_{sw} f_{y,sw,\text{eff}} \frac{1}{u_1d} \sin \alpha = 1.5 \frac{160}{120} 942.48 \cdot 290 \frac{1}{2796 - 160} \sin 90^\circ = 1.222 \frac{\text{N}}{\text{mm}^2}
\]

Punching shear resistance:
\[
v_{Rd,cs} = \min \left[0.75 v_{Rd,c} + v_{Rd,sw}, \frac{1.028}{1.182}\right] = \min \left[0.75 \cdot 0.788 + 1.222, 1.5 \cdot 0.788\right] = 1.182 \text{ N/mm}^2
\]

Shear reinforcement utilization:
\[
\frac{v_{Ed}}{v_{Rd,cs}} = \frac{1.028}{1.182} = 86.97\%
\]

Fig. 9.2.6.2.2 shows the relevant FEM-Design results.

<table>
<thead>
<tr>
<th></th>
<th>Concrete compression</th>
<th>Shear reinforcement</th>
<th>Concrete shear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hand calculation</td>
<td>4.5 N/mm²</td>
<td>1.182 N/mm²</td>
<td>0.788 N/mm²</td>
</tr>
<tr>
<td>FEM-Design calculation</td>
<td>4.5 N/mm²</td>
<td>1.18 N/mm²</td>
<td>0.79 N/mm²</td>
</tr>
<tr>
<td>Difference</td>
<td>0.00%</td>
<td>0.00 %</td>
<td>0.20 %</td>
</tr>
</tbody>
</table>

The differences between the numerical and hand calculations are under 0.5%.
Verification Examples

Figure 9.2.6.2.2 – The punching detailed results in FEM-Design

Shear reinforcement resistance - Part 1.1: 6.4.3

\[ \nu_{Ed} = \frac{\beta \cdot V_{Ed}}{u \cdot d} \] (6.38)

\[ \nu_{fz,sw} = 1.5 \frac{d}{b} \cdot \frac{A_{sw}}{A_{bwat}} \left( \frac{1}{u \cdot d} \cdot \sin(c) \right) \]

\[ \nu_{fz,sw} = \min (0.75 \cdot \nu_{fz,sw} + \nu_{fz,sw} \cdot k_{max}, \nu_{fz,sw}) \]

<table>
<thead>
<tr>
<th>Perimeter index</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Dixt , [mm]} )</td>
<td>320</td>
<td>420</td>
</tr>
<tr>
<td>( \nu_{Ed , [kN]} )</td>
<td>399.69</td>
<td>399.69</td>
</tr>
<tr>
<td>( \beta , [:] )</td>
<td>1.15</td>
<td>1.15</td>
</tr>
<tr>
<td>( d , [mm] )</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td>( \nu_{Ed , [N/mm^2]} )</td>
<td>1.03</td>
<td>0.94</td>
</tr>
<tr>
<td>( \nu_{fz,sw , [N/mm^2]} )</td>
<td>0.79</td>
<td>0.79</td>
</tr>
<tr>
<td>( \nu_{fz,sw , [N/mm^2]} )</td>
<td>1.28</td>
<td>0.99</td>
</tr>
<tr>
<td>( \nu_{fz,sw , [N/mm^2]} )</td>
<td>1.18</td>
<td>1.18</td>
</tr>
<tr>
<td>( \text{Utilization , [%]} )</td>
<td>87</td>
<td>71</td>
</tr>
</tbody>
</table>

Concrete shear resistance - Part 1.1: 6.4.3

LC: ‘Load’

\[ \nu_{Ed} = \frac{\beta \cdot V_{Ed}}{u \cdot d} = \frac{1.15 \cdot 399691.34}{784 \cdot 160} = 3.66 \, N/mm^2 \] (6.38)

\[ \nu_{fz,sw} = \max (C_{fz,sw} \cdot k (100 \cdot \rho_s \cdot f_{cd})^{1/3}, \nu_{min}) + k_1 \cdot \sigma_{sp} = \]

\[ \max (0.12 \cdot 2.00 (100 \cdot 0.0131 - 25.00)^{1/3}, 0.49) + 0.10 \cdot 0.20 = 0.79 \, N/mm^2 \] (6.47)

\[ \nu_{Ed} = 0.68 \, N/mm^2 \leq \nu_{fz,sw} = 0.79 \, N/mm^2 - \text{OK} \]
9.2.6.3 

**Open stirrups**

Inputs:

<table>
<thead>
<tr>
<th>Reinforcement</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{ywk} = 500 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Open stirrup diameter</td>
<td>$\phi_{os} = 4 \text{ mm}$</td>
</tr>
<tr>
<td>Stirrup height</td>
<td>$s_x = 160 \text{ mm}$</td>
</tr>
<tr>
<td>Stirrup width</td>
<td>$s_y = 80 \text{ mm}$</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
</tbody>
</table>

Concrete compression resistance

The calculation and the results are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Shear reinforcement resistance

The results of the shear resistance of the concrete without punching reinforcement are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Thus it means that we need punching reinforcement.

Effective stirrup area:

$$a_{sw} = \frac{2 \phi_{os}^2 \pi}{4} \frac{4^2 \pi}{160 \cdot 80} = 0.001963$$
Effective tensile strength of the open stirrups:

\[
f_{y,sw,eff} = \min \left( \frac{f_{y,wd}}{250 + \frac{d}{4}} \right) = \min \left( \frac{f_{y,swk}}{250 + \frac{d}{4}} \right) = \min \left( \frac{500}{250 + \frac{160}{4}} \right) = \min \left\{ \frac{434.78}{290} \right\} = 290 \ \frac{N}{mm^2}
\]

The punching reinforcement resistance:

\[
v_{Rd,sw} = 1.5 d a_{sw} f_{y,sw,eff} \frac{1}{d} \sin \alpha = 1.5 \cdot 160 \cdot 0.001963 \cdot 290 \cdot \frac{1}{160} \sin 90^\circ = 0.8539 \ \frac{N}{mm^2}
\]

Punching shear resistance:

\[
v_{Rd,cs} = \min \left[ \frac{0.75 v_{Rd,c} + v_{Rd,sw}}{k_{max} v_{Rd,c}} \right] = \min \left[ \frac{0.75 \cdot 0.788 + 0.8539}{1.5 \cdot 0.788} \right] = 1.182 \ \frac{N}{mm^2}
\]

Open stirrups utilization:

\[
\frac{v_{Ed}}{v_{Rd,cs}} = \frac{1.028}{1.182} = 86.97
\]

Fig. 9.2.6.3.2 shows the relevant FEM-Design results.

<table>
<thead>
<tr>
<th></th>
<th>Concrete compression</th>
<th>Shear reinforcement</th>
<th>Concrete shear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hand calculation</td>
<td>4.5 N/mm²</td>
<td>1.182 N/mm²</td>
<td>0.788 N/mm²</td>
</tr>
<tr>
<td>FEM-Design</td>
<td>4.5 N/mm²</td>
<td>1.18 N/mm²</td>
<td>0.79 N/mm²</td>
</tr>
<tr>
<td>Difference</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.20 %</td>
</tr>
</tbody>
</table>

The differences between the numerical and hand calculations are under 0.5%.
**PU.(C.3).1**
**Maximum of load combinations**

C25/30  
<table>
<thead>
<tr>
<th>t</th>
<th>200.00 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>f_{cd,ife}</td>
<td>16.67 N/mm²</td>
</tr>
<tr>
<td>f_{cd,acc,ife}</td>
<td>20.83 N/mm²</td>
</tr>
<tr>
<td>f_{cd}</td>
<td>25.00 N/mm²</td>
</tr>
<tr>
<td>k_{max}</td>
<td>1.50</td>
</tr>
</tbody>
</table>

**Concrete compression resistance - Part 1.1: 6.4.3**

LC: "Load"

\[ v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_{d}} = \frac{1.15 \cdot 399691.34}{784 \cdot 160} = 3.66 \text{ N/mm}^2 \]  \hspace{1cm} (6.38)

\[ v_{Rd,max} = 0.5 \cdot f_{cd} = 0.5 \cdot 16.67 = 8.33 \text{ N/mm}^2 \]

\[ v_{Ed} = 3.66 \text{ N/mm}^2 \leq v_{Rd,max} = 4.50 \text{ N/mm}^2 \]  \hspace{1cm} (6.53) - OK

**Shear reinforcement resistance - Part 1.1: 6.4.3**

\[ v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_{d}} = \frac{1.15 \cdot 399691.34}{5027 \cdot 160} = 0.57 \text{ N/mm}^2 \]  \hspace{1cm} (6.38)

\[ v_{Rd,c} = \min(0.75 v_{Rd,c} + v_{Rd,sw}, k_{max} v_{Rd,c}) \]

<table>
<thead>
<tr>
<th>Parameter index</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dist [mm]</td>
<td>320</td>
<td>420</td>
</tr>
<tr>
<td>V_{Ed} [kN]</td>
<td>399.69</td>
<td>399.69</td>
</tr>
<tr>
<td>( \beta ) [-]</td>
<td>1.15</td>
<td>1.15</td>
</tr>
<tr>
<td>d [mm]</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td>( u ) [m]</td>
<td>2.79</td>
<td>3.42</td>
</tr>
<tr>
<td>V_{Ed} [N/mm²]</td>
<td>1.03</td>
<td>0.84</td>
</tr>
<tr>
<td>V_{Rd,c} [N/mm²]</td>
<td>0.79</td>
<td>0.79</td>
</tr>
<tr>
<td>V_{Rd,sw} [N/mm²]</td>
<td>0.85</td>
<td>0.85</td>
</tr>
<tr>
<td>V_{Rd,sw} [N/mm²]</td>
<td>1.18</td>
<td>1.18</td>
</tr>
<tr>
<td>Utilization [%]</td>
<td>87</td>
<td>71</td>
</tr>
</tbody>
</table>

**Concrete shear resistance - Part 1.1: 6.4.3**

LC: "Load"

\[ v_{Ed} = \frac{\beta \cdot V_{Ed}}{u_{d}} = \frac{1.15 \cdot 399691.34}{5027 \cdot 160} = 0.57 \text{ N/mm}^2 \]  \hspace{1cm} (6.38)

\[ v_{Rd,c} = \max (C_{Rd,c} \cdot k \cdot (0.7 \cdot f_{cd})^{1/3} \cdot \sqrt{v_{Rd,c}} + 0.1 \cdot 0.2) = 0.79 \text{ N/mm}^2 \]  \hspace{1cm} (6.47)

\[ v_{Ed} = 0.57 \text{ N/mm}^2 \leq v_{Rd,c} = 0.79 \text{ N/mm}^2 \] - OK
9.2.6.4 Stud rail general product

Inputs:

<table>
<thead>
<tr>
<th>Reinforcement</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{yw} = 500 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Stud rail diameter</td>
<td>$\phi_s = 10 \text{ mm}$</td>
</tr>
<tr>
<td>Stud rail distances</td>
<td>$s_r = 120 \text{ mm}$</td>
</tr>
<tr>
<td>Stud rail number on one circle</td>
<td>$n = 16 \text{ pcs}$</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
</tbody>
</table>

Concrete compression resistance
The calculation and the results are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Shear reinforcement resistance
The results of the shear resistance of the concrete without punching reinforcement are identical with the relevant part of the former chapter (see Chapter 9.2.6.1).

Thus it means that we need punching reinforcement.

Punching perimeter at $2d$ distance from the edge of the column:

$$u_1 = 2\left(\frac{d_{\text{column}}}{2} + 2d\right)\pi = 2\left(\frac{250}{2} + 2 \cdot 160\right)\pi = 2796 \text{ mm}$$

The area of the 16 stirrups at the $u_1$ perimeter (see Fig. 9.2.6.4.1):
Verification Examples

FEM-Design 19

\[
A_{sw} = 16 \frac{\phi_s^2 \pi}{4} = 16 \frac{10^2 \pi}{4} = 1256 \text{ mm}^2
\]

Effective tensile strength of the open stirrups:

\[
f_{y,sw,eff} = \min \left( \frac{f_{yw}}{250 + \frac{d}{4}} \right) = \min \left( \frac{f_{ywk}}{250 + \frac{d}{4}} \right) = \min \left( \frac{500}{250 + 160} \right) = \min \left( \frac{434.78}{290} \right) = 290 \text{ N/mm}^2
\]

The punching reinforcement resistance:

\[
v_{Rd,sw} = 1.5 \frac{d}{s_c} A_{sw} f_{y,sw,eff} \frac{1}{u_d} \sin \alpha = 1.5 \frac{160}{120} \frac{1}{2796 \cdot 160} \sin 90^\circ = 1.628 \text{ N/mm}^2
\]

Punching shear resistance:

\[
v_{Rd,cs} = \min \left( \frac{0.75 v_{Rd,c} + v_{Rd,sw}}{k_{max} v_{Rd,c}} \right) = \min \left( \frac{0.75 \cdot 0.788 + 1.628}{1.5 \cdot 0.788} \right) = 1.182 \text{ N/mm}^2
\]

Stud rail utilization:

\[
\frac{v_{Ed}}{v_{Rd,cs}} = \frac{1.028}{1.182} = 86.97
\]

Fig. 9.2.6.4.2 shows the relevant FEM-Design results.

<table>
<thead>
<tr>
<th></th>
<th>Concrete compression</th>
<th>Shear reinforcement</th>
<th>Concrete shear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hand calculation</td>
<td>4.5 N/mm²</td>
<td>1.182 N/mm²</td>
<td>0.788 N/mm²</td>
</tr>
<tr>
<td>FEM-Design</td>
<td>4.5 N/mm²</td>
<td>1.18 N/mm²</td>
<td>0.79 N/mm²</td>
</tr>
<tr>
<td>Difference</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.20%</td>
</tr>
</tbody>
</table>

The differences between the numerical and hand calculations are under 0.5%. 
PU.(C.5).1
Maximum of load combinations

C25/30
\[ t = 200.00 \text{ mm} \]
\[ f_{cd,ult.} = 16.67 \text{ N/mm}^2 \]
\[ f_{cd,acc,sec.} = 20.83 \text{ N/mm}^2 \]
\[ f_{ck} = 25.00 \text{ N/mm}^2 \]
\[ k_{\text{max}} = 1.50 \]

<table>
<thead>
<tr>
<th>Perimeter index</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dist [mm]</td>
<td>320</td>
</tr>
<tr>
<td>( V_{Ed} ) [kN]</td>
<td>399.91</td>
</tr>
<tr>
<td>( \beta ) [-]</td>
<td>1.15</td>
</tr>
<tr>
<td>( d ) [mm]</td>
<td>160</td>
</tr>
<tr>
<td>( u ) [m]</td>
<td>2.79</td>
</tr>
<tr>
<td>( V_{Ed} ) [N/mm²]</td>
<td>1.03</td>
</tr>
<tr>
<td>( V_{Rel,c} ) [N/mm²]</td>
<td>0.79</td>
</tr>
<tr>
<td>( V_{Rel,max} ) [N/mm²]</td>
<td>1.63</td>
</tr>
<tr>
<td>( V_{Rel,cr} ) [N/mm²]</td>
<td>1.18</td>
</tr>
<tr>
<td>Utilization [%]</td>
<td>87</td>
</tr>
</tbody>
</table>

Concrete compression resistance - Part 1.1: 6.4.3
LC: 'Load'
\[ V_{Ed} = \frac{\beta \cdot V_{Ed}}{u_0 \cdot d} = \frac{1.15 \cdot 399907.42}{784 \cdot 160} = 3.67 \text{ N/mm}^2 \] (6.38)
\[ V_{Rel,max} = 0.5 \cdot \nu \cdot f_{cd} = 0.5 \cdot 0.54 \cdot 16.67 = 4.50 \text{ N/mm}^2 \]
\[ V_{Ed} = 3.67 \text{ N/mm}^2 \leq V_{Rel,max} = 4.50 \text{ N/mm}^2 \] (6.53) - OK

Shear reinforcement resistance - Part 1.1: 6.4.3
\[ V_{Ed} = \frac{\beta \cdot V_{Ed}}{u_0 \cdot d} \] (6.38)
\[ V_{Rel,cr} = 1.5 \frac{2}{\beta} A_{sw} \frac{f_{max,cr}}{u_0 \cdot d} \sin(\alpha) \]
\[ V_{Rel,cr} = \min(0.75 V_{Rel,c} + V_{Rel,cr}, k_{\text{max}} V_{Rel,c}) \]
9.2.6.5 Stud rail PSB product according to ETA-13/0151

Inputs:

<table>
<thead>
<tr>
<th>Reinforcement</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{yk} = 500 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Stud rail diameter</td>
<td>$\phi_s = 10 \text{ mm}$</td>
</tr>
<tr>
<td>Stud rail distances</td>
<td>$s_r = 120 \text{ mm}$</td>
</tr>
<tr>
<td>Stud rail number on one circle</td>
<td>$n = 8 \text{ pcs}$</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
</tbody>
</table>

Concrete shear resistance

The perimeter of the column:

$u_0 = d_{\text{column}} \pi = 250 \cdot \pi = 785.40 \text{ mm}$

Because this column is an inner column the $\beta$ is:

$\beta = 1.15$

The $u_1$ perimeter:

$u_1 = 2 \left( \frac{d_{\text{column}}}{2} + 2d \right) \pi = 2 \left( \frac{250}{2} + 2 \cdot 160 \right) \pi = 2796 \text{ mm}$

The effective longitudinal reinforcement ratios:
\[
\begin{align*}
\rho_{l,x} &= \frac{\phi_l \pi 1000}{d_{l,x} \cdot 1000} = \frac{20^2 \pi 1000}{4 \cdot 150 \cdot 1000} = 0.01232 ; \quad \rho_{l,y} = \frac{\phi_l \pi 1000}{d_{l,y} \cdot 1000} = \frac{20^2 \pi 1000}{4 \cdot 150 \cdot 1000} = 0.01396 \\
\rho_l &= \min \left( \sqrt{\rho_{l,x} \cdot \rho_{l,y}} \right) = \min \left( \sqrt{0.01232 \cdot 0.01396} \right) = \min \left[ 0.0131 \right] = 0.0131 \\
\end{align*}
\]

The average effective height of the longitudinal reinforcements:
\[
d = \frac{d_{l,x} + d_{l,y}}{2} = \frac{170 + 150}{2} = 160 \text{ mm}
\]

The specific shear force at \( u_1 \):
\[
v_{Ed} = \beta \frac{V_{Ed}}{d u_1} = 1.15 \frac{400000}{160 \cdot 2796} = 1.028 \frac{\text{N}}{\text{mm}^2}
\]

Calculation of \( C_{rd,c} \):
\[
u_0 = \frac{785.4}{160} = 4.91 > 4.0 \quad \text{therefore:}
\]
\[
C_{rd,c} = \frac{0.18}{\gamma_e} = 0.12 ; \quad k_1 = 0.1
\]

Average normal stress in the concrete:
\[
\sigma_{cp} = \min \left( \frac{\sigma_{cx} + \sigma_{cy}}{2} \cdot \frac{n_{x,Ed} + n_{y,Ed}}{1000h} \cdot \frac{0.2}{f_{cd}} \right) = \min \left( \frac{40000 + 40000}{1000 \cdot 200 + 1000 \cdot 200} \cdot \frac{0.2}{25} \cdot \frac{1.5}{1.5} \right) = 0.2 \text{ N/mm}^2
\]

\[
k = \min \left( 1 + \sqrt{\frac{200}{d}} \right) = \min \left( 1 + \sqrt{\frac{200}{160}} \right) = \min \left( 2.12 \right) = 2
\]

\( d \leq 600 \text{ mm} \) therefore:
\[
v_{min} = 0.035 k^{1.5} f_{ck}^{0.5} = 0.035 \cdot 2^{1.5} \cdot 25^{0.5} = 0.495 \text{ N/mm}^2
\]

\[
v_{rd,c} = \max \left( C_{rd,c} k (100 \rho l f_{ck})^{1/2} + k_1 \sigma_{cp} \right) = \max \left( 0.12 \cdot 2 \cdot (100 \cdot 0.01312 \cdot 25)^{1/2} + 0.1 \cdot 0.2 \cdot 0.768 \right) = 0.788 \text{ N/mm}^2
\]
\[ v_{Ed} \geq v_{Rd,c} \geq \nu_{c} \geq \nu_{Rd,c} \geq v_{Ed} \geq v_{Rd,c} \]  
therefore punching reinforcement is needed.

The maximum of punching resistance with punching reinforcement (the example is a slab):

\[ v_{Rd,max} = 1.96 \cdot v_{Rd,c} = 1.96 \cdot 0.788 = 1.544 \frac{N}{mm^2} \]

Thus punching reinforcement is needed.  

\[ v_{Rd,max} \geq v_{Ed} \geq v_{Rd,c} \]  
thus the PSB reinforcement is applicable.

**Shear reinforcement resistance**

The C area is the area closer than 1.125d from the column face (see Fig. 9.2.6.5.1).

- \( m = 8 \) is the number of elements (rows) in the area C.
- \( n = 2 \) is the studs of each element (row) in the area C.
- \( d_a = 10 \text{ mm} \) is the shaft diameter of double headed stud.
- \( \eta = 1.0 \) because \( d < 200 \text{ mm} \).

The total PSB resistance:

\[ V_{Rd,sy} = m \cdot n \cdot d_a^2 \cdot \pi \cdot f_{yk} \cdot \frac{d}{4} \cdot \frac{\gamma_s}{\gamma_y} \cdot \eta = 8 \cdot 2 \cdot 10^2 \cdot \pi \cdot 500 \cdot \frac{\gamma_s}{\gamma_y} = 546.36 \text{ kN} \]

\[ \frac{\beta V_{Ed}}{V_{Rd,sy}} = \frac{1.15 \cdot 400}{546.36} = 84.2 \]  
the PSB resistance is adequate and the detailings are correct.

**Fig. 9.2.6.5.2** shows the relevant FEM-Design results.

<table>
<thead>
<tr>
<th></th>
<th>Concrete shear resistance</th>
<th>Shear reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hand calculation</td>
<td>0.788 N/mm²</td>
<td>546.36 kN</td>
</tr>
<tr>
<td>FEM-Design</td>
<td>0.79 N/mm²</td>
<td>546.36 kN</td>
</tr>
<tr>
<td>Difference</td>
<td>0.20%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

The differences between the numerical and hand calculations are under 0.5%.

Download link to the example file:  
http://download.strusoft.com/FEM-Design/inst180x/models/9.2.6 Punching calculation of a slab.str
Concrete compression resistance - Part 1.1: 6.4.3
Not relevant

Concrete shear resistance (u1) - ETA-13/0151
LC: ‘Load’

\[ v_{Ed} = \frac{\bar{\beta} \cdot V_{Ed}}{u_{Ed} \cdot d} = 1.15 \cdot \frac{399907.42}{2791 \cdot 160} = 1.03 \text{ N/mm}^2 \quad (A1) \]

\[ k = \min \left( 1 + \frac{200}{d} \right)^{0.5}, 2.0 \right) = \min \left( 1 + \frac{200}{160} \right)^{0.5}, 2.0 \right) = 2.00 \]

\[ \rho_i = \min \left( \rho_{1i}, \rho_{2i} \right)^{0.5}, 0.5 \cdot \frac{f_{cs}}{f_{ck}}, 0.02 \right) = \min \left( 0.0131 \cdot 0.0131 \right)^{0.5}, 0.5 \cdot 16.67 / 434.78, 0.02 \right) = 0.0131 \]

\[ v_{Rd,c} = \max \left( C_{Rd,c} \cdot k \cdot \left( 100 \cdot \rho_i \cdot v_{cs} \right)^{1/3}, v_{cs} \right) + k_i \cdot \sigma_{pp} = \max \left( 0.12 \cdot 2.00 \cdot \left( 100 - 0.0131 \cdot 25.00 \right)^{1/3}, 0.49 \right) + 0.10 \cdot 0.20 = 0.79 \text{ N/mm}^2 \quad (A3) \]

\[ v_{Ed} = 0.78 \text{ N/mm}^2 \leq v_{Rd,c} = 0.79 \text{ N/mm}^2 \quad - \text{OK} \]

\[ v_{Ed} = 1.03 \text{ N/mm}^2 \geq v_{Rd,max} = 1.54 \text{ N/mm}^2 \quad - \text{OK} \]

Reinforcement shear resistance (Area C) - ETA-13/0151
LC: ‘Load’

\[ \eta = \max \left( 1.0, \min \left( 1.6, 1.0 + \left( d - 200 \right) \cdot \frac{1.6 - 1.0}{800 - 200} \right) \right) = \max \left( 1.0, \min \left( 1.6, 1.0 + \left( 160 - 200 \right) \cdot \frac{1.6 - 1.0}{800 - 200} \right) \right) = 1.00 \]

\[ V_{Rd,ly} = \eta \cdot u_{Out,C} \cdot A_{p} \cdot f_{yd} / \eta = 16 \cdot 78.54 \cdot 434.78 / 1.00 = 546.36 \text{ kN} \quad (A7) \]

\[ \beta \cdot V_{Ed} = 1.15 \cdot 399.91 = 459.89 \text{ kN} \leq V_{Rd,ly} = 546.36 \text{ kN} \quad - \text{OK} \]

Concrete shear resistance (uOut) - ETA-13/0151
LC: ‘Load’

\[ v_{Ed} = \frac{u_{Out} \cdot V_{Ed}}{u_{Out} \cdot d} = 1.15 \cdot \frac{399907.42}{3669 \cdot 160} = 0.78 \text{ N/mm}^2 \quad (A4) \]

\[ k = \min \left( 1 + \frac{200}{d} \right)^{0.5}, 2.0 \right) = \min \left( 1 + \frac{200}{160} \right)^{0.5}, 2.0 \right) = 2.00 \]

\[ \rho_i = \min \left( \rho_{1i}, \rho_{2i} \right)^{0.5}, 0.5 \cdot \frac{f_{cs}}{f_{ck}}, 0.02 \right) = \min \left( 0.0131 \cdot 0.0131 \right)^{0.5}, 0.5 \cdot 16.67 / 434.78, 0.02 \right) = 0.0131 \]

\[ v_{Rd,c} = \max \left( C_{Rd,c} \cdot k \left( 100 \cdot \rho_i \cdot f_{cy} \right)^{1/3}, v_{cs} \right) + k_i \cdot \sigma_{pp} = \max \left( 0.12 \cdot 2.00 \cdot \left( 100 - 0.0131 \cdot 25.00 \right)^{1/3}, 0.49 \right) + 0.10 \cdot 0.20 = 0.79 \text{ N/mm}^2 \quad (A3) \]

\[ v_{Ed} = 0.78 \text{ N/mm}^2 \leq v_{Rd,c} = 0.79 \text{ N/mm}^2 \quad - \text{OK} \]
9.2.7 Interaction of normal force and biaxial bending in a column

In this section we will calculate the utilization and the load-bearing capacity of a cantilever column under biaxial bending and normal force (see Fig. 9.2.7.1) according to non-linear concrete and reinforcing steel material model.

Inputs:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Concrete characteristic compressive strength</td>
<td>$f_{ck} = 20 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Partial factor of concrete</td>
<td>$\gamma_c = 1.50$</td>
</tr>
<tr>
<td>Effective creep factor</td>
<td>$\phi_{ef} = 2$</td>
</tr>
<tr>
<td>Partial factor of concrete Young's modulus</td>
<td>$\gamma_{cE} = 1.2$</td>
</tr>
<tr>
<td>Reinforcing steel characteristic yield strength</td>
<td>$f_{yk} = 500 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Partial factor of reinforcing steel</td>
<td>$\gamma_s = 1.15$</td>
</tr>
<tr>
<td>Ultimate limit strain of reinforcing steel</td>
<td>$\varepsilon_{uk} = 2.5 %$, $\varepsilon_{ud} = 2.25 %$</td>
</tr>
<tr>
<td>Slope of plastic part (material model)</td>
<td>$k = 1.05$</td>
</tr>
<tr>
<td>Longitudinal bar diameter</td>
<td>$\phi = 20 \text{ mm}$</td>
</tr>
<tr>
<td>Stirrup diameter</td>
<td>$\phi_s = 8 \text{ mm}$</td>
</tr>
<tr>
<td>Concrete cover (on stirrups)</td>
<td>$c = 20 \text{ mm}$</td>
</tr>
<tr>
<td>Column height</td>
<td>$L = 3.0 \text{ m}$</td>
</tr>
<tr>
<td>Column width (square section)</td>
<td>$b = 300 \text{ mm}; h = 300 \text{ mm}$</td>
</tr>
<tr>
<td>The effective depth</td>
<td>$d = 300 - 20 - 8 - 20/2 = 262 \text{ mm}$</td>
</tr>
<tr>
<td>Normal force</td>
<td>$N_{Ed} = 500 \text{ kN}$</td>
</tr>
<tr>
<td>Bending moment in $y'$ direction</td>
<td>$M_{Ed,y} = 10 \text{ kNm}$</td>
</tr>
<tr>
<td>Bending moment in $z'$ direction</td>
<td>$M_{Ed,z} = 10 \text{ kNm}$</td>
</tr>
</tbody>
</table>

*Figure 9.2.7.1 – Cantilever column with the external forces and moments and the RC section - the x direction is the axis of the column, y and z are respect to the moment directions*
In Eurocode 2 there are different methods to calculate the load-bearing capacity of the column. One of them is the *Nominal stiffness method* and another one is the *Nominal curvature method*. We will calculate the utilizations and the load-bearing capacities with independent “hand” calculations respect to both mentioned methods then we will compare the results with FEM-Design calculations.

### 9.2.7.1 Nominal stiffness method

Here we will calculate the nominal bending stiffness of the column and then increase the acting moments to consider second order effects. The Eurocode 2 suggests to increase the moment only in the unfavorable principal direction. In FEM-Design to ensure the most unfavourable result the first order moments will be increased with the effect of the imperfections and second order effects in both principal directions to be on the safe side and get the most unfavourable condition.

The nominal bending stiffness of the column:

\[
EI = K_c E_{cd} I_c + K_s E_s I_s = 0.0566 \cdot 25000 \cdot 6.75 \cdot 10^8 + 1 \cdot 200000 \cdot 1.576 \cdot 10^7 = 4.107 \cdot 10^{12} \text{Nmm}^2,
\]

where:

\[
K_c = \frac{k_1 k_2}{1 + \varphi e} = \frac{1 \cdot 0.1698}{1 + 2} = 0.0566 \quad \text{is a factor for effects of cracking, creep etc.}
\]

\[
k_1 = \sqrt{\frac{f_{ck}}{20}} = \sqrt{\frac{20}{20}} = 1 \quad \text{is a factor which depends on concrete strength.}
\]

\[
k_2 = \min \left\{ \frac{n \lambda}{170}, 0.2 \right\} = \min \left\{ \frac{0.4167 \cdot 69.282}{170}, 0.2 \right\} = 0.1698 \quad \text{is a factor which depends on axial force and slenderness.}
\]

\[
n = \frac{N_{Ed}}{A_c f_{cd}} = \frac{N_{Ed}}{b^2 f_{ck}} = \frac{500000}{300^2 \cdot \frac{20}{1.5}} = 0.4167 \quad \text{is the relative axial force.}
\]

\[
\lambda = \frac{I_0}{i} = \frac{2L}{i} = \frac{2 \cdot 3000}{86.603} = 69.282 \quad \text{is the slenderness ratio.}
\]

\[
i = \sqrt{\frac{I_c}{A_c}} = \sqrt{\frac{b^4}{12}} = \sqrt{\frac{300^4}{12 \cdot 300^2}} = 86.603 \text{mm} \quad \text{is the radius of gyration of the uncracked concrete.}
\]
section.

\[ I_c = \frac{b^4}{12} = \frac{300^4}{12} = 6.75 \cdot 10^8 \text{ mm}^4 \] is the inertia of the uncracked concrete section.

\[ E_{cd} = E_{cm} / \gamma_c = 30000 / 1.2 = 25000 \text{ MPa} \] is the design value of the modulus of elasticity of concrete.

\[ K_s = 1 \] is a factor for contribution of reinforcement.

\[ I_s = A_s \left( d - \frac{b}{2} \right)^2 = 4 \frac{\varphi^2 \pi}{4} \left[ d - \frac{b}{2} \right]^2 = 4 \frac{20^2 \pi}{4} \left[ \frac{262 - 300}{2} \right]^2 = 1.576 \cdot 10^7 \text{ mm}^4 \]

is the second moment of area of reinforcement, about the centre of area of the concrete.

After this we need to increase the first order moments in both direction due to imperfection and second order effects.

Considering the effect of the imperfection (see the former underlined comment also):

\[ M_{0,Ed,y} = M'_{Ed,y} + e_i N_{Ed} = 10000000 + 15 \cdot 500000 = 17.5 \text{ kNm} \]

\[ M_{0,Ed,z} = M'_{Ed,z} + e_i N_{Ed} = 10000000 + 15 \cdot 500000 = 17.5 \text{ kNm} \]

The eccentricity according to the imperfection in both direction:

\[ e_i = \frac{l_0}{400} = \frac{2 L}{400} = \frac{2 \cdot 3000}{400} = 15 \text{ mm} \]

NOTE: In Eurocode 2 the \( \frac{l_0}{400} \) value may be reduced in the function of the column height, but FEM-Design do not consider this effect thus we are on the safe side.

The increased design moment values according to the second order effects (see the former underlined comment also):

\[ M_{Ed,y} = \frac{M_{0,Ed,y}}{1 - \frac{N_{Ed}}{N_B}} = \frac{17.5}{1 - \frac{500}{1126.2}} = 31.47 \text{ kNm} \quad \text{and} \]

\[ M_{Ed,z} = \frac{M_{0,Ed,z}}{1 - \frac{N_{Ed}}{N_B}} = \frac{17.5}{1 - \frac{500}{1126.2}} = 31.47 \text{ kNm} \]
where $N_B$ is the buckling load based on nominal stiffness of the column:

$$N_B = \frac{\pi^2 EI}{L_0^2} = \frac{\pi^2 \times 4.107 \times 10^{12}}{(2 \cdot 3000)^2} = 1126000 \text{ N} = 1126 \text{ kN}$$

Now we need to calculate the resistance (load-bearing capacity). For this the M-N interaction curve (bending moment – normal force interaction curve) in the principal directions or the M-N interaction surface is necessary. In Eurocode 2 there is a simplified method to check the utilization with the aid of the M-N interaction curves in the principal directions but there is a more accurate solution for the problem with the help of the M-N interaction surface. Let’s see what is the difference between these calculation methods and then we show you what is the FEM-Design solution for this problem.

First of all with independent numerical calculation we provided the M-N interaction curve in the principal directions. According to the square and double-symmetric cross section these two curves will be identical each other (see Fig. 9.2.7.2).

At the given design normal force value ($N_{Ed} = 500 \text{ kN}$) the moment resistance with numerical calculation in both $y$ and $z$ direction is: $M_{Rd,y} = M_{Rd,z} = 104.1 \text{ kNm}$ (see Fig. 9.2.7.2).

The utilization based the approximation formula of Eurocode 2 is the following:

$$\left(\frac{M_{Ed,y}}{M_{Rd,y}}\right)^a + \left(\frac{M_{Ed,z}}{M_{Rd,z}}\right)^a = \left(\frac{31.47}{104.1}\right)^{1.155} + \left(\frac{31.47}{104.1}\right)^{1.155} = 50.23\%$$

Figure 9.2.7.2 – The M-N interaction curve in the x-y or in the x-z plain
The normal force utilization is:

\[
\frac{N_{Ed}}{N_{Rd}} = \frac{A_c f_{cd} + A_s f_{yd} \rho_s}{b^2 f_{cd} + 4 \frac{\Phi^2 \pi}{4} f_{yd}} = \frac{500000}{300^2 13.33 + 4 \left( \frac{20^2 \pi}{4} \right) 435} = 0.2863
\]

The “a” value can be linearly interpolated based on the following table:

<table>
<thead>
<tr>
<th>( \frac{N_{Ed}}{N_{Rd}} )</th>
<th>0.1</th>
<th>0.7</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
</tr>
</tbody>
</table>

The M-N curves were calculated based on the material models what you can see in Fig. 9.2.7.3. The strain values for concrete coincide with the values what were mentioned in Chapter 9.2.4.

NOTE: Due to numerical stability FEM-Design uses a bit modified concrete material model according to Fig. 9.2.4.3 but by ultimate limit state with the design stress values.

Concrete:

\[
\sigma_c (\varepsilon_c) = \begin{cases} 
  f_{cd} & \text{if } 0 \leq \varepsilon_c \leq \varepsilon_{c2} \\
  f_{cd} \left[ 1 - \left( \frac{\varepsilon_c}{\varepsilon_{c2}} \right)^2 \right] & \text{if } \varepsilon_{c2} < \varepsilon_c \leq \varepsilon_{cu2}
\end{cases}
\]

Rebars:

\[
\sigma_s (\varepsilon_s) = \begin{cases} 
  \varepsilon_s E_s & \text{if } \varepsilon_s \leq \frac{f_{yd}}{E_s} \\
  f_{yd} + (k-1) \frac{\varepsilon_s f_{yd}}{E_s} & \text{if } \frac{f_{yd}}{E_s} < \varepsilon_s \leq \varepsilon_{ad}
\end{cases}
\]

Figure 9.2.6.3 – The material models for the M-N curve calculations
left: concrete (only compression), right: rebars (both tension and compression)
Fig. 9.2.7.4 shows a slice of the M-N surface in the direction of 45° of angle between \(y\) and \(z\) axes. Based on this figure we can calculate the utilization by hand in a more accurate way. At the \(N_{Ed} = 500\) kN level the maximum moment capacity is 81.96 kNm for the 45° direction (see Fig. 9.2.7.4). Based on this moment capacity we can calculate the maximum moment capacity in the principal directions according to the equal design moments in \(y\) and \(z\) directions:

\[
M_{Rd, y}^{\text{precise}}(N_{Ed}) = M_{Rd, z}^{\text{precise}}(N_{Ed}) = \frac{81.96}{\sqrt{2}} = 57.95\text{ kNm}
\]

Due to this value we can calculate the “\(a\)” value more precisely based on the fact that in this example the increased moment values are the same in the two directions. The following expression must be true:

\[
2 \left( \frac{57.95}{104.1} \right)^{a^{\text{precise}}} = 1.00 \quad \text{and based on this:} \quad a^{\text{precise}} = 1.183
\]

Thus the more precise utilization is:

\[
\left( \frac{M_{Ed, y}}{M_{Rd, y}} \right)^{a^{\text{precise}}} + \left( \frac{M_{Ed, z}}{M_{Rd, z}} \right)^{a^{\text{precise}}} = \left( \frac{31.47}{104.1} \right)^{1.183} + \left( \frac{31.47}{104.1} \right)^{1.183} = 48.57\%
\]

This means that the Eurocode formula is on the safe side based on this utilization type.

---

*Figure 9.2.7.4 – A slice of the M-N surface between \(y\) and \(z\) axes with 45° of angle*
Fig. 9.2.7.4 shows a slice of the M-N surface in the 45° of angle direction between y and z axes (which is the relevant direction in this example, see the increased design moment values in the two directions). In FEM-Design the utilization is the following:

\[ N_{Ed} = \eta N_{Rd} \quad ; \quad M_{Ed,y} = \eta M_{Rd,y} \quad ; \quad M_{Ed,z} = \eta M_{Rd,z} \]

where \( \eta \) is the utilization, it means that if we increase the normal force and the two bending moments linearly at the same time we will reach the failure surface of the cross-section (see the red line in Fig. 9.2.7.4). With other words the eccentricity is constant.

According to this scenario the utilization with the independent “hand” calculation is the following:

The red line breaks through the M-N curve at (see Fig. 9.2.7.4):

\[ N_{Rd} = 816.0 \, \text{kN} \quad , \quad M_{Rd} = 72.63 \, \text{kNm} \]

The utilization of the column based on the red line geometrical point of view according to Fig. 9.2.7.4:

\[ \eta = \frac{\sqrt{M_{Ed,y}^2 + M_{Ed,z}^2 + N_{Ed}^2}}{\sqrt{M_{Rd}^2 + N_{Rd}^2}} = \frac{\sqrt{31.47^2 + 31.47^2 + 500^2}}{\sqrt{72.63^2 + 816^2}} = 61.27\% \]

Utilization based on the FEM-Design detailed results is (see Fig. 9.2.7.5): \( \eta = 64\% \)

The difference between the “hand” and numerical calculations is less than 5%. This difference comes from the fact that FEM-Design uses a bit modified concrete material model (see. Fig. 9.2.4.3, but here the stress values were replaced by the design values) to ensure the numerical stability for every case.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.2.7 Interaction of normal force and biaxial bending in a column.str
Figure 9.2.7.5 - The detailed results of the RC column in FEM-Design with nominal stiffness method.

### Section utilization for axial effects (Part 1.1: 5.8, 5.1)

<table>
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<tr>
<th>Sections</th>
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<tr>
<td>Φ</td>
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</tr>
<tr>
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<td>Φ''</td>
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<tr>
<td>Φ''''</td>
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<tr>
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<tr>
<td>(E_d, [kN/m³])</td>
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<tr>
<td>(E_d, [kN/m³])</td>
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</tr>
<tr>
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<tr>
<td>(E_d, [kN/m³])</td>
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<td>(E_d, [kN/m³])</td>
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<td>112.16</td>
</tr>
<tr>
<td>(E_d, [kN/m³])</td>
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</tr>
<tr>
<td>(E_d, [kN/m³])</td>
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</tr>
<tr>
<td>(E_d, [kN/m³])</td>
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</tr>
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</table>

### Ultimate limit state

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<td>Φ''''''</td>
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<tr>
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<td>4.1078e+12</td>
</tr>
<tr>
<td>(E_d, [kN/m³])</td>
<td>112.16</td>
</tr>
<tr>
<td>(E_d, [kN/m³])</td>
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<td>(E_d, [kN/m³])</td>
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<tr>
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<tr>
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<tr>
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</tr>
<tr>
<td>(E_d, [kN/m³])</td>
<td>-31.47</td>
</tr>
</tbody>
</table>

### Section utilization

**Ultimate internal forces:** $N_u = N_{bu}, M_{ud1} = M_{bu1}, M_{ud2} = M_{bu2}$

**Utilization:** $U_N$

---

### Verification Examples

**FEM-Design 19**

3rd order effects are considered according to nominal stiffness method (Part 1.1.5.8.7)

### Tables

<table>
<thead>
<tr>
<th>Sections</th>
<th>Y</th>
</tr>
</thead>
<tbody>
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<td>LC</td>
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<tr>
<td>Φ</td>
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</tr>
<tr>
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<td>(E_d, [kN/m³])</td>
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**Sections**

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<td>Φ''</td>
<td>0.17</td>
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<tr>
<td>Φ''''</td>
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</tr>
<tr>
<td>(E_d, [kN/m³])</td>
<td>-31.47</td>
</tr>
</tbody>
</table>

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**Ultimate internal forces:** $N_u = N_{bu}, M_{ud1} = M_{bu1}, M_{ud2} = M_{bu2}$

**Utilization:** $U_N$
**9.2.7.2 Nominal curvature method**

Here we will calculate the nominal curvature of the column and then increase the acting moments based on this curvature to consider second order effects. The Eurocode 2 suggests to increase the moment only in the unfavorable principal direction. In FEM-Design to ensure the most unfavourable result the first order moments will be increased with the effect of the imperfections and second order effects in both principal directions to be on the safe side and get the most unfavourable condition.

Considering the effect of the imperfection:

\[ M_{0,Ed,y} = M_{Ed,y} + e_iN_{Ed} = 10000000 + 15 \cdot 500000 = 17.5 \text{kNm} \]

\[ M_{0,Ed,z} = M_{Ed,z} + e_iN_{Ed} = 10000000 + 15 \cdot 500000 = 17.5 \text{kNm} \]

The eccentricity according to the imperfection in both direction:

\[ e_i = \frac{l_0}{400} = \frac{2L}{400} = \frac{2 \cdot 3000}{400} = 15 \text{mm} \quad \text{(see also the relevant NOTE in Chapter 9.2.6.1).} \]

The second order eccentricity:

\[ e_2 = \frac{1}{r} \frac{r^2}{c} = \frac{1.816 \cdot 10^{-5} \cdot 6000^2}{10} = 65.38 \text{mm} \]

where the curvature:

\[ \frac{1}{r} = K_r K_r \frac{1}{r_0} = 0.9842 \cdot 1.845 \cdot 10^{-5} = 1.816 \cdot 10^{-5} \frac{1}{\text{mm}} \]

where:

\[ K_r = \min \left\{ \frac{n_u - n}{n_u - n_{bal}} \right\} = \frac{1.4554 - 0.4167}{1.4554 - 0.4} = 0.9842 \quad \text{is a correction factor depending on axial load.} \]

\[ n_u = 1 + \omega = 1 + 0.4554 = 1.4554 \quad \omega = \frac{A_i f_{yd}}{A_c f_{cd}} = \frac{4 \frac{\phi^2 \pi f_{yk}}{Y_s}}{4 \frac{4}{1.15} \frac{500}{300} \frac{20}{1.5}} = 0.4554 \]
The relative axial normal force:

\[
n = \frac{N_{Ed}}{A_c f_{cd}} = \frac{500000}{\frac{b^2 f_{ck}}{Y_c}} = 0.4167 \quad n_{bal} = 0.4
\]

\[
K_v = \max\left\{1 + \beta \frac{f_{ck}}{\phi_{c f}}, 1\right\} = 0.9762 \quad \text{is a factor for taking account of creep.}
\]

Where:

\[
\beta = 0.35 + \frac{f_{ck}}{200} - \frac{\lambda}{150} = 0.35 + \frac{20}{200} - \frac{69.282}{150} = -0.01188
\]

\[
\lambda = \frac{l_0}{i} = \frac{2 \cdot 3000}{86.603} = 69.282 \quad \text{the slenderness ratio (see Chapter 9.2.7.1 also).}
\]

The increased design moment values according to the second order effects:

\[
M_{Ed,y} = M_{0,Ed,y} + N_{Ed} e_2 = 17.5 + 500 \cdot 0.06538 = 50.19 \text{kNm} \quad \text{and}
\]

\[
M_{Ed,z} = M_{0,Ed,z} + N_{Ed} e_2 = 17.5 + 500 \cdot 0.06538 = 50.19 \text{kNm}
\]

*Figure 9.2.7.6 – A slice of the M-N surface between y and z axes with 45° of angle*
Fig. 9.2.7.6 shows a slice of the M-N surface in the 45° of angle direction between \( y \) and \( z \) axes (which is the relevant direction in this example, see the increased design moment values in the two directions). In FEM-Design the utilization is the following:

\[
N_{Ed} = \eta N_{Rd} \quad ; \quad M_{Ed,y} = \eta M_{Rd,y} \quad ; \quad M_{Ed,z} = \eta M_{Rd,z}
\]

where \( \eta \) is the utilization, it means that if we increase the normal force and the two bending moments linearly at the same time we will reach the failure surface of the cross-section (see the red line in Fig. 9.2.7.6). With other words the eccentricity is constant.

According to this scenario the utilization with the independent “hand” calculation is the following:

The red line breaks through the M-N curve at (see Fig. 9.2.7.6):

\[
N_{Rd} = 566.0 \text{kN} , M_{Rd} = 80.35 \text{kNm}
\]

The utilization of the column based on the red line geometrical point of view according to Fig. 9.2.7.6:

\[
\eta = \frac{\sqrt{M_{Ed,y}^2 + M_{Ed,z}^2 + N_{Ed}^2}}{\sqrt{M_{Rd}^2 + N_{Rd}^2}} = \frac{\sqrt{50.19^2 + 50.19^2 + 500^2}}{\sqrt{80.35^2 + 566.0^2}} = 88.33\%
\]

Utilization based on the FEM-Design detailed results (see Fig. 9.2.7.7): \( \eta = 92.7\% \)

The difference between the “hand” and numerical calculations is around 5%. This difference comes from the fact that FEM-Design uses a bit modified concrete material model (see. Fig. 9.2.4.3, but here the stress values were replaced by the design values) to ensure the numerical stability for every case.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.2.7 Interaction of normal force and biaxial bending in a column.str
Verification Examples

Figure 9.2.7.7 – The detailed results of the RC column in FEM-Design with nominal stiffness method

Section utilization for axial effects (Part 1.1.6.8.6.1)

v_i = \frac{1}{v} \quad (5.14)

2nd order effects is considered according to nominal curvature method (Part 1.1.6.8.9)

\alpha = \frac{1}{7.5} \quad (5.30)

n = \frac{N_{uk}}{(A_{s} \cdot f_{y})}

n_{w} = 0.4

\beta = 0.36 \quad (5.31)

\nu_{w} = \max\left(1, \frac{\beta \cdot n_{w}}{1.0}\right) \quad (5.37)

\frac{a_{w}}{a} = \frac{1}{0.25} \quad (6.27)

M_{w} = M_{uk} \cdot n_{w}

\sigma_{sw} = \max(20 \text{ mm}, h_{c} / 50) = \max(20 \text{ mm}, h_{c} / 30) = 20 \text{ mm}

\sigma_{sw} = \max(30 \text{ mm}, h_{c} / 30) = \max(30 \text{ mm}, h_{c} / 30) = 20 \text{ mm}

[NM_{w} \leq \nu_{w} \cdot M_{w} \rightarrow N_{w} = n_{w} \cdot N_{uk}]

Section utilization

Ultimate internal forces: \nu_{w} = N_{w}, M_{uk} = M_{uk}, M_{uk} = M_{uk}

Ultimate load: \nu_{w} = \frac{1}{\nu_{w}}

Stresses and strains (Part 1.1.6.12, 6.18, 3.1.7)

\nu_{w} = \frac{1}{\nu_{w}}
9.2.8 Calculation of a statically indeterminate beam with post tensioned cables

Inputs:

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Characteristic tensile strength of steel</td>
<td>( f_{pk} = 1860 \text{ MPa} )</td>
</tr>
<tr>
<td>Cross-section of one strand</td>
<td>( A_p = 150 \text{ mm}^2 )</td>
</tr>
<tr>
<td>Number of strands</td>
<td>( n = 2 )</td>
</tr>
<tr>
<td>Curvature coefficient</td>
<td>( \mu = 0.05 )</td>
</tr>
<tr>
<td>Wobble coefficient</td>
<td>( \kappa = 0.007 \text{ 1/m} )</td>
</tr>
<tr>
<td>Anchorage set slip</td>
<td>( g = 4 \text{ mm} )</td>
</tr>
<tr>
<td>Young's modulus of the strand</td>
<td>( E_p = 195 \text{ MPa} )</td>
</tr>
<tr>
<td>Jacking side</td>
<td>point A</td>
</tr>
<tr>
<td>Young's modulus of concrete when post tensioning applied</td>
<td>( E_{cm} = E_{cm}(t) = 30 \text{ GPa} )</td>
</tr>
<tr>
<td>The final value of creep coefficient</td>
<td>( \phi(t,t_0) = 2.00 )</td>
</tr>
<tr>
<td>Final value of shrinkage</td>
<td>( \varepsilon_{cs} = 0.4 \text{ ‰} )</td>
</tr>
<tr>
<td>Relaxation class of the strands</td>
<td>Class 2</td>
</tr>
<tr>
<td>Value of relaxation loss</td>
<td>( \rho_{1000} = 2.5 % )</td>
</tr>
<tr>
<td>The cross-section of the beam is rectangle</td>
<td>( b = 300 \text{ mm}, h = 600 \text{ mm} )</td>
</tr>
</tbody>
</table>

In this example we would like to calculate the equivalent forces of a post tensioned cable system before and after the long term stress losses and compare these results with FEM-Design results. For these calculations we need the data which were indicated above in the table and we need to know the shape of the cables. The geometry of the statically indeterminate beam and the shape of the cables are shown in Fig. 9.2.8.1.

The angular deviation of the cables is a function starting at point A and it depends on the shape of the cables (e.g. base points and inflections). Now we have parabolic shapes on each parts (see Fig. 9.2.8.1).
The angular deviation function: \( \alpha(x) \)

The values of the angular deviation function at some typical points:

\[
\begin{align*}
\alpha(A) &= 0 \text{ rad} ; \\
\alpha(B) &= \alpha(A) + 2 \arctan \left( \frac{4 f_1}{L_1} \right) = 0.6039 \text{ rad} ; \\
\alpha(C) &= \alpha(B) + 2 \arctan \left( \frac{4 f_2}{L_2} \right) = 1.215 \text{ rad} ; \\
\alpha(D) &= \alpha(C) + 2 \arctan \left( \frac{4 f_1}{L_1} \right) = 1.819 \text{ rad}
\end{align*}
\]

Based on these values the function of the angular deviation, see Fig. 9.2.8.2.

Let's be the stress at the active jacking end (point A) during post tensioning:

\[
\sigma_{p0} = 0.8 f_{pk} = 1488 \text{ MPa}
\]

NOTE:

By a real design task according to the Eurocode 2 we need to consider an upper bound for this value, e.g.: \( \sigma_{p0,max} = \min(0.8 f_{pk}, 0.9 f_{p0,\text{uk}}) \), but by this example we do not consider it because it does not affect the method of the following calculation. In FEM-Design the user should give the \( \sigma_{p0} \) value and all of the calculations will consider this input stress as jacking stress.

Before the calculation of the equivalent forces which come from the post tensioning of the beam, first of all we need to calculate the different stress losses.

Firstly we consider the stress losses which come from the technology of post tensioning.

The stresses in the strands are also a function considering the losses due to friction.
The function of the stresses considering losses due to friction: $\sigma_{pl}(x)$

The values of the function of the stresses considering losses due to friction at some typical points:

$\sigma_{pl}(A) = \sigma_{p0} = 1488$ MPa

$\sigma_{pl}(B) = \sigma_{p0}e^{-\mu(\alpha + \kappa L_s)} = 1488e^{-0.05(0.6039+0.007\cdot4.24)} = 1442$ MPa

$\sigma_{pl}(C) = \sigma_{p0}e^{-\mu(\alpha + \kappa (L_s+L_1))} = 1488e^{-0.05(1.215+0.007\cdot4.24+1.52)} = 1397$ MPa

$\sigma_{pl}(D) = \sigma_{p0}e^{-\mu(\alpha + \kappa (L_s+L_1+L_2))} = 1488e^{-0.05(1.819+0.007\cdot4.24+1.52+4.24)} = 1354$ MPa

Based on these values the function of the stresses considering the losses due to friction, see Fig. 9.2.8.3.

![Figure 9.2.8.3 – The function of the stresses considering losses due to friction $[\sigma_{pl}(x)]$](image)

Now we need to calculate the stresses considering the losses due to the anchorage set slip.

The jacking is applied at the start point A.

The length of the effect of the anchorage set slip ($L_{sl}$) comes from the following equation:

$$g \cdot E_p = 2 \int_0^{L_{sl}} (\sigma_{pl}(x) - \sigma_{pl}(L_{sl})) dx = 2 \int_0^{L_{sl}} (\sigma_{pl}(x) - \sigma_{p0}e^{-\mu(\alpha + \kappa L_s)}) dx$$

The solution of this equation after some iterations is:

$L_{sl} = 6.667$ m

The stress loss at the active jacking side due to anchorage set slip:

$$\Delta \sigma_{sl} = 2 \left[ \sigma_{p0} \left[ 1 - e^{-\mu(\alpha + \kappa L_s)} \right] \right] = 2 \left[ 1488 \left[ 1 - e^{-0.05(1.344+0.007\cdot6.667)} \right] \right] = 200$ MPa

The stresses in the strands are also a function considering the anchorage set slip.

The function of the stresses considering losses due to anchorage set slip and friction: $\sigma_{p2}(x)$
The values of this function at some typical points:

\[ \sigma_{p2}(A) = -\sigma_{p1}(A) + 2\sigma_{p0} - \Delta\sigma_{s1} = -1488 + 2 \cdot 1488 - 200 = 1288 \text{ MPa} \]

\[ \sigma_{p2}(B) = -\sigma_{p1}(B) + 2\sigma_{p0} - \Delta\sigma_{s1} = -1442 + 2 \cdot 1488 - 200 = 1334 \text{ MPa} \]

\[ \sigma_{p2}(C) = -\sigma_{p1}(C) + 2\sigma_{p0} - \Delta\sigma_{s1} = -1397 + 2 \cdot 1488 - 200 = 1379 \text{ MPa} \]

\[ \sigma_{p2}(L_{s1}) = -\sigma_{p1}(L_{s1}) + 2\sigma_{p0} - \Delta\sigma_{s1} = -1388 + 2 \cdot 1488 - 200 = 1388 \text{ MPa} \]

\[ \sigma_{p2}(D) = \sigma_{p1}(D) = 1354 \text{ MPa} \]

Based on these values the function of the stresses considering the losses due to anchorage set slip and friction, see Fig. 9.2.8.4.

The average stress value in the strands after the losses due to anchorage set slip and friction:

\[ \sigma_m = \frac{\int_0^{2L} \sigma_{p2}(x) \, dx}{2L} = \frac{13445}{10} = 1344.5 \text{ MPa} \]

NOTE:

By a real design task according to the Eurocode 2 we need to consider an upper bound for this value, e.g.: \( \sigma_{m,\max} = \min(0.75 f_{pk}, 0.85 f_{p0,1k}) \), but by this example we do not consider it because it does not affect the remaining calculation. In FEM-Design the program is also calculating an average stress after considering the losses due to anchorage set slip and friction and using this value to estimate the short and long term stress losses.

We need to consider the short and long term stress losses in the strands after the jacking.
Next to the frictional and anchorage set slip losses another short term loss is the losses due to elastic shortening.

The average normal stress in the concrete cross-section at the level of the anchorages:

\[
\sigma_c = \frac{n \sigma_m A_p}{A_c} + \frac{n \sigma_m A_p e_1}{I_c} = \frac{2 \cdot 1344.5 \cdot 150}{300 \cdot 600} + \frac{2 \cdot 1344.5 \cdot 150 \cdot 100}{300 \cdot 600^{3/12}} \approx 2.988 \text{ MPa}
\]

The average elastic stress losses according to Eurocode 2:

\[
\Delta \sigma_{el} = \frac{n-1}{2n} \sigma_c \frac{E_p}{E_{cm}(t)} = \frac{2-1}{2} \cdot \frac{2.988}{30} = 4.856 \text{ MPa}
\]

The average stress in the strands after the elastic shortening:

\[
\sigma_u = \sigma_m - \Delta \sigma_{el} = 1344.5 - 4.856 = 1341.5 \text{ MPa}
\]

The average stress in concrete at the level of the anchorages considering the elastic shortening:

\[
\sigma_{c,u} = \frac{n \sigma_u A_p}{A_c} + \frac{n \sigma_u A_p e_1}{I_c} = \frac{2 \cdot 1341.5 \cdot 150}{300 \cdot 600} + \frac{2 \cdot 1341.5 \cdot 150 \cdot 100}{300 \cdot 600^{3/12}} \approx 2.981 \text{ MPa}
\]

After the short term stress losses we can calculate the long term stress losses in the strands.

The stress losses due to creep:

\[
\Delta \sigma_{cr} = \frac{E_p}{E_{cm}} \varphi(t, t_0) \sigma_{c,u} = \frac{195}{30} \cdot 2.00 \cdot 2.981 = 38.75 \text{ MPa}
\]

The stress losses due to shrinkage:

\[
\Delta \sigma_s = E_p \epsilon_{cs} = 195 \cdot 0.4 = 78 \text{ MPa}
\]

The stress losses due to relaxation:

\[
\mu_p = \frac{\sigma_u}{f_{pk}} = \frac{1341.5}{1860} = 0.72
\]

\[
\Delta \sigma_{pr} = 0.66 \rho_{1000} e^{9.1 \mu_p} \left( \frac{t}{1000} \right)^{0.75(1-\mu_p)} 10^{-2} \sigma_u = 0.66 \cdot 2.5 e^{9.1 \cdot 0.72} \left( \frac{500000}{1000} \right)^{0.75(1-0.72)} 10^{-5} \cdot 1341.5
\]

(Relaxation class: 2.)

\[
\Delta \sigma_{pr} = 57.19 \text{ MPa}
\]
These three effects, namely the creep, shrinkage and the relaxation are in interaction. The stress losses due to the interaction:

\[
\Delta \sigma_{p,c+s+r} = \frac{\Delta \sigma_{c} + 0.8 \Delta \sigma_{pr} + \Delta \sigma_{cr}}{1 + \frac{E_p}{E_{cm}} \frac{n A_p}{A_c} \left(1 + \frac{A_c}{I_c} \frac{z_{cp}}{z_{cp}}\right) [1 + 0.8 q(t, t_0)]}
\]

\[
= \frac{78 + 0.8 \cdot 57.19 + 38.75}{1 + \frac{195}{30} \cdot \frac{2 \cdot 150}{300 \cdot 600} \left(1 + \frac{300 - 600}{300 \cdot 600^2 / 12}\right) [1 + 0.8 \cdot 2.00]} = 156.6 \text{ MPa}
\]

The average stress in the strands before the long term losses (T0):

\[\sigma_{ai} = \sigma_{m} - \Delta \sigma_{el} = 1344.5 - 4.856 = 1341.5 \text{ MPa}\]

The average stress in the strands after the long term losses (T∞):

\[\sigma_{i} = \sigma_{ai} - \Delta \sigma_{p,c+s+r} = 1344.5 - 156.6 = 1187.9 \text{ MPa}\]

Based on these values we can calculate the equivalent forces which will represent the effect of the post tensioning on the statically indeterminate beam.

The equivalent forces at T0 time before the long term losses:

The concentrated forces at the ends (at the centroid of the concrete cross-section):

\[P_{0} = n A_p \sigma_{ai} = 2 \cdot 150 \cdot 1341.5 = 402.5 \text{ kN}\]

The angle of the tangent of the cable at the ends:

\[\alpha = \tan \frac{4 f_1}{L_1} = \tan \frac{4 \cdot 330}{4240} = 17.29^\circ\]

The horizontal and vertical components of the concentrated forces at the ends:

\[P_{0H} = P_{0} \cos \alpha = 402.5 \cdot \cos 17.29^\circ = 384.3 \text{ kN}\]

\[P_{0V} = P_{0} \sin \alpha = 402.5 \cdot \sin 17.29^\circ = 119.5 \text{ kN}\]

The concentrated moments at the ends according to the eccentricities at the ends:

\[M_{0} = P_{0} e_1 = 384.3 \cdot 0.1 = 38.4 \text{ kNm}\]
The intensity of the distributed load according to the different parabola shapes:

\[ u_{01} = \frac{8 P_0 f_1}{L_1^2} = \frac{8 \cdot 402.5 \cdot 0.33}{4.24^2} = 59.1 \text{ kN/m} \]

\[ u_{02} = \frac{8 P_0 f_2}{L_2^2} = \frac{8 \cdot 402.5 \cdot 0.12}{1.52^2} = 167.3 \text{ kN/m} \]

The equivalent forces at \( T_\infty \) time after the long term losses:

The concentrated forces at the ends (at the centroid of the concrete cross-section):

\[ P_\infty = n A_p \alpha = 2 \cdot 150 \cdot 1187.9 = 356.4 \text{ kN} \]

The horizontal and vertical components of the concentrated forces at the ends:

\[ P_{\infty H} = P_\infty \cos \alpha = 356.4 \cdot \cos 17.29^\circ = 340.3 \text{ kN} \]

\[ P_{\infty V} = P_\infty \sin \alpha = 356.4 \cdot \sin 17.29^\circ = 105.9 \text{ kN} \]

The concentrated moments at the ends according to the eccentricities at the ends:

\[ M_\infty = P_{\infty H} e_1 = 340.3 \cdot 0.1 = 34.0 \text{ kNm} \]

The intensity of the distributed load according to the different parabola shapes:

\[ u_{\infty 1} = \frac{8 P_\infty f_1}{L_1^2} = \frac{8 \cdot 356.4 \cdot 0.33}{4.24^2} = 52.3 \text{ kN/m} \]

\[ u_{\infty 2} = \frac{8 P_\infty f_2}{L_2^2} = \frac{8 \cdot 356.4 \cdot 0.12}{1.52^2} = 148.1 \text{ kN/m} \]

Fig. 9.2.8.5 shows the equivalent forces on the statically indeterminate beam before and after the long term stress losses in the strands.
Considering the equivalent forces after the long term stress losses ($T\infty$, Fig. 9.2.8.5 below) the camber (at the mid-span) of the statically indeterminate beam (without detailed calculation according to the theory of elasticity) is:

$$e_{p,\text{camber}} = 2.86 \text{ mm}$$

By this camber calculation we considered only the effect of the equivalent forces (Fig. 9.2.8.5 below) and the effective modulus of elasticity of concrete: $E_{c,\text{eff}} = E_c/(1+\phi(t,t_0)) = 10 \text{ GPa}$.

Fig. 9.2.8.6. shows the shape of the post tensioned cable in FEM-Design.
We can say that the results of the hand calculation and the automatic post tensioned cable calculation of FEM-Design are identical. See the FEM-Design results about the equivalent forces in Fig. 9.2.8.7 and the camber in Fig. 9.2.8.8.

Keep in mind that FEM-Design post tensioned cable modul calculates the equivalent forces in more precise way than this hand calculation and considers the curvatures of the shape of the cable in more accurate way. Theoretically the calculated equivalent forces are in equilibrium but the presented hand calculation method does not consider the friction force (and its eccentricities), but FEM-Design is checking the equilibrium of the equivalent forces and automatically applying a distributed axial force and a distributed bending moment on the beam if it is necessary. These forces and moments are also indicated in the program by the post tensioned equivalent load cases.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.2.8 Calculation of a statically indeterminate beam with post tensioned cables.str
9.3 Steel design

9.3.1 Interaction of normal force, bending moment and shear force

In this sub-chapter an IPE 200 beam will be investigated under the interaction of normal force, shear force and bending moment around its strong axis (see Fig. 9.3.1.1). Stability analysis will not be considered here only strength resistance calculations.

Inputs:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield strength of structural steel</td>
<td>( f_y = 235 \text{ N/mm}^2 )</td>
</tr>
<tr>
<td>Cross-sectional width</td>
<td>( b = 100 \text{ mm} )</td>
</tr>
<tr>
<td>Cross-sectional height</td>
<td>( h = 200 \text{ mm} )</td>
</tr>
<tr>
<td>Flange thickness</td>
<td>( t_f = 8.5 \text{ mm} )</td>
</tr>
<tr>
<td>Web thickness</td>
<td>( t_w = 5.6 \text{ mm} )</td>
</tr>
<tr>
<td>Web height</td>
<td>( h_w = 159 \text{ mm} )</td>
</tr>
<tr>
<td>Radius of root fillet</td>
<td>( r = 12 \text{ mm} )</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>( A = 2848 \text{ mm}^2 )</td>
</tr>
<tr>
<td>Plastic cross-sectional modulus</td>
<td>( W_{pl,y} = 220638 \text{ mm}^3 )</td>
</tr>
<tr>
<td>Normal force</td>
<td>( N_{Ed} = 70 \text{ kN (compression)} )</td>
</tr>
<tr>
<td>Bending moment around strong (y') axis</td>
<td>( M_{Ed} = 37.5 \text{ kNm} )</td>
</tr>
<tr>
<td>Shear force</td>
<td>( V_{Ed} = 150 \text{ kN} )</td>
</tr>
</tbody>
</table>

Figure 9.3.1.1 – The statical system and internal forces
First we need to make the classification of the cross section individually for every internal forces:

The coefficient depending on $f_y$:

$$\epsilon = \sqrt{\frac{235}{f_y}} = \sqrt{\frac{235}{235}} = 1$$

**Classification due to normal force**

**Flanges:**

$$c = \frac{b - t_w - 2r}{2} = \frac{100 - 5.6 - 2 \cdot 12}{2} = 35.2 \text{ mm} \quad ; \quad \frac{c}{t} = \frac{35.2}{8.5} = 4.14 \quad ; \quad \frac{c}{t} < 9 \epsilon$$

Because $4.14 < 9$ thus the flanges are in Class 1.

**Web:**

$$\frac{c}{t} = \frac{h_w}{t_w} = \frac{159}{5.6} = 28.39 \quad ; \quad \frac{c}{t} < 33 \epsilon$$

Because $28.39 < 33$ thus the web is in Class 1.

Therefore the cross section is in Class 1 under normal force.

**Classification due to bending moment around strong axis**

**Flange:**

$$c = \frac{b - t_w - 2r}{2} = \frac{100 - 5.6 - 2 \cdot 12}{2} = 35.2 \text{ mm} \quad ; \quad \frac{c}{t} = \frac{35.2}{8.5} = 4.14 \quad ; \quad \frac{c}{t} < 9 \epsilon$$

because $4.14 < 9$ thus the flange is in Class 1.

**Web:**

$$\frac{c}{t} = \frac{h_w}{t_w} = \frac{159}{5.6} = 28.39 \quad ; \quad \frac{c}{t} < 72 \epsilon$$

because $28.39 < 72$ thus the web is in Class 1.

Therefore the cross section is in Class 1 under bending moment around strong axis.
Interaction of bending, shear and axial force

Normal force resistance under compression:

\[ N_{c,Rd} = A \frac{f_y}{f_{y,M0}} = \frac{2848}{235} \frac{235}{1.0} = 669.3 \text{kN} \]

Bending moment resistance around strong axis:

\[ M_{c,Rd} = W_{pl,y} \frac{f_y}{f_{y,M0}} = \frac{220638}{235} \frac{235}{1.0} = 51.85 \text{kNm} \]

Shear resistance:

\[ A_v = A - 2bt_f + (t_w + 2r)t_f = 2848 - 2 \cdot 100 \cdot 8.5 + (5.6 + 2 \cdot 12) \cdot 8.5 = 1400 \text{ mm}^2 \]

\[ V_{pl,Rd} = A_v \frac{f_y}{\sqrt{3}} = \frac{1400}{\sqrt{3}} \frac{235}{1.0} = 189.9 \text{kN} \]

The moment resistance should be reduced if \( V_{Ed} > 0.5V_{pl,Rd} \).

Now \( 150 \text{kN} > 94.95 \text{kN} \) therefore the reduction factor is:

\[ \rho = \left( \frac{2V_{Ed}}{V_{pl,Rd}} - 1 \right)^2 = \left( \frac{2 \cdot 150}{189.9} - 1 \right)^2 = 0.3361 \]

The reduced bending resistance:

\[ M_{y,V,Rd} = \left( W_{pl,y} - \frac{\rho A_w^2}{4t_w} \right) \frac{f_y}{f_{y,M0}} = \left( \frac{220638 - 0.3361 \cdot 1400^2}{4 \cdot 5.6} \right) \frac{235}{1.0} = 44.94 \text{kNm} \]

The utilization according to the interaction:

\[ \frac{N_{Ed}}{N_{c,Rd}} + \frac{M_{y,Ed}}{M_{y,V,Rd}} = \frac{70}{669.3} + \frac{37.5}{44.94} = 0.939 < 1.0 \text{ thus the resistance is adequate.} \]

Fig. 9.3.1.2 shows the statical system and the internal forces in FEM-Design.
The detailed results with the interaction utilization based on FEM-Design are shown in Fig. 9.3.1.3.
The difference between the two calculations is 3%.

Download link to the example file:
9.3.2 Buckling of a doubly symmetric I section

The buckling stability analysis will be investigated in an IPE 240 simply supported beam (see Fig. 9.3.2.1).

Inputs:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield strength of structural steel</td>
<td>( f_y = 355 \text{ N/mm}^2 )</td>
</tr>
<tr>
<td>Cross-sectional width</td>
<td>( b = 120 \text{ mm} )</td>
</tr>
<tr>
<td>Cross-sectional height</td>
<td>( h = 240 \text{ mm} )</td>
</tr>
<tr>
<td>Flange thickness</td>
<td>( t_f = 9.8 \text{ mm} )</td>
</tr>
<tr>
<td>Web thickness</td>
<td>( t_w = 6.2 \text{ mm} )</td>
</tr>
<tr>
<td>Web height</td>
<td>( h_w = 190.4 \text{ mm} )</td>
</tr>
<tr>
<td>Radius of root fillet</td>
<td>( r = 15 \text{ mm} )</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>( A = 3912 \text{ mm}^2 )</td>
</tr>
<tr>
<td>Inertia around strong axis</td>
<td>( I_y = 38916273 \text{ mm}^4 )</td>
</tr>
<tr>
<td>Inertia around weak axis</td>
<td>( I_z = 2836341 \text{ mm}^4 )</td>
</tr>
<tr>
<td>St. Venant torsional constant</td>
<td>( I_t = 127368 \text{ mm}^4 )</td>
</tr>
<tr>
<td>Warping constant</td>
<td>( I_\omega = 3668029708 \text{ mm}^6 )</td>
</tr>
<tr>
<td>Buckling length in both directions</td>
<td>( L_{cr} = 6.0 \text{ m} )</td>
</tr>
</tbody>
</table>

First of all we need to make the classification of the cross section for normal force:

The coefficient depending on \( f_y \):
\[ \varepsilon = \sqrt{\frac{235}{f_y}} = \sqrt{\frac{235}{355}} = 0.8136 \]

**Classification due to normal force**

Flanges:

\[
c = \frac{b - (t_w - 2r)}{2} = \frac{120 - 6.2 - 2 \cdot 15}{2} = 41.9 \text{ mm} \quad ; \quad \frac{c}{t} = \frac{c}{t_f} = \frac{41.9}{9.8} = 4.276 \quad ; \quad \frac{c}{t} < 9 \varepsilon
\]

Because \( 4.276 < 7.322 \) thus the flanges are in Class 1.

Web:

\[
\frac{c}{t} = \frac{h_w}{t_w} = \frac{190.4}{6.2} = 30.71 \quad ; \quad \frac{c}{t} < 38 \varepsilon
\]

Because \( 30.71 < 30.92 \) thus the web is in Class 2.

Therefore the cross section is in Class 2 under normal force.

**Flexural buckling around strong axis**

The radius of gyration (y-y axis):

\[
i_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{38916273}{3912}} = 99.74 \text{ mm}
\]

The non-dimensional slenderness:

\[
\bar{\lambda}_y = \frac{L_{cr}}{i_y} \frac{1}{\lambda_1} = \frac{6000}{99.74} \frac{1}{76.41} = 0.7873
\]

where \( \lambda_1 = \pi \sqrt{\frac{E_s}{f_y}} = \pi \sqrt{\frac{210000}{355}} = 76.41 \)

The imperfection \( \alpha \) factor value based on EN 1993-1-1 Table 6.2:

\[
\frac{h}{b} = \frac{240}{120} = 2 > 1.2 \quad \text{rolled section (y-y axis)} \quad \text{and} \quad t_f = 9.8 \text{ mm} < 40 \text{ mm} \quad \text{therefore “a” buckling curve is relevant thus the imperfection factor is} \quad \alpha_y = 0.21 \quad .
\]

\[
\Phi_y = 0.5 \left[ 1 + \alpha_y \left( \bar{\lambda}_y - 0.2 \right) + \bar{\lambda}_y^2 \right] = 0.5 \left[ 1 + 0.21 \left( 0.7873 - 0.2 \right) + 0.7873^2 \right] = 0.8716
\]

Reduction factor:

\[
\chi_y = \frac{1}{\Phi_y + \sqrt{\Phi_y^2 - \bar{\lambda}_y^2}} = \frac{1}{0.8716 + \sqrt{0.8716^2 - 0.7873^2}} = 0.8029
\]
Flexural buckling resistance: \( N_{b,y,Rd} = \frac{X_y A f_y}{Y_{MI}} = \frac{0.8029 \cdot 3912 - 355}{1} = 1115 \text{kN} \)

**Flexural buckling around weak axis**

The radius of gyration (z-z axis): \( i_z = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{2836341}{3912}} = 26.93 \text{mm} \)

The non-dimensional slenderness: \( \bar{\lambda}_z = \frac{L_{cr}}{i_z} \left( \frac{1}{\lambda_1} \right) = \frac{6000}{26.93} \left( \frac{1}{76.41} \right) = 2.916 \), where \( \lambda_1 = \frac{\pi}{\sqrt{E_s f_y}} = \frac{\pi}{\sqrt{210000 \cdot 355}} = 76.41 \)

The imperfection \( \alpha \) factor value based on EN 1993-1-1 Table 6.2:

\[
\frac{h}{b} = \frac{240}{120} = 2 > 1.2 \quad \text{rolled section (z-z axis) and} \quad t_f = 9.8 \text{mm} < 40 \text{mm} \quad \text{therefore “b” buckling curve is relevant thus the imperfection factor is} \quad \alpha_z = 0.34 .
\]

\( \Phi_z = 0.5 \left( 1 + \alpha_z \left( \bar{\lambda}_z - 0.2 \right) + \bar{\lambda}_z^2 \right) = 0.5 \left( 1 + 0.34 \left( 2.916 - 0.2 \right) + 2.916^2 \right) = 5.213 \)

Reduction factor: \( \chi_z = \frac{1}{\Phi_z + \sqrt{\Phi_z^2 - \bar{\lambda}_z^2}} = \frac{1}{5.213 + \sqrt{5.213^2 - 2.916^2}} = 0.1049 \)

Flexural buckling resistance: \( N_{b,z,Rd} = \frac{X_z A f_y}{Y_{MI}} = \frac{0.1049 \cdot 3912 - 355}{1} = 145.7 \text{kN} \)

**Torsional buckling**

The elastic torsional buckling critical force:

\[
N_{cr,T} = \frac{1}{i_0} \left( G I_i + \frac{\pi^2 E I_{wT}}{L_{cr}^2} \right) = \frac{1}{103.3} \left( 80769 \cdot 127368 + \frac{\pi^2 \cdot 210000 \cdot 36680292708}{6000^2} \right) = 1162 \text{kN}
\]

where \( i_0 \) is the polar radius of gyration:

\[
i_0 = \sqrt{i_y^2 + i_z^2 + y_0^2 + z_0^2} = \sqrt{99.74^2 + 26.93^2 + 0^2 + 0^2} = 103.3 \text{ mm} ,
\]

where \( y_0 \) and \( z_0 \) are the distances between the center of gravity and the shear center of the cross-section respect to the principal directions.

The torsional-flexural buckling is not relevant because the cross-section is doubly symmetric.
The non-dimensional slenderness for torsional buckling: 
\[ \tilde{\lambda}_T = \sqrt{\frac{A_f}{N_{cr,T}}} = \sqrt{\frac{3912 \cdot 355}{1162000}} = 1.093 \]

The imperfection \( \alpha \) factor value based on EN 1993-1-1 Table 6.2:
\[ \frac{h}{b} = \frac{240}{120} = 2 > 1.2 \quad \text{rolled section (z-z axis) and} \quad t_f = 9.8 \text{mm} < 40 \text{mm} \quad \text{therefore “b” buckling curve is relevant thus the imperfection factor is} \quad \alpha_T = 0.34 \ . \]

\[ \Phi_T = 0.5 \left( 1 + \alpha_T \left( \tilde{\lambda}_T - 0.2 \right) + \tilde{\lambda}_T^2 \right) = 0.5 \left( 1 + 0.34 \left( 1.093 - 0.2 \right) + 1.093^2 \right) = 1.249 \]

Reduction factor:
\[ \chi_T = \frac{1}{\Phi_T + \sqrt{\Phi_T^2 - \tilde{\lambda}_T^2}} = \frac{1}{1.249 + \sqrt{1.249^2 - 1.093^2}} = 0.5395 \]

Torsional buckling resistance:
\[ N_{b,T,Rd} = \frac{\chi_T A_f}{Y_{M1}} = \frac{0.5395 \cdot 3912 \cdot 355}{1} = 749.2 \text{ kN} \]

The statical system and the internal forces shown in Fig. 9.3.2.2.

Fig. 9.3.2.3 shows the detailed results about flexural and torsional buckling in FEM-Design.

The numerical result are almost identical with the hand calculations. The difference is less than 0.5%.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.3.2 Buckling of a doubly symmetric I section.str
Verification Examples

Figure 9.3.2.3 – Detailed results based on FEM-Design

**Flexural buckling, 1-1 • Part 1-1: 6.3.1**

LC: \( x = 3500 \) mm

Class, \( \text{Class}_{\text{1}} = 1 \), \( \text{Class}_{\text{2}} = 1 \)

\[ P = 922 \text{ mm} \]

\[ A = 3912 \text{ mm}^2 \]

\[ l_\text{y} = 3.892 \times 10^4 \text{ mm} \]

\[ l_\text{z} = 2.836 \times 10^4 \text{ mm} \]

\[ l_\text{t} = 3.892 \times 10^4 \text{ mm} \]

\[ \ell = 2.836 \times 10^4 \text{ mm} \]

\[ W_{\text{y}1} = 3.665 \times 10^5 \text{ mm}^3 \]

\[ W_{\text{z}2} = 7.395 \times 10^4 \text{ mm}^3 \]

\[ W_{\text{t}1} = 3.243 \times 10^5 \text{ mm}^3 \]

\[ W_{\text{t}2} = 4.727 \times 10^4 \text{ mm}^3 \]

\[ l_1 = 103 \text{ mm} \]

\[ l_2 = 27 \text{ mm} \]

\[ l_3 = 1.274 \times 10^5 \text{ mm} \]

\[ f_y = 3.668 \times 10^5 \text{ mm}^3 \]

\[ \varepsilon = 0.81 \]

\[ \lambda_1 = 76.40 \]

\[ f_y = 355 \text{ N/mm}^2 \]

\[ \varepsilon = 0.81 \]

\[ \lambda_1 = 76.40 \]

\[ N_{\text{y}1,1} = 140.00 \]

\[ N_{\text{y}1,1} = 1114.85 \]

\[ 0.13 \leq 1.00 \] (6.46) - OK

**Flexural buckling, 2-2 • Part 1-1: 6.3.1**

LC: \( x = 3500 \) mm

Class, \( \text{Class}_{\text{1}} = 1 \), \( \text{Class}_{\text{2}} = 1 \)

\[ \lambda_1 = \frac{l_1}{l_2} \cdot \lambda_2 = 27.76 \times 10^4 \] (6.50)

\[ \varepsilon = 0.34 \] (Buckling curve: b)

\[ \psi = 0.5 \left[ 1 + a_t \cdot \left( \lambda_1 - 0.2 \right) + \lambda_2 \right] = 0.5 \left[ 1 + 0.21 \cdot (0.79 - 0.2) + 0.79 \right] = 0.87 \]

\[ X_1 = \min \left\{ \frac{1}{\psi + \sqrt{\psi \cdot \lambda_1 \cdot \lambda_2}} \right\} \cdot \min \left\{ \frac{1}{\psi + \sqrt{\psi \cdot \lambda_1 \cdot \lambda_2}} \right\} = 0.83 \] (6.49)

\[ N_{\text{y}1,2} = 0.80 \times 392 \times 355 = 1114.85 \text{ kN} \] (6.47)

\[ N_{\text{y}1,2} = 1.00 \]

\[ N_{\text{y}1,2} = 1114.85 \]

\[ 0.13 \leq 1.00 \] (6.46) - OK

**Torsional-flexural buckling • Part 1-1: 6.3.1**

LC: \( x = 3500 \) mm

Class, \( \text{Class}_{\text{1}} = 1 \), \( \text{Class}_{\text{2}} = 1 \)

\[ I_z = \sqrt{l_1^2 + l_2^2 + \psi_1 \cdot l_1^2 + \psi_2 \cdot l_2^2} = \sqrt{21000 \times 3.668 \times 10^5} = 103 \text{ mm} \]

\[ N_{\text{y}1,1} = \frac{1}{103^3} \left( \frac{1}{60769} \cdot 1.274 \times 10^5 + \psi_1 \cdot \psi_2 \cdot l_1 \cdot l_2 \right) = \frac{1}{103^3} \left( \frac{1}{60769} \cdot 1.274 \times 10^5 + \psi_1 \cdot \psi_2 \cdot l_1 \cdot l_2 \right) = 1161.62 \text{ kN} \]

\[ N_{\text{y}1,1} = 1161.62 \text{ kN} \]

\[ N_{\text{y}1,1} = \min \{ 1161.62, 1, 1161.62 \} = 1161.62 \text{ kN} \]

\[ \lambda_t = \sqrt{\frac{l_1}{l_2}} \cdot \frac{392 \times 355}{1115.2} = 1.09 \] (6.53)

\[ a_t = 0.34 \] (Buckling curve: b)

\[ \psi = 0.5 \left[ 1 + a_t \cdot \left( \lambda_t - 0.2 \right) + \lambda_t \right] = 0.5 \left[ 1 + 0.34 \cdot (1.09 - 0.2) + 1.09 \right] = 1.25 \]

\[ X_2 = \min \left\{ \frac{1}{\psi + \sqrt{\psi \cdot \lambda_1 \cdot \lambda_2}} \right\} \cdot \min \left\{ \frac{1}{\psi + \sqrt{\psi \cdot \lambda_1 \cdot \lambda_2}} \right\} = 0.54 \] (6.49)

\[ N_{\text{y}1,2} = 0.54 \times 392 \times 355 = 748.75 \text{ kN} \] (6.47)

\[ N_{\text{y}1,2} = 748.75 \]

\[ N_{\text{y}1,2} = 140.00 \]

\[ N_{\text{y}1,2} = 748.75 \]

\[ N_{\text{y}1,2} = \frac{1.99 \leq 1.00}{1.00} \] (6.46) - OK
9.3.3 Buckling of a doubly symmetric + section

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield strength of structural steel</td>
<td>$f_y = 355 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Cross sectional width</td>
<td>$b = 200 \text{ mm}$</td>
</tr>
<tr>
<td>Thickness of the parts</td>
<td>$t = 4 \text{ mm}$</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>$A = 1584 \text{ mm}^2$</td>
</tr>
<tr>
<td>Inertia around strong axis</td>
<td>$I_y = 2667712 \text{ mm}^4$</td>
</tr>
<tr>
<td>Inertia around weak axis</td>
<td>$I_z = 2667712 \text{ mm}^4$</td>
</tr>
<tr>
<td>St. Venant torsional constant</td>
<td>$I_t = 8529 \text{ mm}^4$</td>
</tr>
<tr>
<td>Warping constant</td>
<td>$I_{\omega} = 7097545 \text{ mm}^6$</td>
</tr>
<tr>
<td>Buckling length</td>
<td>$L_{cr} = 2 \text{ m}$</td>
</tr>
</tbody>
</table>

First of all we need to make the classification of the cross section for normal force:

The coefficient depending on $f_y$:

$$\varepsilon = \sqrt{\frac{235}{f_y}} \sqrt{\frac{235}{355}} = 0.8136$$

Classification due to normal force

Outstand flanges:

$$c = \frac{b - t}{2} = \frac{200 - 4}{2} = 98 \text{ mm} \quad ; \quad \frac{c}{t} = \frac{98}{4} = 24.45 \quad ; \quad \frac{c}{t} > 14 \varepsilon$$

Because $24.45 > 11.39$ thus the flanges (and the whole section) are in Class 4.
**Calculation of the effective cross-section**

\[ k_o = 0.43 \]

\[ \bar{\lambda}_p = \frac{c}{l} = \frac{98}{28.4 \cdot \sqrt{k_o}} = 0.8136 \]

because \( \bar{\lambda}_p = 1.617 > 0.748 \) thus: \( \rho = \frac{\bar{\lambda}_p - 0.188}{\lambda_p^2} = 0.5465 \)

\[ b_{eff} = \rho c = 0.5465 \cdot 98 = 53.56 \text{ mm} \]

\[ A_{eff} = t^2 + 4b_{eff} = 4^2 + 4 \cdot 53.56 = 872.9 \text{ mm}^2 \]

**Flexural buckling**

The radius of gyration (based on the gross section): \( i_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{2667712}{1584}} = 41.04 \text{ mm} \)

The non-dimensional slenderness: \( \bar{\lambda} = \frac{L_{cr}}{i_y} \sqrt{A} \left( \frac{A_{eff}}{A} \right) = \frac{2000}{41.04} \left( \frac{1}{76.41} \right) \sqrt{\frac{872.9}{1584}} = 0.4733 \), where

\[ \lambda_1 = \pi \sqrt{\frac{E}{f_y}} = \pi \sqrt{\frac{210000}{355}} = 76.41 \]

In EN 1993-1-1 Table 6.2 this type of section is not included thus “c” buckling curve was chosen. The imperfection factor is: \( \alpha = 0.49 \).

\[ \Phi = 0.5 \left[ 1 + \alpha (\bar{\lambda} - 0.2) + \bar{\lambda}^2 \right] = 0.5 \left[ 1 + 0.49(0.4733 - 0.2) + 0.4733^2 \right] = 0.6790 \]

Reduction factor: \( \chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} = \frac{1}{0.6790 + \sqrt{0.6790^2 - 0.4733^2}} = 0.8577 \)

Flexural buckling resistance: \[ N_{b, Rd} = \chi A_{eff} f_y Y_M = \frac{0.8577 \cdot 872.9 \cdot 355}{1} = 265.8 \text{ kN} \]

**Torsional buckling**

The elastic torsional buckling critical force:

\[ N_{cr, T} = \frac{1}{i_0^2} \left( G I_i + \frac{\pi^2 E I_o}{L_{cr}^2} \right) = \frac{1}{58.04^2} \left( 80769 \cdot 8529 + \frac{\pi^2 \cdot 210000 \cdot 7097545}{200^2} \right) = 205.6 \text{ kN} \]

where \( i_0 \) is the polar radius of gyration:

\[ i_0 = \sqrt{i_y^2 + i_z^2 + y_0^2 + z_0^2} = \sqrt{41.04^2 + 41.04^2 + 0^2 + 0^2} = 58.04 \text{ mm} \)
where $y_0$ and $z_0$ are the distances between the center of gravity and the shear center of the cross-section with respect to the principal directions.

The torsional-flexural buckling is not relevant because the cross-section is doubly symmetric.

The non-dimensional slenderness for torsional buckling:

$$\bar{\lambda}_T = \sqrt{\frac{A_{eff} f_y}{N_{cr, T}}} = \sqrt{\frac{872.9 \cdot 355}{205600}} = 1.228$$

In EN 1993-1-1 Table 6.2 this type of section is not included thus “c” buckling curve was chosen. The imperfection factor is: $\alpha_T = 0.49$.

$$\Phi_T = 0.5 \left( 1 + \alpha_T \left( \bar{\lambda}_T - 0.2 \right) + \bar{\lambda}_T^2 \right) = 0.5 \left( 1 + 0.49 \left( 1.228 - 0.2 \right) + 1.228^2 \right) = 1.506$$

Reduction factor:

$$\chi_T = \frac{1}{\Phi_T + \sqrt{\Phi_T^2 - \bar{\lambda}_T^2}} = \frac{1}{1.506 + \sqrt{1.506^2 - 1.228^2}} = 0.4206$$

Torsional buckling resistance:

$$N_{b, T, Rd} = \frac{\chi_T A_{eff} f_y}{\gamma_{MI}} = \frac{0.4206 \cdot 872.9 \cdot 355}{1} = 130.3 \text{ kN}$$

The statical system and the normal forces shown in Fig. 9.3.3.2.

![Figure 9.3.3.2 – The statical system and the normal force diagram (kN) in FEM-Design](image)

Fig. 9.3.3.3 shows the detailed results about flexural and torsional buckling in FEM-Design.

The numerical result are almost identical with the hand calculations. The difference is less than 0.1%.
Verification Examples

Figure 9.3.3.3 – Detailed results based on FEM-Design

### Flexural buckling, 1.1 - Part 1-1: 6.3.1
LC: "1", x = 1500 mm
Class 4, Class 2, Class 0 = 4

- $\lambda = \frac{W_e}{A} \sqrt{\frac{E}{v}} = \frac{2000}{41} \sqrt{\frac{200}{1054}} = 0.47$ (6.51)
- $\alpha = 0.49$ (Buckling curve: c)

\[
x_i = \min \left( \frac{1}{\sqrt{n_i + \sqrt{n_i - \lambda_i}}}, \frac{1}{1.06 + \sqrt{0.66^2 - 0.47^2}} \right) = 0.66 \quad (6.49)
\]

\[
N_{M,3} = \frac{1}{1.06} \cdot 355 \cdot 355 = 265.80 \text{ kN} \quad (6.48)
\]

\[
\frac{N_{M,3}}{N_{M,3}} = 0.49 \leq 1.00 \quad (6.48) - \text{OK}
\]

### Flexural buckling, 2.2 - Part 1-1: 6.3.1
LC: "1", x = 1500 mm
Class 4, Class 2, Class 0 = 4

- $\lambda = \frac{W_e}{A} \sqrt{\frac{E}{v}} = \frac{2000}{41} \sqrt{\frac{200}{1054}} = 0.47$ (6.51)
- $\alpha = 0.49$ (Buckling curve: c)

\[
x_i = \min \left( \frac{1}{\sqrt{n_i + \sqrt{n_i - \lambda_i}}}, \frac{1}{1.06 + \sqrt{0.66^2 - 0.47^2}} \right) = 0.66 \quad (6.49)
\]

\[
N_{M,3} = \frac{1}{1.06} \cdot 355 \cdot 355 = 265.80 \text{ kN} \quad (6.48)
\]

\[
\frac{N_{M,3}}{N_{M,3}} = 0.49 \leq 1.00 \quad (6.48) - \text{OK}
\]

### Torsional-Rxial buckling - Part 1-1: 6.3.1
LC: "1", x = 1500 mm
Class 4, Class 2, Class 0 = 4

- $y = \sqrt{x^2 + y_0^2 + z_0^2} = \sqrt{x^2 + y^2 + z^2} = 50 \text{ mm}$

\[
N_{M,R} = \frac{1}{6} \left( \frac{1}{6} \cdot \frac{\pi^2 E I_w}{y^2} \right) = \frac{1}{6} \cdot \frac{80769 \cdot 8.5290 \cdot 0.03 \cdot \pi^2 \cdot 1000 \cdot 7.0385 \cdot 1000}{205.01 \text{ kN}} = 205.01 \text{ kN}
\]

\[
y^2 \left( N - N_{M,1} \right) \left( N - N_{M,2} \right) \left( N - N_{M,3} \right) - y^2 \frac{N_{M,1}}{N_{M,3}} - y^2 \frac{N_{M,2}}{N_{M,3}} - y^2 \left( 1 - N_{M,3} \right) = 0
\]

- $N_{M,3} = 205.01 \text{ kN}$

\[
N_{M,1} = N_{M,2} + N_{M,3} = \min(205.01, 205.01) = 205.01 \text{ kN}
\]

\[
\lambda = \frac{\frac{1}{E I_w} \cdot \frac{1}{N_{M,3}}}{\frac{1}{E I_w} \cdot \frac{1}{N_{M,3}}} = \frac{673.355}{205.61} = 3.3 \quad (6.52)
\]

- $\alpha = 0.49$ (Buckling curve: c)

\[
x_i = \min \left( \frac{1}{\sqrt{n_i + \sqrt{n_i - \lambda_i}}}, \frac{1}{1.06 + \sqrt{0.66^2 - 0.47^2}} \right) = 0.66 \quad (6.49)
\]

\[
N_{M,3} = \frac{1}{1.06} \cdot 355 \cdot 355 = 265.80 \text{ kN} \quad (6.48)
\]

\[
\frac{N_{M,3}}{N_{M,3}} = 0.49 \leq 1.00 \quad (6.48) - \text{OK}
\]
Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.3.3 Buckling of a doubly symmetric + section.str
9.3.4 Buckling of a mono-symmetric channel section

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield strength of structural steel</td>
<td>$f_y = 275 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>Thickness of the parts</td>
<td>$t = 20 \text{ mm}$</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>$A = 9200 \text{ mm}^2$</td>
</tr>
<tr>
<td>Inertia around strong axis</td>
<td>$I_y = 35158841 \text{ mm}^4$</td>
</tr>
<tr>
<td>Inertia around weak axis</td>
<td>$I_z = 13426667 \text{ mm}^4$</td>
</tr>
<tr>
<td>St. Venant torsional constant</td>
<td>$I_t = 1217153 \text{ mm}^4$</td>
</tr>
<tr>
<td>Warping constant</td>
<td>$I_\omega = 54039544948 \text{ mm}^6$</td>
</tr>
<tr>
<td>Buckling length</td>
<td>$L_{cr} = 6 \text{ m}$</td>
</tr>
</tbody>
</table>

First of all we need to make the classification of the cross section for normal force. The coefficient depending on $f_y$: 

$N_{Ed} = 500 \text{ kN}$

$N_{Ed} = 500 \text{ kN}$
\[ \varepsilon = \sqrt{\frac{235}{f_y}} = \sqrt{\frac{235}{275}} = 0.9244 \]

**Classification due to normal force**

**Flanges:**

\[ c = 200 - 20 = 180 \text{ mm} \quad ; \quad \frac{c}{t} = \frac{180}{20} = 9.0 \quad ; \quad 9 \varepsilon < \frac{c}{t} < 10 \varepsilon \]

Because \( 7.395 < 9.0 < 9.244 \) thus the flanges are in Class 2.

**Web:**

\[ c = 100 - 20 - 20 = 60 \text{ mm} \quad ; \quad \frac{c}{t} = \frac{60}{20} = 3.0 \quad ; \quad \frac{c}{t} < 33 \varepsilon \]

Because \( 3.0 < 30.51 \) thus the web is in Class 1.

According to these calculations the section is in Class 2 due to normal force.

**Flexural buckling around strong axis**

The radius of gyration (y-y axis, see Fig. 9.3.4.1): \( i_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{35158841}{9200}} = 61.82 \text{ mm} \)

The non-dimensional slenderness: \( \bar{\lambda}_y = \frac{L_{cr}}{i_y} \cdot \frac{1}{\lambda_1} = \frac{6000}{61.82} \cdot \frac{1}{86.81} = 1.118 \) where \( \lambda_1 = \pi \sqrt{\frac{E}{f_y}} = \pi \sqrt{\frac{210000}{275}} = 86.81 \)

According to EN 1993-1-1 Table 6.2 “c” buckling curve was chosen. The imperfection factor is: \( \alpha_y = 0.49 \).

\[ \Phi_y = 0.5 \left[ 1 + \alpha_y \left( \bar{\lambda}_y - 0.2 \right) + \bar{\lambda}_y^2 \right] = 0.5 \left[ 1 + 0.49 \left( 1.118 - 0.2 \right) + 1.118^2 \right] = 1.350 \]

Reduction factor: \( \chi_y = \frac{1}{\Phi_y + \sqrt{\Phi_y^2 - \bar{\lambda}_y^2}} = \frac{1}{1.350 + \sqrt{1.350^2 - 1.118^2}} = 0.4747 \)

Flexural buckling resistance:

\[ N_{b,y,rd} = \frac{\chi_y A f_y}{\gamma_{ML}} = \frac{0.4747 \cdot 9200 \cdot 275}{1} = 1201 \text{ kN} \]
**Flexural buckling around weak axis**

The radius of gyration (z-z axis, see Fig. 9.3.4.1): \[ i_z = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{13426667}{9200}} = 38.20 \text{ mm} \]

The non-dimensional slenderness: \[ \bar{\lambda}_z = \frac{L_{cr}}{i_z} \left( \frac{1}{\lambda_1} \right) = \frac{6000}{38.20} \frac{1}{86.81} = 1.809 \]

According to EN 1993-1-1 Table 6.2 “c” buckling curve was chosen. The imperfection factor is: \[ \alpha_z = 0.49 \]

\[ \Phi_z = 0.5 \left( 1 + \alpha_z \bar{\lambda}_z - 0.2 + \bar{\lambda}_z^2 \right) = 0.5 \left( 1 + 0.49 \left( 1.809 - 0.2 \right) + 1.809^2 \right) = 2.530 \]

Reduction factor: \[ \chi_z = \frac{1}{\Phi_z + \sqrt{\Phi_z^2 - \bar{\lambda}_z^2}} = \frac{1}{2.530 + \sqrt{2.530^2 - 1.809^2}} = 0.2326 \]

Flexural buckling resistance: \[ N_{b,z, Rd} = \frac{\chi_z A f_y \gamma_{MI} \cdot 9200 \cdot 275}{1} = 588.5 \text{ kN} \]

**Torsional buckling**

The elastic torsional buckling critical force: \[ N_{cr,T} = \frac{1}{i_0^2} \left( G I_t + \pi^2 E I_{gt} \right) = \frac{1}{176.0^2} \left( 80769 \cdot 1217153 + \frac{\pi^2 \cdot 210000 \cdot 54039544948}{6000^2} \right) = 3274 \text{ kN} \]

where \( i_0 \) is the polar radius of gyration:

\[ i_0 = \sqrt{i_y^2 + i_z^2 + y_0^2 + z_0^2} = \sqrt{61.82^2 + 38.20^2 + 0^2 + (68.3 + 92.0)^2} = 176.0 \text{ mm} \]

where \( y_0 \) and \( z_0 \) are the distances between the center of gravity and the shear center of the cross-section respect to the principal directions (see Fig. 9.3.4.1).

The non-dimensional slenderness for torsional buckling: \[ \bar{\lambda}_T = \sqrt{\frac{A f_t}{N_{cr,T}}} = \sqrt{\frac{9200 \cdot 275}{3274000}} = 0.8791 \]

According to EN 1993-1-1 Table 6.2 “c” buckling curve was chosen. The imperfection factor is: \[ \alpha_T = 0.49 \]

\[ \Phi_T = 0.5 \left( 1 + \alpha_T \bar{\lambda}_T - 0.2 + \bar{\lambda}_T^2 \right) = 0.5 \left( 1 + 0.49 \left( 0.8791 - 0.2 \right) + 0.8791^2 \right) = 1.053 \]
Reduction factor: \[ \chi_T = \frac{1}{\Phi_T + \sqrt{\Phi_T^2 - \lambda_T^2}} = \frac{1}{1.053 + \sqrt{1.053^2 - 0.8791^2}} = 0.6125 \]

Torsional buckling resistance: \[ N_{b,T,Rd} = \chi_T A f_y \frac{\gamma_{MI}}{1} = 0.6125 \cdot 9200 \cdot 275 \cdot 1 = 1550 \text{kN} \]

**Torsional-flexural buckling.**

The torsional-flexural buckling could be relevant by a mono symmetric section.

We need to find the roots of the following equation:

\[ i_0^2 (N - N_{cr,y})(N - N_{cr,z})(N - N_{cr,T}) - N^2 y_0^2 (N - N_{cr,z}) - N^2 z_0^2 (N - N_{cr,y}) = 0 \]

Where in addition the quantities already calculated:

\[ N_{cr,y} = \frac{\pi^2 EI_y}{L_{cr}^2} = \frac{\pi^2 \cdot 210000 \cdot 35158841}{6000^2} = 2024 \text{kN} \quad \text{and} \]

\[ N_{cr,z} = \frac{\pi^2 EI_z}{L_{cr}^2} = \frac{\pi^2 \cdot 210000 \cdot 13426667}{6000^2} = 773.0 \text{kN} \]

\[ y_0 = 0 \text{mm} \quad ; \quad z_0 = 68.3 + 92 = 160.3 \text{mm} \quad \text{the distance between the gravity center and shear center respect to the principal directions (see Fig. 9.3.4.1).} \]

\[ i_0 = \sqrt{i_y^2 + i_z^2 + y_0^2 + z_0^2} = \sqrt{61.82^2 + 38.20^2 + 0^2 + 160.3^2} = 176.0 \text{mm} \]

Thus the third degree polynomial:

\[ 176^2 (N - 2024)(N - 773)(N - 3274) - N^2 0^2 (N - 773) - N^2 160.3^2 (N - 2024) = 0 \]

The roots of the equation (see Fig. 9.3.4.3):

\[ N_1 = 642.7 \text{kN} \quad ; \quad N_2 = 2024 \text{kN} \quad ; \quad N_3 = 23097 \text{kN} \]
Because the smallest root is smaller than the smallest clear critical elastic force thus the torsional-flexural buckling is relevant in this case.

\[ N_1 = 642.7 \text{kN} < N_{cr,z} = 773.0 \text{kN} < N_2 = N_{cr,y} = 2024 \text{kN} < N_{cr,T} = 3274 \text{kN} < N_3 = 23097 \text{kN} \]

Therefore:

\[ N_{cr,TF} = N_1 = 642.7 \text{kN} \]

The non-dimensional slenderness for torsional-flexural buckling:

\[
\lambda_{TF}^{-} = \sqrt{\frac{Af_y}{N_{cr,TF}}} = \sqrt{\frac{9200 \cdot 275}{642700}} = 1.984
\]

\[
\Phi_{TF} = 0.5 \left( 1 + \alpha_{TF} \left( \lambda_{TF}^{-} - 0.2 \right) + \lambda_{TF}^{-2} \right) = 0.5 \left( 1 + 0.49 \left( 1.984 - 0.2 \right) + 1.984^2 \right) = 2.905
\]
Reduction factor:

\[ \chi_{TF} = \frac{1}{\Phi_{TF} + \sqrt{\Phi_{TF}^2 - \lambda_{TF}^2}} = \frac{1}{2.905 + \sqrt{2.905^2 - 1.984^2}} = 0.1989 \]

Torsional-flexural buckling resistance:

\[ N_{b,TF,Rd} = \frac{\chi_{TF} A f_y}{Y_M} = \frac{0.1989 \cdot 9200 \cdot 275}{1} = 503.2 \text{kN} \]

The statical system and the normal forces shown in Fig. 9.3.4.4.

Fig. 9.3.4.5 shows the detailed results about torsional-flexural and torsional buckling in FEM-Design.

The numerical result are almost identical with the hand calculations. The difference is less than 0.1%.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/9.3.4 Buckling of a mono symmetric channel section.str
Flexural buckling - 1-1 - Part 1-1: 6.3.1

LC: 1-1, \( x = 3500 \) mm

Class \( h_9 = 2, \) Class \( h_{11} = 2, \) Class \( h_{12} = 1 \)

\( f_1 = \frac{L_1}{L_2} = \frac{L_1}{L_2} = \frac{6000}{L_2} = 68.80 \) (6.50)

\( a_1 = 0.49 \) (Buckling curve: c)

\( q_1 = 0.5 \left( 1 + \frac{a_1}{a_2} \left( 1 - \frac{L_2}{L_1} \right) \right) \left( 1 + 0.49 \left( 0.12 - 0.2 \right) \right) = 1.35 \)

\( x_1 = \left( \frac{x_1}{l_1} \right) \left( 1 + 0.49 \left( 0.12 - 0.2 \right) \right) = 1.61 \) (6.50)

\( N_{n1} = \frac{f_1}{a_1} \left( 1.15 + \frac{1.35}{1.12 - 1.61} \right) = 0.47 \) (6.47)

\( N_{n1,R1} = 500.00 \text{ kN} \leq 1.00 \) (6.46) - OK

Flexural buckling - 2-2 - Part 1-1: 6.3.1

LC: 1-1, \( x = 3500 \) mm

Class \( h_9 = 2, \) Class \( h_{11} = 2, \) Class \( h_{12} = 1 \)

\( f_1 = \frac{L_1}{L_2} = \frac{L_1}{L_2} = \frac{6000}{L_2} = 68.80 \) (6.50)

\( a_1 = 0.49 \) (Buckling curve: c)

\( q_2 = 0.5 \left( 1 + \frac{a_1}{a_2} \left( 1 - \frac{L_2}{L_1} \right) \right) \left( 1 + 0.49 \left( 0.12 - 0.2 \right) \right) = 2.53 \)

\( x_2 = \left( \frac{x_2}{l_1} \right) \left( 1 + 0.49 \left( 0.12 - 0.2 \right) \right) = 1.61 \) (6.50)

\( N_{n2} = \frac{f_1}{a_1} \left( 1.15 + \frac{2.53}{1.12 - 1.61} \right) = 0.23 \) (6.47)

\( N_{n2,R1} = 500.00 \text{ kN} \leq 0.85 \leq 1.00 \) (6.46) - OK

Torsional-flexural buckling - Part 1-1: 6.3.1

LC: 1-1, \( x = 3500 \) mm

Class \( h_9 = 2, \) Class \( h_{11} = 2, \) Class \( h_{12} = 1 \)

\( t = \sqrt{1 + \left( \frac{L_2}{L_1} \right)^2} + \sqrt{x^2 + x_2^2} = \sqrt{1 + 0.2^2} + \sqrt{1 + 0.6^2} = 1.76 \) mm

\( N_{n1} = \frac{t^2}{L_2} \cdot \frac{E}{\beta_b} \cdot \frac{L_2}{L_1} = \frac{1.76^2}{68.80} \cdot \frac{210000}{3518841} = 2024.19 \text{ kN} \)

\( N_{n1,R1} = \frac{t^2}{L_2} \cdot \frac{E}{\beta_a} \cdot \frac{L_2}{L_1} = \frac{1.76^2}{68.80} \cdot \frac{210000}{3518841} = 773.01 \text{ kN} \)

\( N_{n2} = \frac{t^2}{L_2} \cdot \frac{E}{\beta_b} \cdot \frac{L_2}{L_1} = \frac{1.76^2}{68.80} \cdot \frac{210000}{3518841} = 2727.88 \text{ kN} \)

\( l^2 (N - N_{n1}) (N - N_{n2}) (N - N_{n1}) (N - N_{n2}) = 0 \)

Smallest root of the above equation:

\( N_{n2} = 642.72 \text{ kN} \)

\( N_{n1} = \min(N_{n1,R1}, N_{n2}) = \min(2024.19, 642.72) = 642.72 \text{ kN} \)

\( t = \frac{t}{\sqrt{1 + \left( \frac{L_2}{L_1} \right)^2}} = \frac{1.76}{\sqrt{1 + 0.2^2}} = 1.08 \) (6.53)

\( a_1 = 0.49 \) (Buckling curve: c)

\( q_2 = 0.5 \left( 1 + \frac{a_1}{a_2} \left( 1 - \frac{L_2}{L_1} \right) \right) \left( 1 + 0.49 \left( 0.12 - 0.2 \right) \right) = 2.91 \)

\( x_2 = \left( \frac{x_2}{l_1} \right) \left( 1 + 0.49 \left( 0.12 - 0.2 \right) \right) = 1.61 \) (6.50)

\( N_{n2} = \frac{t^2}{L_2} \cdot \frac{E}{\beta_b} \cdot \frac{L_2}{L_1} = \frac{1.76^2}{68.80} \cdot \frac{210000}{3518841} = 2727.88 \text{ kN} \)

\( N_{n2,R1} = \frac{t^2}{L_2} \cdot \frac{E}{\beta_a} \cdot \frac{L_2}{L_1} = \frac{1.76^2}{68.80} \cdot \frac{210000}{3518841} = 773.01 \text{ kN} \)

\( N_{n2} = 642.72 \text{ kN} \)

\( N_{n1} = \min(N_{n1,R1}, N_{n2}) = \min(2024.19, 642.72) = 642.72 \text{ kN} \)

\( N_{n2} = 642.72 \text{ kN} \)

\( N_{n1} = \min(N_{n1,R1}, N_{n2}) = \min(2024.19, 642.72) = 642.72 \text{ kN} \)
9.3.5 Lateral torsional buckling of a doubly symmetric I section

In this sub-chapter we will calculate the lateral torsional buckling resistance of an IPE 240 beam under concentrated load. We will calculate according to EN1993-1-1:6.3.2.2 (general case) and EN1993-1-1:6.3.2.4 (simplified assessment). The section is in Class 1 due to pure bending.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield strength of structural steel</td>
<td>$f_y = 355$ MPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.3$</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>$E = 210$ GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>$G = \frac{E}{2(1+\nu)} = 80.77$ GPa</td>
</tr>
<tr>
<td>Distance between lateral restraints</td>
<td>$L = 6.0$ m</td>
</tr>
<tr>
<td>Height of the cross-section</td>
<td>$h = 240$ mm</td>
</tr>
<tr>
<td>Inertia around weak axis</td>
<td>$I_z = 2836341$ mm$^4$</td>
</tr>
<tr>
<td>St. Venant torsional constant</td>
<td>$I_t = 127368$ mm$^4$</td>
</tr>
<tr>
<td>Warping constant</td>
<td>$I_\omega = 36680292708$ mm$^6$</td>
</tr>
<tr>
<td>Plastic cross-sectional modulus around strong axis</td>
<td>$W_{pl,y} = 366645$ mm$^3$</td>
</tr>
</tbody>
</table>

**General case EN1993-1-1:6.3.2.2**

$k = 1.0$ ; $k_\omega = 1.0$  free to rotate about z axis and restraint against movements and free to warp but restraint against rotation about the longitudinal axis.

The $C_i$ coefficients are depending on the loading and end restraint conditions:

$C_1 = 1.35 ; C_2 = 0.63 ; C_3 = 1.73$

Because the cross section is doubly symmetric: $z_j = 0$ .

**Figure 9.3.5.1 – The statical system and the cross-section**

$F_{Ed} = 25$ kN

$M_{Ed} = 37.5$ kNm
The distance between the point of load application and the shear centre (this value has a sign):
\[ z_g = \frac{h}{2} = \frac{240}{2} = 120 \text{ mm} \] see Fig. 9.3.5.1.

The calculation of the elastic critical moment:
\[
\begin{align*}
M_{cr} &= C_1 \frac{\pi^2 EI_z}{(kL)^2} \left( \sqrt{\frac{k}{k_\omega}} \frac{I_w}{I_z} + \frac{(kL)^2 G I_t}{\pi^2 EI_z} + (C_2 z_g - C_3 z_j)^2 - (C_2 z_g - C_3 z_j) \right) \\
Z &= C_2 z_g - C_3 z_j = 0.63 \cdot 120 - 1.73 \cdot 0 = 75.6 \text{ mm}
\end{align*}
\]

\[
M_{cr} = 1.35 \frac{\pi^2 210000 \cdot 2836341}{(1 \cdot 6000)^2} \left( \sqrt{\frac{1}{1}} \frac{3.668 \cdot 10^{10}}{2836341} + \frac{(1 \cdot 6000)^2 80769 \cdot 127368}{\pi^2 210000 \cdot 2836341} + 75.6^2 - 75.6 \right)
\]

\[ M_{cr} = 46.32 \text{ kNm} \]

Non-dimensional slenderness:
\[ \lambda_{LT} = \sqrt{\frac{W_{pl,y} f_y}{M_{cr}}} = \sqrt{\frac{366645 \cdot 355}{46320000}} = 1.676 \]

\[ \Phi_{LT} = 1 + \alpha_{LT} (\lambda_{LT} - 0.2) + \lambda_{LT}^2 = 1 + 0.21 (1.676 - 0.2) + 1.676^2 = 2.059 \]

where the imperfection \( \alpha_{LT} \) factor value based on EN 1993-1-1 Table 6.4:

because \[ \frac{h}{b} = \frac{240}{120} = 2 \leq 2 \] rolled section therefore “a” buckling curve is relevant thus the imperfection factor is \( \alpha_{LT} = 0.21 \).

Reduction factor for lateral torsional buckling:
\[ \chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \lambda_{LT}^2}} = \frac{1}{2.059 + \sqrt{2.059^2 - 1.676^2}} = 0.3072 \]

The lateral torsional buckling resistance:
\[ M_{b,Rd} = \chi_{LT} W_{pl,y} \frac{f_y}{M_I} = 0.3072 \frac{366645 \cdot 355}{1.0} = 39.98 \text{ kNm} \]
Simplified assessment EN1993-1-1:6.3.2.4

In this method we check the buckling of an equivalent T section, where the compressed flange is the same as in the original section, the web height is third of the compressed web height.

Note: We simplify the equivalent section, we are not considering roundings.

The height of the section: \( 220.4/2/3 + 9.8 = 46.5 \text{ mm} \) (see Fig. 9.3.5.1-2).

\[
A_f = 9.8 \cdot 120 + 6.2 \cdot (46.5 - 9.8) = 1404 \text{ mm}^2
\]
\[
I_{f,z} = 9.8 \cdot 120^3 / 12 + (46.5 - 9.8) \cdot 6.2^3 / 12 = 1412000 \text{ mm}^4
\]

The relevant radius of gyration of the equivalent T section:
\[
i_{f,z} = \sqrt{\frac{I_{f,z}}{A_f}} = \sqrt{\frac{1412000}{1403}} = 31.72 \text{ mm}
\]

Non-dimensional slenderness:
\[
\tilde{\lambda}_f = \frac{k_c L}{i_f \lambda_1} = \frac{0.86 \cdot 6000}{31.72 \cdot 76.41} = 2.129
\]
where \( \lambda_1 = \pi \sqrt{\frac{E}{f_y}} = \pi \sqrt{\frac{210000}{355}} = 76.41 \) and \( k_c \) depends on the moment distribution.

\[
\Phi_f = 1 + \alpha_f (\tilde{\lambda}_f - 0.2) + \tilde{\lambda}_f^2 = \frac{1 + 0.49(2.129 - 0.2) + 2.129^2}{2} = 3.239
\]
where the imperfection factor comes from buckling curve “c”: \( \alpha_f = 0.49 \)
Reduction factor for lateral torsional buckling:

\[ \chi_{LT} = \frac{1}{\Phi_f + \sqrt{\Phi_f^2 - \bar{\lambda}_f^2}} = \frac{1}{3.239 + \sqrt{3.239^2 - 2.129^2}} = 0.1760 \]

The lateral torsional buckling resistance:

\[ M_{b, Rd} = k_f \chi_{LT} W_{pl, y} f_y = 1.1 \cdot 0.1760 \cdot 366645 \cdot \frac{355}{1.0} = 25.20 \text{kNm} \]

Fig. 9.3.5.3 shows the FEM-Design statical system and the bending moment diagram.

The differences between the hand and FEM-Design calculations are less than 2%.

See Fig. 9.3.5.4 about the detailed results of FEM-Design.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.3.5 Lateral torsional buckling of a doubly symmetric I section.str
Figure 9.3.5.4 – The detailed results about the lateral torsional buckling based on FEM-Design

Lateral torsional buckling - Part 1-1: 6.3.2.2
LC: 1st, x = 3000 mm
Class = 2, Class\textsubscript{c} = 1, Class\textsubscript{b} = 1
\lambda = 10.5 (buckling curve a)
\psi = 0.5 (1 + \rho \cdot \frac{\lambda}{\pi})\frac{\lambda}{\pi} = 0.5 (1 + 0.21 (1.68 - 0.2))\frac{\lambda}{\pi} = 1.98
\lambda_{cr} = \frac{\pi}{\sqrt{\psi}} = 4.36 \text{ degrees}
\psi_{cr} = \frac{\lambda}{\lambda_{cr}} = \frac{10.5}{4.36} = 2.4 \text{ degrees}

IPE 240
\begin{align*}
P &= 922 \text{ mm} \\
A &= 3,972 \text{ mm}^2 \\
J &= 2.836 \times 10^3 \text{ mm}^4 \\
I &= 3.892 \times 10^7 \text{ mm}^4 \\
W &= 7.396 \times 10^4 \text{ mm}^3 \\
W' &= 3.243 \times 10^3 \text{ mm}^3 \\
W'' &= 4.727 \times 10^5 \text{ mm}^3 \\
I_1 &= 100 \text{ mm} \\
I_2 &= 27 \text{ mm} \\
I_3 &= 1.274 \times 10^3 \text{ mm}^3 \\
I_4 &= 3.668 \times 10^4 \text{ mm}^3 \\
S &= 210000 \text{ N/mm}^2 \\
E &= 20199 \text{ N/mm}^2 \\
V_{y,1} &= 1.00 \\
V_{y,10} &= 1.00 \\
V_{y,12} &= 1.25 \\
V_{y,14} &= 1.00 \\
\gamma &= 0.81 \\
\nu &= 76.40 \\
\varepsilon &= 355 \text{ N/mm}^2 \\
r &= 25\sqrt{r} = 25\sqrt{2} = 35.36 \text{ degrees}
\end{align*}

Lateral torsional buckling - Part 1-1: 6.3.2.4
LC: 1st, x = 3000 mm
Class = 2, Class\textsubscript{c} = 1, Class\textsubscript{b} = 1
\lambda = 10.5 (buckling curve a)
\psi = 0.5 (1 + \rho \cdot \frac{\lambda}{\pi})\frac{\lambda}{\pi} = 0.5 (1 + 0.49 (2.11 - 0.2))\frac{\lambda}{\pi} = 3.19
\lambda_{cr} = \frac{\pi}{\sqrt{\psi}} = 4.36 \text{ degrees}
\psi_{cr} = \frac{\lambda}{\lambda_{cr}} = \frac{10.5}{4.36} = 2.4 \text{ degrees}

\begin{align*}
\lambda &= \text{min}\left(\frac{1}{\psi}, \frac{1}{\psi_{cr}}, \frac{1}{\psi_{cr}}, \frac{1}{\psi_{cr}}\right) = \text{min}\left(\frac{1}{0.5}, \frac{1}{2.4}, \frac{1}{2.4}, \frac{1}{2.4}\right) = 1.00 \text{ (6.56)}

M_{y,axi} = \frac{1}{\lambda_{cr}} \frac{1}{2} \frac{I}{E} = \frac{1}{4.36} \frac{1}{2} \frac{2.836 \times 10^3}{20199} = 39.96 \text{ kNm} \text{ (6.55)}

M_{y,axi} = 37.60 \text{ kNm} \text{ (5.64) - OK}

M_{y,axi} = 39.96 \text{ kNm} \text{ (6.54) - OK}

\end{align*}
9.3.6 Interaction of biaxial bending and axial compression in an RHS section

In this sub-chapter the stability interaction of an RHS section (KKR 200x100x10, cold formed hollow section) will be investigated. The statical system and the design load values are indicated in Fig. 9.3.6.1, furthermore the other general input data are in the table below.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield strength of structural steel</td>
<td>$f_y = 355$ MPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.3$</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>$E = 210$ GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>$G = E / (2(1+\nu)) = 80.77$ GPa</td>
</tr>
<tr>
<td>Span length</td>
<td>$L = 6.0$ m</td>
</tr>
<tr>
<td>Cross-sectional height</td>
<td>$h = 200$ mm</td>
</tr>
<tr>
<td>Cross-sectional width</td>
<td>$b = 100$ mm</td>
</tr>
<tr>
<td>Cross-sectional thickness</td>
<td>$t = 10$ mm</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>$A = 5257$ mm$^2$</td>
</tr>
<tr>
<td>Inertia around strong axis</td>
<td>$I_y = 24443956$ mm$^4$</td>
</tr>
<tr>
<td>Inertia around weak axis</td>
<td>$I_x = 8177434$ mm$^4$</td>
</tr>
<tr>
<td>St. Venant torsional constant</td>
<td>$I_n = 21571134$ mm$^4$</td>
</tr>
<tr>
<td>Warping constant</td>
<td>$L_n = 4313360830$ mm$^6$</td>
</tr>
<tr>
<td>Elastic cross-sectional modulus around strong axis</td>
<td>$W_{el,y} = 244440$ mm$^3$</td>
</tr>
<tr>
<td>Elastic cross-sectional modulus around weak axis</td>
<td>$W_{el,x} = 163549$ mm$^3$</td>
</tr>
<tr>
<td>Plastic cross-sectional modulus around strong axis</td>
<td>$W_{pl,y} = 318082$ mm$^3$</td>
</tr>
<tr>
<td>Plastic cross-sectional modulus around weak axis</td>
<td>$W_{pl,x} = 195250$ mm$^3$</td>
</tr>
</tbody>
</table>

According to the EN1993-1-1 the section is in Class 1 due to pure bending in both directions and also under normal force. The calculation will be performed with EN1993-1-1 Annex A – Method 1 and with Annex B – Method 2 regarding to get $k_{ij}$ interaction factors.

The characteristic normal force resistance:

$$N_{Rk} = A f_y = 5257 \cdot 355 = 1866 \text{kN}$$

The characteristic bending moment resistance around strong axis:

$$M_{y', Rk} = W_{pl, y'} f_y = 318082 \cdot 355 = 112.9 \text{kNm}$$

The characteristic bending moment resistance around weak axis:

$$M_{z', Rk} = W_{pl, z'} f_y = 195250 \cdot 355 = 69.31 \text{kNm}$$
### Flexural buckling around strong axis

The radius of gyration (strong axis):

\[ i_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{24443956}{5257}} = 68.19 \text{ mm} \]

The non-dimensional slenderness:

\[ \bar{\lambda} = \frac{L_{cr}}{i_y} = \frac{6000}{68.19} \cdot \frac{1}{76.41} = 1.152 \]

where

\[ \lambda_1 = \pi \sqrt{\frac{E}{f_y}} = \pi \sqrt{\frac{210000}{355}} = 76.41 \].

The imperfection \( \alpha \) factor value based on EN 1993-1-1 Table 6.2:

cold formed hollow section therefore “c” buckling curve is relevant thus the imperfection factor is \( \alpha_y = 0.49 \).

\[ \Phi_y = 0.5 \left[ 1 + \alpha_y \left( \bar{\lambda}_y - 0.2 \right) + \bar{\lambda}_y^2 \right] = 0.5 \left[ 1 + 0.49 \left( 1.152 - 0.2 \right) + 1.152^2 \right] = 1.386 \]

Reduction factor:

\[ \chi_y = \frac{1}{\Phi_y + \sqrt{\Phi_y^2 - \bar{\lambda}_y^2}} = \frac{1}{1.386 + \sqrt{1.386^2 - 1.152^2}} = 0.4637 \]

\[ F_{Ed} = 20 \text{ kN} \]

\[ q_{Ed} = 8 \text{ kN/m} \]

\[ N_{Ed} = 80 \text{ kN} \]

\[ M_{Ed} = 36 \text{ kNm} \]

\[ M_{Ed} = 30 \text{ kNm} \]

---

*Figure 9.3.6.1 – The statical system, the cross-section and the design loads and internal forces in the global system*
Flexural buckling around weak axis

The radius of gyration (weak axis): \( i_z = \sqrt{\frac{I_z'}{A}} = \sqrt{\frac{8177434}{5257}} = 39.44 \text{ mm} \)

The non-dimensional slenderness: \( \bar{\lambda}_z = \frac{L_{cr}}{i_z} \frac{1}{\lambda_1} = \frac{6000}{39.44} \frac{1}{76.41} = 1.991 \), where \( \lambda_1 = \pi \sqrt{\frac{E_s}{f_y}} = \pi \sqrt{\frac{210000}{355}} = 76.41 \).

The imperfection \( \alpha \) factor value based on EN 1993-1-1 Table 6.2: cold formed hollow section therefore “c” buckling curve is relevant thus the imperfection factor is \( \alpha_z = 0.49 \).

\[ \Phi_z = 0.5 \left( 1 + \alpha_z \left( \bar{\lambda}_z - 0.2 \right) + \bar{\lambda}_z^2 \right) = 0.5 \left( 1 + 0.49 \left( 1.991 - 0.2 \right) + 1.991^2 \right) = 2.921 \]

Reduction factor: \( \chi_z = \frac{1}{\Phi_z + \sqrt{\Phi_z^2 - \bar{\lambda}_z^2}} = \frac{1}{2.921 + \sqrt{2.921^2 - 1.991^2}} = 0.1977 \)

Lateral torisalonal buckling general case according to EN1993-1-1:6.3.2.2

The major axis bending (around strong axis) is relevant according to EN1993-1-1:6.3.2.1

\( k = 1.0 \); \( k_\omega = 1.0 \) free to rotate about weak axis and restraint against movements and free to warp but restraint against rotation about the longitudinal axis.

The \( C_i \) coefficients are depending on the loading and end restraint conditions:

\( C_1 = 1.35 \); \( C_2 = 0.63 \); \( C_3 = 1.73 \)

Because the cross section is doubly symmetric: \( z_j = 0 \).

The distance between the point of load application and the shear centre in the relevant direction (this value has a sign): \( z_g = \frac{h}{2} = \frac{200}{2} = 100 \text{ mm} \) see Fig. 9.3.6.1.

The calculation of the elastic critical moment:

\[
M_{cr} = C_1 \frac{\pi^2 E I_z}{(k L)^2} \left[ \left( \frac{k}{k_\omega} \right)^2 \frac{I_{zz}}{I_z} + \left( \frac{k L}{k_\omega} \right)^2 G I_z \left( C_2 z_g - C_3 z_j \right)^2 - \left( C_2 z_g - C_3 z_j \right)^2 \right]
\]

\[
Z = C_2 z_g - C_3 z_j = 0.63 \cdot 100 - 1.73 \cdot 0 = 63.0 \text{ mm}
\]

\[
M_{cr} = 1.35 \frac{\pi^2}{(1 \cdot 6000)^2} \frac{210000 \cdot 8177434}{8177434} \left[ \left( \frac{1}{1} \right)^2 4.313 \cdot 10^6 + \left( 1 \cdot 6000 \right)^2 80769 \cdot 21571134 \right] \left[ \frac{\pi^2}{210000 \cdot 8177434} + 63.0^2 - 63.0 \right]
\]
$M_{cr} = 1183 \text{kNm}$

Non-dimensional slenderness: 

$$\lambda_{LT} = \sqrt{\frac{W_{pl,y} f_y}{M_{cr}}} = \sqrt{\frac{318082 \cdot 355}{1183000000}} = 0.3090$$

$$\Phi_{LT} = \frac{1 + \alpha_{LT}(\lambda_{LT} - 0.2) + \lambda_{LT}^2}{2} = 1 + 0.76(0.3090 - 0.2) + 0.3090^2 = 0.5892$$

where the imperfection $\alpha_{LT}$ factor value based on EN 1993-1-1 Table 6.4 thus “d” buckling curve is relevant. The imperfection factor is $\alpha_{LT} = 0.76$.

Reduction factor for lateral torsional buckling:

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \lambda_{LT}^2}} = \frac{1}{0.5892 + \sqrt{0.5892^2 - 0.3090^2}} = 0.9167$$

**Interaction factors according to EN1993-1-1 Annex A – Method 1**

Auxiliary terms:

$$\lambda_{\max} = \max \left[ \lambda_{y}, \lambda_{z} \right] = \max \left[ 1.152, 1.991 \right] = 1.991$$

Calculation of the non-dimensional slenderness due to uniform bending moment:

$$M_{cr} = \frac{\pi^2 EI^2}{L^2} \left[ \frac{I_w + L^2 GI_z}{\pi^2 EI_z} \right] = \frac{\pi^2}{600^2} \left( 210000 \cdot 8177434 \right) = 905.7 \text{kNm}$$

Non-dimensional slenderness: 

$$\lambda_0 = \sqrt{\frac{W_{pl,y} f_y}{M_{cr}}} = \sqrt{\frac{318082 \cdot 355}{905700000}} = 0.3531$$

The elastic flexural buckling forces:

$$N_{cr,y} = \frac{\pi^2 EI_y}{L^2} = \frac{\pi^2}{600^2} \left( 210000 \cdot 24443956 \right) = 1407 \text{kN}$$

$$N_{cr,z} = \frac{\pi^2 EI_z}{L^2} = \frac{\pi^2}{600^2} \left( 210000 \cdot 8177434 \right) = 470.8 \text{kN}$$
Because the section is doubly symmetric the elastic torsional-flexural buckling force:
\[ N_{cr,TF} = N_{cr,T} \]

The elastic critical torsional force:
\[ N_{cr,T} = \frac{1}{I_y^2 + I_z^2} \left( \frac{G I_t}{L_{cr}} + \frac{\pi^2 E I_0}{L_{cr}^2} \right) = \frac{1}{68.19^2 + 39.44^2} \left( 80769 \cdot 21571134 + \frac{\pi^2 \cdot 210000 \cdot 4.313 \cdot 10^9}{6000^2} \right) \]
\[ N_{cr,TF} = N_{cr,T} = 280800 \text{ kN} \]

Because:
\[ \tilde{\lambda}_0 = 0.3531 > 0.2 \sqrt{C_1} \left( 1 - \frac{N_{Ed}}{N_{cr,z}} \right) \left( 1 - \frac{N_{Ed}}{N_{cr,TF}} \right) = 0.2 \sqrt{1.35} \left( 1 - \frac{80}{470.8} \right) \left( 1 - \frac{80}{280800} \right) = 0.2218 \]

Thus:
\[ C_{my,0} = 1 - 0.18 \frac{N_{Ed}}{N_{cr,y}} = 1 - 0.18 \frac{80}{1407} = 0.9898 \]
\[ a_{LT} = 1 - \frac{J_t}{J_y} = 1 - \frac{21571134}{24443956} = 0.1175 \]
\[ \varepsilon_y = \frac{M_{y,Ed}}{N_{Ed}} \frac{A}{W_{el,y}} = 30 \cdot 10^6 \frac{5257}{80 \cdot 10^3 244440} = 8.065 \]
\[ C_{my} = C_{my,0} + (1 - C_{my,0}) \sqrt{\varepsilon_y} a_{LT} = 0.9898 + (1 - 0.9898) \frac{\sqrt{8.065} \cdot 0.1175}{1 + \sqrt{8.065} \cdot 0.1175} = 0.9924 \]
\[ C_{mz} = C_{mz,0} = 1 + 0.03 \frac{N_{Ed}}{N_{cr,z}} = 1 + 0.03 \frac{80}{470.8} = 1.005 \]
\[ C_{mLT} = \max \left[ \left( 1 - \frac{N_{Ed}}{N_{cr,z}} \right) \left( 1 - \frac{N_{Ed}}{N_{cr,T}} \right) \right] = \max \left[ \frac{0.9924^2}{1 + \frac{80}{470.8} \left( 1 - \frac{80}{280800} \right)} \right] \]
\[ C_{mLT} = \max \left[ \frac{0.1270}{1} \right] = 1 \]
\[ b_{LT} = 0.5 a_{LT} \tilde{\lambda}_0^2 \frac{M_{y,Ed}}{M_{pl,y,Rd}} \frac{M_{z,Ed}}{M_{pl,z,Rd}} = 0.5 \cdot 0.1175 \cdot 0.3531^2 \frac{30}{0.9167} \frac{36}{112.9} \frac{69.31}{353} = 0.001103 \]
\[ c_{LT} = 10 a_{LT} \tilde{\lambda}_0^2 \frac{M_{y,Ed}}{C_{my} X_{LT} M_{pl,y,Rd}} = 10 \cdot 0.1175 \cdot 0.3531^2 \frac{30}{5 + 1.991^2} \frac{0.9924}{0.9167} \frac{112.9}{69.31} = 0.002066 \]
\[ d_{LT} = 2 a_{LT} \tilde{\lambda}_0 \frac{M_{y,Ed}}{0.1 + \tilde{\lambda}_z^4} \frac{M_{z,Ed}}{C_{mz} X_{LT} M_{pl,z,Rd}} = 2 \cdot 0.1175 \cdot 0.3531 \frac{30}{0.9167} \frac{112.9}{69.31} \frac{69.31}{353} = 0.002512 \]
\[
\begin{align*}
d_{LT} &= 2.0 \cdot 0.1175 \cdot \frac{0.3531}{0.1 + 1.991^4} \cdot \frac{30}{0.9924 \cdot 0.9167 \cdot 112.9} \cdot \frac{36}{1.005 \cdot 69.31} = 0.0007921 \\
e_{LT} &= 1.7 \cdot a_{LT} \cdot \frac{\bar{\lambda}_w}{0.1 + \bar{\lambda}_z} \cdot \frac{M_{y, Ed}}{M_{y, Rd}} = 1.7 \cdot 0.1175 \cdot 0.3531 \cdot \frac{30}{0.1 + 1.991^4} \cdot \frac{0.9924 \cdot 0.9167 \cdot 112.9}{1.005 \cdot 69.31} = 0.001303 \\
w_y &= \min \left[ \frac{W_{pl, \ast, y}}{W_{el, \ast, y}} \right] = \min \left[ \frac{318082}{244440} \right] = \min \left[ \frac{1.301}{1.5} \right] = 1.301 \\
w_z &= \min \left[ \frac{W_{pl, \ast, z}}{W_{el, \ast, z}} \right] = \min \left[ \frac{195250}{163549} \right] = \min \left[ \frac{1.194}{1.5} \right] = 1.194 \\
n_{pl} &= \frac{N_{Ed}}{N_{Rd} / y_{ML}} = \frac{80}{1866/1.0} = 0.04287 \\
C_{yy} &= \max \left[ 1 + (w_y - 1) \left( 2 - \frac{1.6}{w_y} \frac{C_{my}^2 \bar{\lambda}_{max}}{w_y} - \frac{1.6}{w_y} \frac{C_{my}^2 \bar{\lambda}_{max}^2}{w_y} \right) n_{pl} - b_{LT} \right] \\
C_{yy} &= \max \left[ 1 + (1.301 - 1) \left( 2 - \frac{1.6}{1.301} \frac{0.9924^2 \cdot 1.991 - 1.6}{1.301} \frac{0.9924^2 \cdot 1.991^2}{0.04287 - 0.001103} \right) \right] \\
C_{yy} &= \max \left[ \frac{0.9324}{0.7685} \right] = 0.9324 \\
C_{yz} &= \max \left[ 1 + (w_z - 1) \left( 2 - 14 \frac{C_{mz}^2 \bar{\lambda}_{max}^2}{w_z} \right) n_{pl} - c_{LT} \right] \\
C_{yz} &= \max \left[ 1 + (1.194 - 1) \left( 2 - 14 \frac{1.005^2 \cdot 1.991^2}{1.194^5} \right) \frac{0.04287 - 0.002066}{0.8241} \right] = \max \left[ \frac{0.8241}{0.4815} \right] = 0.8241 \\
C_{yz} &= \max \left[ \frac{0.8241}{0.4815} \right] = 0.8241 \\
C_{yz} &= \max \left[ \frac{0.8241}{0.4815} \right] = 0.8241 \\
C_{yz} &= \max \left[ \frac{0.8241}{0.4815} \right] = 0.8241 \\
C_{yz} &= \max \left[ \frac{0.8241}{0.4815} \right] = 0.8241 \\
C_{yz} &= \max \left[ \frac{0.8241}{0.4815} \right] = 0.8241 \\
C_{yz} &= \max \left[ \frac{0.8241}{0.4815} \right] = 0.8241
\end{align*}
\]
\[ C_{zy} = \text{max} \left[ 1 + \left( \frac{w_y}{w_y - 1} \right) \left( 2 - 14 \frac{C_{my}^2 \lambda_{\text{max}}^2}{w_y^5} \right) n_{pl} - d_{LT} \right] \]

\[ C_{zy} = \text{max} \left[ 1 + \left( 1.301 - 1 \right) \left( 2 - 14 \frac{0.9924^2 \cdot 1.991^2}{1.301^3} \right) 0.04287 - 0.0007921 \right] \]

\[ C_{zy} = \text{max} \left[ 0.8363 \quad \frac{1}{0.4813} = 0.8363 \right] \]

\[ C_{zz} = \text{max} \left[ 1 + \left( \frac{w_z}{w_z - 1} \right) \left( 2 - \frac{1.6}{w_z} C_{mz}^2 \lambda_{\text{max}}^2 - \frac{1.6}{w_z} C_{mz}^2 \lambda_{\text{max}}^2 \right) n_{pl} - e_{LT} \right] \]

\[ C_{zz} = \text{max} \left[ 1 + \left( 1.194 - 1 \right) \left( 2 - \frac{1.6}{1.194} \frac{1.005^2 \cdot 1.991 - 1.6}{1.005^2 \cdot 1.991^2} 0.04287 - 0.001303 \right) \right] \]

\[ C_{zz} = \text{max} \left[ 0.9493 \quad \frac{1}{0.8376} = 0.9493 \right] \]

\[ \mu_y = \frac{1 - \frac{N_{Ed}}{N_{cr,y}}}{1 - \frac{80}{1407}} = 0.9687 \]

\[ \mu_z = \frac{1 - \frac{N_{Ed}}{N_{cr,z}}}{1 - \frac{80}{470.8}} = 0.8589 \]
And finally the interaction factors based on the auxiliary terms:

\[ k_{yy} = C_{my} C_{my LT} \frac{\mu_y}{N_{Ed}} \frac{1}{C_{yy}} = 0.9924 \cdot 1 \frac{0.9687}{1 - \frac{80}{1407}} = 1.093 \]

\[ k_{yz} = C_{mz} \frac{\mu_y}{N_{Ed}} \frac{1}{C_{yz}} 0.6 \sqrt{\frac{w_z}{w_y}} = 1.005 \frac{0.9687}{1 - \frac{80}{470.8}} = 0.6768 \]

\[ k_{zy} = C_{my} C_{my LT} \frac{\mu_z}{N_{Ed}} \frac{1}{C_{zy}} 0.6 \sqrt{\frac{w_y}{w_z}} = 0.9924 \cdot 1 \frac{0.8589}{1 - \frac{80}{112.9}} = 0.6768 \]

\[ k_{zz} = C_{mz} \frac{\mu_z}{N_{Ed}} \frac{1}{C_{zz}} = 1.005 \frac{0.8589}{1 - \frac{80}{470.8}} = 1.095 \]

The interaction formulas:

\[ \frac{N_{Ed}}{\chi_y N_{Rk}} + k_{yy} \frac{M_{y,Ed}}{\chi_{LT} M_{y, Rk}} + k_{yz} \frac{M_{z,Ed}}{\chi_{MT} M_{z, Rk}} = \frac{80}{1.0} + \frac{1.093}{0.916} + \frac{0.818}{0.936} = 0.8342 \]

\[ \frac{N_{Ed}}{\chi_z N_{Rk}} + k_{zy} \frac{M_{y,Ed}}{\chi_{LT} M_{y, Rk}} + k_{zz} \frac{M_{z,Ed}}{\chi_{MT} M_{z, Rk}} = \frac{80}{1.0} + \frac{0.677}{0.916} + \frac{0.980}{0.936} = 0.9818 \]

The hand calculation and FEM-Design calculation are almost identical to each other considering Annex A (Method 1). The difference is less than 0.5%. See Fig. 9.3.6.2 about the detailed FEM-Design results.

**Interaction factors according to EN1993-1-1 Annex B – Method 2**

Due to the concentrated load in the weak direction (bending around major axis) EN 1993-1-1 Table B.3:

\[ C_{my} = 0.9 \]

Due to the uniform load in the strong direction (bending around minor axis) EN 1993-1-1 Table B.3:

\[ C_{mz} = 0.95 \]
The interaction factors according to EN 1993-1-1 Table B.1:

\[
\begin{align*}
  k_{yy} &= \min \left[ C_{my} \left( 1 + \left( \frac{\chi_y}{1} - 0.2 \right) \frac{N_{Ed}}{N_{Rk}} \frac{M_y}{Y_{M1}} \right) \right] = \min \left[ 0.9 \left( 1 + (1.152 - 0.2) \frac{80}{0.4637 \cdot 1866} \frac{0.1}{1.0} \right) \right] = 0.9666 \\
  k_{zz} &= \min \left[ C_{mz} \left( 1 + \left( \frac{\chi_z}{1} - 0.2 \right) \frac{N_{Ed}}{N_{Rk}} \frac{M_z}{Y_{M1}} \right) \right] = \min \left[ 0.95 \left( 1 + (1.991 - 0.2) \frac{80}{0.1977 \cdot 1866} \frac{0.1}{1.0} \right) \right] = 1.115 \\
  k_{yz} &= 0.6 \cdot k_{zz} = 0.6 \cdot 1.115 = 0.669; \quad k_{zy} = 0.6 \cdot k_{yy} = 0.6 \cdot 0.9666 = 0.580
\end{align*}
\]

The interaction formulas:

\[
\begin{align*}
  \frac{N_{Ed}}{X_y N_{Rk}} + k_{yy} \frac{M_{y,Ed}}{X_L M_{y,Rk}} + k_{zy} \frac{M_{z,Ed}}{Y_{M1}} &= \frac{80}{0.4637 \cdot 1866} + \frac{0.9666 \cdot 30}{1.0} + \frac{0.669 \cdot 36}{1.0} = 0.7201 \\
  \frac{N_{Ed}}{X_z N_{Rk}} + k_{zy} \frac{M_{y,Ed}}{X_L M_{y,Rk}} + k_{zz} \frac{M_{z,Ed}}{Y_{M1}} &= \frac{80}{0.1977 \cdot 1866} + \frac{0.580 \cdot 30}{1.0} + \frac{1.115 \cdot 36}{1.0} = 0.9641
\end{align*}
\]

The hand calculation and FEM-Design calculation are identical to each other considering Annex B (Method 2). See Fig. 9.3.6.3 about the detailed FEM-Design results.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst170x/models/9.3.6 Interaction of biaxial bending and axial compression in an RHS section.str
Verification Examples

**Fig. 9.3.6.2 - The FEM-Design results about the interaction according to Annex A**

<table>
<thead>
<tr>
<th>Verification Type</th>
<th>Equation</th>
<th>Parameters</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Flexural buckling, 1-1 - Part 1: 6.3.1</strong></td>
<td>$T_{b1} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
<tr>
<td><strong>Lateral torsional buckling - Part 1: 6.3.2.2</strong></td>
<td>$N_{b1} = 1000 \times 10 \times 10 \times 10$</td>
<td>$1000, 10, 10, 10$</td>
<td>$2000 \times 100 \times 100$</td>
</tr>
<tr>
<td><strong>Interaction between normal force and bending 1. - Part 1: 6.3.3</strong></td>
<td>$N_{b1} = 1000 \times 10 \times 10 \times 10$</td>
<td>$1000, 10, 10, 10$</td>
<td>$2000 \times 100 \times 100$</td>
</tr>
</tbody>
</table>

**Normal capacity - Part 1-1: 6.2**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
<th>Parameters</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b1} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b2} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
</tbody>
</table>

**Flexural buckling, 1-1 - Part 1: 6.3.1**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
<th>Parameters</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b1} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b2} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
</tbody>
</table>

**Lateral torsional buckling - Part 1: 6.3.2.2**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
<th>Parameters</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b1} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b2} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
</tbody>
</table>

**Interaction between normal force and bending 1. - Part 1: 6.3.3**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
<th>Parameters</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b1} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b2} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
</tbody>
</table>

**Interaction between normal force and bending 2. - Part 1: 6.3.3**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
<th>Parameters</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b1} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
<tr>
<td>$C_{b1} = 1$, $C_{b2} = 1$, $C_{b3} = 1$</td>
<td>$\phi_{b2} = \frac{1}{2} \left( F_{b1} + F_{b2} \right)$</td>
<td>$F_{b1}, F_{b2}$</td>
<td>$1.15 \times 10^6$ Nmm²</td>
</tr>
</tbody>
</table>
Verification Examples

Fig. 9.3.6.3 – The FEM-Design results about the interaction according to Annex B

Normal capacity - Part 1-4: 6.2
LC 1'- 'x = 3000 mm
Class 1 = 1. Class 1 = 1. Class 1 = 1
\( V_{N,d} = 0.90 \times V_{N,KN} = 0.90 \times 293 = 263.7\) kN \( p_{\ell} = 0.00 \)
\( V_{N,KN} = 1.00 \times V_{N,KN} = 1.00 \times 293 = 293 \) kN \( p_{\ell} = 0.00 \)
\( N_{N,d} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1
\( N_{N,KN} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1

Flexural buckling, 1-4 - Part 1-4: 6.3.1
LC 1'- 'x = 3000 mm
Class 1 = 1. Class 1 = 1. Class 1 = 1
\( \gamma_{M} = 1.50 \) for method 1
\( N_{N,d} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1
\( N_{N,KN} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1

Interaction between normal force and bending 1.
LC 1'- 'x = 3000 mm
Class 1 = 1. Class 1 = 1. Class 1 = 1
\( V_{N,d} = 0.90 \times V_{N,KN} = 0.90 \times 293 = 263.7\) kN \( p_{\ell} = 0.00 \)
\( V_{N,KN} = 1.00 \times V_{N,KN} = 1.00 \times 293 = 293 \) kN \( p_{\ell} = 0.00 \)
\( N_{N,d} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1
\( N_{N,KN} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1

Lateral torsional buckling - Part 1-4: 6.3.2
LC 1'- 'x = 3000 mm
Class 1 = 1. Class 1 = 1. Class 1 = 1
\( N_{N,d} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1
\( N_{N,KN} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1

Interaction between normal force and bending 2.
LC 1'- 'x = 3000 mm
Part 1-4: 6.3.3
Class 1 = 1. Class 1 = 1. Class 1 = 1
\( V_{N,d} = 0.90 \times V_{N,KN} = 0.90 \times 293 = 263.7\) kN \( p_{\ell} = 0.00 \)
\( V_{N,KN} = 1.00 \times V_{N,KN} = 1.00 \times 293 = 293 \) kN \( p_{\ell} = 0.00 \)
\( N_{N,d} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1
\( N_{N,KN} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1

Flexural buckling, 2-2 - Part 1-1: 6.3.1
LC 1'- 'x = 3000 mm
Class 1 = 1. Class 1 = 1. Class 1 = 1
\( \gamma_{M} = 1.50 \) for method 1
\( N_{N,d} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1
\( N_{N,KN} = 80 + 0.00 = 80 \) kN \( \gamma_{M} = 1.50 \) for method 1

Interaction between normal force and bending 2.
LC 1'- 'x = 3000 mm
Part 1-1: 6.3.3
Class 1 = 1. Class 1 = 1. Class 1 = 1
\( V_{N,d} = 0.90 \times V_{N,KN} = 0.90 \times 293 = 263.7\) kN \( p_{\ell} = 0.00 \)
\( V_{N,KN} = 1.00 \times V_{N,KN} = 1.00 \times 293 = 293 \) kN \( p_{\ell} = 0.00 \)
9.3.7 Interaction calculation with a Class 4 section

This chapter is unfinished.
9.4 Timber design

This chapter is unfinished.

9.4.1 Design of timber bars
9.4.2 CLT panel design with laminated composite shell theory

9.4.2.1 Calculation of the homogenized shell material stiffness matrix

In the next sub-chapters of this example we will calculate the homogenized stiffness values with different settings using the layer composition what you can see in Fig. 18.

<table>
<thead>
<tr>
<th>No</th>
<th>Material</th>
<th>Thickness [mm]</th>
<th>Theta [°]</th>
<th>Ex [N/mm²]</th>
<th>Ey [N/mm²]</th>
<th>Fxy [°]</th>
<th>Gxy [N/mm²]</th>
<th>Gyz [N/mm²]</th>
<th>Gxz [N/mm²]</th>
<th>Eho [N/mm²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C24</td>
<td>45.00</td>
<td>0.00</td>
<td>11600</td>
<td>459</td>
<td>0-40</td>
<td>690</td>
<td>690</td>
<td>100</td>
<td>420</td>
</tr>
<tr>
<td>2</td>
<td>C16</td>
<td>40.00</td>
<td>-90</td>
<td>8000</td>
<td>270</td>
<td>0-40</td>
<td>500</td>
<td>500</td>
<td>50</td>
<td>370</td>
</tr>
<tr>
<td>3</td>
<td>C24</td>
<td>35.00</td>
<td>0.00</td>
<td>11600</td>
<td>459</td>
<td>0-40</td>
<td>690</td>
<td>690</td>
<td>100</td>
<td>420</td>
</tr>
</tbody>
</table>

Figure 18 – The adjusted layer composition

In the next calculations (and also by FEM-Design “Display stiffness” button) we will calculate the stiffnesses relate to the mid-surface (center) of the shell layout composition.

The co-ordinates of the edge of the different layers related to the mid-surface:

\[
{z} = \begin{bmatrix} 0.045 \\ 0.030 \\ -0.010 \\ -0.045 \end{bmatrix} \text{ m}
\]

If someone adjusts the \(k_{def}\) deformation factors by the panel properties in the different limit states then the homogenization method will be the same but before the calculation the Young's moduli and shear moduli will be divided with \(1+k_{def}\) and these reduced moduli will be the input values of the homogenization method as it was stated in Chapter 5.5.

9.4.2.1.1 Calculation with shear coupling

In this sub-chapter we will show the stiffness calculation in case of shear coupling between the layers when they are working together. In the first Chapter 6.1.1.1 we will show the calculation with the glue at narrow side option, then in Chapter 6.1.1.2 with no glue at narrow side option.
9.4.2.1.1 Glue at narrow side

First of all we should calculate the material stiffness matrices of the layers interpreted in the grain directions.

Material of layer No. 1 and 3 are the same therefore these matrices are the same:

In-plane part:

\[
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix} = \begin{bmatrix}
\frac{E_x}{1-\nu_{xy}^2} & \nu_{xy} \frac{E_y}{E_x} & 0 \\
\nu_{xy} \frac{E_y}{E_x} & \frac{E_y}{1-\nu_{xy}^2} & 0 \\
0 & 0 & G_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
11600 \\
1-0.4^2 \frac{450}{11600} \\
0
\end{bmatrix}
\begin{bmatrix}
0.4 \\
1-0.4^2 \frac{450}{11600} \\
0
\end{bmatrix}
\begin{bmatrix}
450 \\
1-0.4^2 \frac{450}{11600} \\
0
\end{bmatrix}
\begin{bmatrix}
0.4 \\
1-0.4^2 \frac{450}{11600} \\
0
\end{bmatrix}
\begin{bmatrix}
450 \\
1-0.4^2 \frac{450}{11600} \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1-0.4^2 \frac{450}{11600} \\
0
\end{bmatrix}
\begin{bmatrix}
11672 \\
181.1 \\
0
\end{bmatrix} \text{MPa}
\]

Transverse part:

\[
\begin{bmatrix}
Q_{55} & 0 \\
0 & Q_{44}
\end{bmatrix} = \begin{bmatrix}
G_{xz} & 0 \\
0 & G_{yz}
\end{bmatrix} = \begin{bmatrix}
690 & 0 \\
0 & 100
\end{bmatrix} \text{MPa}
\]

Layer No. 2:

In-plane part:

\[
\begin{bmatrix}
8000 \\
1-0.4^2 \frac{270}{8000} \\
0.4 \\
1-0.4^2 \frac{270}{8000} \\
0
\end{bmatrix}
\begin{bmatrix}
0.4 \\
1-0.4^2 \frac{270}{8000} \\
0.4 \\
1-0.4^2 \frac{270}{8000} \\
0
\end{bmatrix}
\begin{bmatrix}
270 \\
270 \\
270 \\
270 \\
0
\end{bmatrix}
\begin{bmatrix}
0.4 \\
1-0.4^2 \frac{270}{8000} \\
0.4 \\
1-0.4^2 \frac{270}{8000} \\
0
\end{bmatrix}
\begin{bmatrix}
0.4 \\
1-0.4^2 \frac{270}{8000} \\
0.4 \\
1-0.4^2 \frac{270}{8000} \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1-0.4^2 \frac{270}{8000} \\
0.4 \\
1-0.4^2 \frac{270}{8000} \\
0
\end{bmatrix}
\begin{bmatrix}
500 \\
0 \\
500 \\
0 \\
500
\end{bmatrix}
\begin{bmatrix}
8043 \\
108.6 \\
0.6 \\
0 \\
0
\end{bmatrix} \text{MPa}
\]

Transverse part:

\[
\begin{bmatrix}
Q_{55} & 0 \\
0 & Q_{44}
\end{bmatrix} = \begin{bmatrix}
500 & 0 \\
0 & 50
\end{bmatrix} \text{MPa}
\]
According to Fig. 18 the grain direction of layer No. 1 and 3 are in the local $x'$ direction of the shell ($0^\circ$ orientation). It means that the previously calculated material stiffness will be the one what we need to use during the homogenization:

$$\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_{1&3} = 
\begin{bmatrix}
11672 & 181.1 & 0 \\
181.1 & 452.8 & 0 \\
0 & 0 & 690
\end{bmatrix} \text{ MPa}
$$

$$\begin{bmatrix}
\bar{Q}_{15} & \bar{Q}_{45} \\
\bar{Q}_{45} & \bar{Q}_{44}
\end{bmatrix}_{1&3} = 
\begin{bmatrix}
690 & 0 \\
0 & 100
\end{bmatrix} \text{ MPa}
$$

But by layer No. 2 we can see that the grain direction is perpendicular to the $x'$ local system direction ($90^\circ$ orientation). Thus we should transform the material stiffness into the local $x'$ direction. With the transformation matrix in the theoretical description we will get:

$$\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_{2} = 
\begin{bmatrix}
271.5 & 108.6 & 0 \\
108.6 & 8043 & 0 \\
0 & 0 & 500
\end{bmatrix} \text{ MPa}
$$

$$\begin{bmatrix}
\bar{Q}_{15} & \bar{Q}_{45} \\
\bar{Q}_{45} & \bar{Q}_{44}
\end{bmatrix}_{2} = 
\begin{bmatrix}
50 & 0 \\
0 & 500
\end{bmatrix} \text{ MPa}
$$

After these steps we can calculate the ABD and shear part of the stiffness matrix.

**The bending part:**

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k^j - z_k^{j+1}) \quad (i, j = 1, 2, 6)$$

$$D_{11} = \frac{1}{3} \left[ 11672(0.045^3 - 0.03^3) + 271.5(0.03^3 - (-0.01)^3) + 11672((-0.01)^3 - (-0.045)^3) \right] =
= 0.6027 \frac{\text{MNm}^2}{m} = 602.7 \text{kNm}
$$

$$D_{12} = \frac{1}{3} \left[ 181.1(0.045^3 - 0.03^3) + 108.6(0.03^3 - (-0.01)^3) + 181.1((-0.01)^3 - (-0.045)^3) \right] =
= 0.01033 \frac{\text{MNm}^2}{m} = 10.33 \text{kNm}
$$

$$D_{22} = \frac{1}{3} \left[ 452.8(0.045^3 - 0.03^3) + 8043(0.03^3 - (-0.01)^3) + 452.8((-0.01)^3 - (-0.045)^3) \right] =
= 0.09835 \frac{\text{MNm}^2}{m} = 98.35 \text{kNm}
$$

$$D_{66} = \frac{1}{3} \left[ 690(0.045^3 - 0.03^3) + 500(0.03^3 - (-0.01)^3) + 690((-0.01)^3 - (-0.045)^3) \right] =
= 0.04014 \frac{\text{MNm}^2}{m} = 40.14 \text{kNm}
$$
In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:

\[ D_{16} = D_{26} = 0. \]

**The membrane part:**

\[ A_{ij} = \sum_{k=1}^{N} \left( \overline{O}_{ij} \right)_k (z_k - z_{k+1}) \quad (i, j = 1, 2, 6) \]

\[ A_{11} = 11672(0.045 - 0.03) + 271.5(0.03 - (-0.01)) + 11672((-0.01) - (-0.045)) = 594.46 \text{ MN/m} = 594460 \text{ kN/m} \]

\[ A_{12} = 181.1(0.045 - 0.03) + 108.6(0.03 - (-0.01)) + 181.1((-0.01) - (-0.045)) = 13.40 \text{ MN/m} = 13400 \text{ kN/m} \]

\[ A_{22} = 452.8(0.045 - 0.03) + 8043(0.03 - (-0.01)) + 452.8((-0.01) - (-0.045)) = 344.4 \text{ MN/m} = 344400 \text{ kN/m} \]

\[ A_{66} = 690(0.045 - 0.03) + 500(0.03 - (-0.01)) + 690((-0.01) - (-0.045)) = 54.5 \text{ MN/m} = 54500 \text{ kN/m} \]

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:

\[ A_{16} = A_{26} = 0. \]

**Eccentricity part:**

\[ B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\overline{O}_{ij})_k \left(z_k^2 - z_{k+1}^2 \right) \quad (i, j = 1, 2, 6) \]

\[ B_{11} = \frac{1}{2} \left[ 11672(0.045^2 - 0.03^2) + 271.5(0.03^2 - (-0.01)^2) + 11672((-0.01)^2 - (-0.045)^2) \right] = -4.560 \text{ MNm/m} = -4560 \text{ kN} \]

\[ B_{12} = \frac{1}{2} \left[ 181.1(0.045^2 - 0.03^2) + 108.6(0.03^2 - (-0.01)^2) + 181.1((-0.01)^2 - (-0.045)^2) \right] = -0.029 \text{ MNm/m} = -29 \text{ kN} \]

\[ B_{22} = \frac{1}{2} \left[ 452.8(0.045^2 - 0.03^2) + 8043(0.03^2 - (-0.01)^2) + 452.8((-0.01)^2 - (-0.045)^2) \right] = 3.036 \text{ MNm/m} = 3036 \text{ kN} \]
Verification Examples

\[
B_{66} = \frac{1}{2} \left[ 690(0.045^2 - 0.03^2) + 500(0.03^2 - (-0.01)^2) + 690((-0.01)^2 - (-0.045)^2) \right] = -0.076 \text{ MNm/m} = -76 \text{ kN}
\]

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:

\[
B_{16} = B_{26} = 0.
\]

Shear part:

In this case the calculation of transverse shear stiffnesses is much easier than by a general case (see the theory description also about this topic).

Here:

\[
\begin{bmatrix}
S_{55} & S_{45} \\
S_{45} & S_{44}
\end{bmatrix} =
\begin{bmatrix}
\overline{S}_{55} & 0 \\
0 & \overline{S}_{44}
\end{bmatrix}
\]

With the aim of the following equations first of all we should calculate the shear correction factors in the main stiffness direction (and perpendicular to it) which now coincide with the local system of the shell (see Fig. 18).

The shear correction factors:

\[
\rho_{13} = \frac{R_1^2}{d_1 \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{g_1^z(z)}{\overline{Q}_{55}(z)} \, dz} = 0.1638 \quad ; \quad \rho_{23} = \frac{R_2^2}{d_2 \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{g_2^z(z)}{\overline{Q}_{44}(z)} \, dz} = 0.8528
\]

Based on these the shear stiffnesses:

\[
\overline{S}_{55} = \rho_{13} \sum_{k=1}^{N} (\overline{Q}_{55})_k (z_k - z_{k+1}) =
\]

\[
= 0.1638 \left[ 690(0.045 - 0.03) + 500(0.03 - (-0.01)) + 690((-0.01) - (-0.045)) \right] = 5979 \frac{\text{kN}}{\text{m}}
\]

\[
\overline{S}_{44} = \rho_{23} \sum_{k=1}^{N} (\overline{Q}_{44})_k (z_k - z_{k+1}) =
\]

\[
= 0.8528 \left[ 100(0.045 - 0.03) + 500(0.03 - (-0.01)) + 100((-0.01) - (-0.045)) \right] = 21320 \frac{\text{kN}}{\text{m}}
\]
Thus the final homogenized laminated shell stiffnesses (in kNm, kN and kN/m):

\[
\begin{bmatrix}
602.7 & 10.33 & 0 & -4560 & -29 & 0 & 0 & 0 \\
10.33 & 98.35 & 0 & -29 & 3036 & 0 & 0 & 0 \\
0 & 0 & 40.14 & 0 & 0 & -76 & 0 & 0 \\
-4560 & -29 & 0 & 569460 & 13400 & 0 & 0 & 0 \\
-29 & 3036 & 0 & 13400 & 344400 & 0 & 0 & 0 \\
0 & 0 & -76 & 0 & 0 & 54500 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5979 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 21320 & 0
\end{bmatrix}
\]

Fig. 19 shows the values based on FEM-Design. The values are the same.
9.4.2.1.1.2 No glue at narrow side

This calculation only differs from Chapter 6.1.1.1 that in the whole homogenization method the Young's moduli of the layers in perpendicular direction to the grains assumed to zero independently from the input data, see Fig. 18.

Material of layer No. 1 and 3 are the same therefore these matrices are the same:

**In-plane part:**

\[
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix} =
\begin{bmatrix}
\frac{E_x}{1-\nu_{xy}^2} & \frac{E_y}{1-\nu_{xy}^2} & 0 \\
\nu_{xy} \frac{E_y}{E_x} & \frac{E_y}{E_x} & 0 \\
0 & 0 & \frac{G_{xy}}{E_x}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
11600 & 0 & 0 \\
1-0.4^2 & 0 & 11600 \\
0 & 0 & 0 \\
0.4 & 0 & 11600 \\
0 & 0 & 0 \\
1-0.4^2 & 0 & 11600 \\
0 & 0 & 690
\end{bmatrix} \text{MPa}
\]

**Transverse part:**

\[
\begin{bmatrix}
Q_{55} & 0 \\
0 & Q_{44}
\end{bmatrix} =
\begin{bmatrix}
G_{xz} & 0 \\
0 & G_{yz}
\end{bmatrix} = \begin{bmatrix} 11600 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 690 \end{bmatrix} \text{MPa}
\]

**Layer No. 2:**

**In-plane part:**

\[
\begin{bmatrix}
\frac{8000}{1-0.4^2} & 0 & 0 \\
0 & \frac{8000}{1-0.4^2} & 0 \\
0.4 & 0 & 8000 \\
0 & 0.4 & 8000 \\
1-0.4^2 & 0 & 8000 \\
0 & 1-0.4^2 & 8000 \\
0 & 0 & 500 \\
0 & 0 & 0
\end{bmatrix} =
\begin{bmatrix}
8000 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 500 \end{bmatrix} \text{MPa}
\]

**Transverse part:**

\[
\begin{bmatrix}
Q_{55} & 0 \\
0 & Q_{44}
\end{bmatrix} = \begin{bmatrix} 500 & 0 \\
0 & 50 \end{bmatrix} \text{MPa}
\]

According to Fig. 18 the grain direction of layer No. 1 and 3 are in the local \(x'\) direction of the
shell (0° orientation). It means that the previously calculated material stiffness will be the one what we need to use during the homogenization.

\[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}_{1&3} = \begin{bmatrix}
11600 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 690
\end{bmatrix} \text{ MPa}
\]

\[
\begin{bmatrix}
Q_{45} & Q_{45} \\
Q_{45} & Q_{44}
\end{bmatrix}_{1&3} = \begin{bmatrix}
690 & 0 \\
0 & 100
\end{bmatrix} \text{ MPa}
\]

But by layer No. 2 we can see that the grain direction is perpendicular to the \(x'\) local system direction (90° orientation). Thus we should transform the former material stiffness into the local \(x'\) direction. With the transformation matrix in the theoretical description we will get:

\[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}_{2} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 8000 & 0 \\
0 & 0 & 500
\end{bmatrix} \text{ MPa}
\]

\[
\begin{bmatrix}
Q_{45} & Q_{45} \\
Q_{45} & Q_{44}
\end{bmatrix}_{2} = \begin{bmatrix}
50 & 0 \\
0 & 500
\end{bmatrix} \text{ MPa}
\]

After these steps we can calculate the ABD and shear part of the stiffness matrix.

The bending part:

\[
D_{ij} =\frac{1}{3} \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k^3 - z_{k+1}^3) \quad (i, j = 1, 2, 6)
\]

\[
D_{11} = \frac{1}{3} [11600(0.045^3 - 0.03^3) + 0(0.03^3 - (-0.01)^3) + 11600((-0.01)^3 - (-0.045)^3)] = 0.5964 \text{ MNm}^2/m = 596.4 \text{ kNm}
\]

\[
D_{22} = \frac{1}{3} [0(0.045^3 - 0.03^3) + 8000(0.03^3 - (-0.01)^3) + 0((-0.01)^3 - (-0.045)^3)] = 0.07467 \text{ MNm}^2/m = 74.67 \text{ kNm}
\]

\[
D_{66} = \frac{1}{3} [690(0.045^3 - 0.03^3) + 500(0.03^3 - (-0.01)^3) + 690((-0.01)^3 - (-0.045)^3)] = 0.04014 \text{ MNm}^2/m = 40.14 \text{ kNm}
\]

\[
D_{12} = D_{16} = D_{26} = 0
\]
The membrane part:

\[ A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k - z_{k+1}) \quad (i,j = 1,2,6) \]

\[ A_{11} = 11600(0.045 - 0.03) + 0(0.03 - (-0.01)) + 11600((-0.01) - (-0.045)) = 580 \frac{MN}{m} = 580000 \frac{kN}{m} \]

\[ A_{22} = 0(0.045 - 0.03) + 8000(0.03 - (-0.01)) + 0((-0.01) - (-0.045)) = 320 \frac{MN}{m} = 320000 \frac{kN}{m} \]

\[ A_{66} = 690(0.045 - 0.03) + 500(0.03 - (-0.01)) + 690((-0.01) - (-0.045)) = 54.5 \frac{MN}{m} = 54500 \frac{kN}{m} \]

\[ A_{12} = A_{16} = A_{26} = 0 \]

Eccentricity part:

\[ B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k^2 - z_{k+1}^2) \quad (i,j = 1,2,6) \]

\[ B_{11} = \frac{1}{2} \left[ 11600(0.045^2 - 0.03^2) + 0(0.03^2 - (-0.01)^2) + 11600((-0.01)^2 - (-0.045)^2) \right] = -4.64 \frac{MNm}{m} = -4640 \frac{kN}{m} \]

\[ B_{22} = \frac{1}{2} \left[ 0(0.045^2 - 0.03^2) + 8000(0.03^2 - (-0.01)^2) + 0((-0.01)^2 - (-0.045)^2) \right] = 3.2 \frac{MNm}{m} = 3200 \frac{kN}{m} \]

\[ B_{66} = \frac{1}{2} \left[ 690(0.045^2 - 0.03^2) + 500(0.03^2 - (-0.01)^2) + 690((-0.01)^2 - (-0.045)^2) \right] = -0.076 \frac{MNm}{m} = -76 \frac{kN}{m} \]

\[ B_{12} = B_{16} = B_{26} = 0 \]

Shear part:

In this case the calculation of the shear correction is much easier than by a general case (see the theory description also about this topic).

Here:

\[
\begin{bmatrix}
S_{55} & S_{45} \\
S_{45} & S_{44}
\end{bmatrix}
= \begin{bmatrix}
\overline{S}_{55} & 0 \\
0 & \overline{S}_{44}
\end{bmatrix}
\]
With the aim of the following equations first of all we should calculate the shear correction factors in the main stiffness direction (and perpendicular to it) which now coincide with the local system of the shell (see Fig. 18).

The shear correction factors:

$$\rho_{13} = \frac{R_1^2}{\frac{t}{2} \int R_{55}^2(z) \, dz} = 0.1640 \quad ; \quad \rho_{23} = \frac{R_2^2}{\frac{t}{2} \int R_{44}^2(z) \, dz} = 0.6667$$

Based on these the shear stiffnesses:

$$\bar{S}_{55} = \rho_{13} \sum_{k=1}^{N} (\overline{R}_{55})_k (z_k - z_{k+1}) =$$

$$= 0.1640 \left[ 690 \left( 0.045 - 0.03 \right) + 50 \left( 0.03 - (-0.01) \right) + 690 \left( (-0.01) - (-0.045) \right) \right] = 5986 \text{ kN/m}$$

$$\bar{S}_{44} = \rho_{23} \sum_{k=1}^{N} (\overline{R}_{44})_k (z_k - z_{k+1}) =$$

$$= 0.6667 \left[ 100 \left( 0.045 - 0.03 \right) + 500 \left( 0.03 - (-0.01) \right) + 100 \left( (-0.01) - (-0.045) \right) \right] = 16668 \text{ kN/m}$$

Thus the final homogenized laminated shell stiffnesses (in kNm, kN and kN/m):

$$\begin{bmatrix}
596.4 & 0 & 0 & -4640 & 0 & 0 & 0 & 0 \\
0 & 74.67 & 0 & 0 & 3200 & 0 & 0 & 0 \\
0 & 0 & 40.14 & 0 & 0 & -76 & 0 & 0 \\
-4640 & 0 & 0 & 580000 & 0 & 0 & 0 & 0 \\
0 & 3200 & 0 & 0 & 320000 & 0 & 0 & 0 \\
0 & 0 & -76 & 0 & 0 & 54500 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5986 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 16668 & 0
\end{bmatrix}$$

Fig. 20 shows the values based on FEM-Design. The values are the same.
Figure 20 – The relevant stiffness values based on FEM-Design
9.4.2.1.2 Calculation without shear coupling

In this sub-chapter we will show the stiffness calculation in case of without shear coupling between the layers when they are NOT working together. In Chapter 6.1.2.1 we will show the calculation with the glue at narrow side option, then in Chapter 6.1.2.2 with no glue at narrow side option.

9.4.2.1.2.1 Glue at narrow side

The material stiffness properties are the same layer-by-layer in the shell local system just like in Chapter 6.1.1.1.

Layer No. 1 and 3:

\[
\begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \\
\end{bmatrix}_{1,3} = \begin{bmatrix}
11672 & 181.1 & 0 \\
181.1 & 452.8 & 0 \\
0 & 0 & 690 \\
\end{bmatrix} \text{MPa}
\]

\[
\begin{bmatrix}
\overline{Q}_{45} & \overline{Q}_{45} \\
\overline{Q}_{45} & \overline{Q}_{44} \\
\end{bmatrix}_{1,3} = \begin{bmatrix}
690 & 0 \\
0 & 100 \\
\end{bmatrix} \text{MPa}
\]

Layer No. 2:

\[
\begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \\
\end{bmatrix}_{2} = \begin{bmatrix}
271.5 & 108.6 & 0 \\
108.6 & 8043 & 0 \\
0 & 0 & 500 \\
\end{bmatrix} \text{MPa}
\]

\[
\begin{bmatrix}
\overline{Q}_{45} & \overline{Q}_{45} \\
\overline{Q}_{45} & \overline{Q}_{44} \\
\end{bmatrix}_{2} = \begin{bmatrix}
50 & 0 \\
0 & 500 \\
\end{bmatrix} \text{MPa}
\]

Based on these information the ABD and shear part of the stiffness matrix can be calculated.

The bending part:

\[
D_{ij} = \frac{\sum_{k=1}^{N} (\bar{Q})_{ij,k} (z_k - z_{k+1})^3}{12}
\]

\[
D_{11} = \frac{1}{12} \left[ 11672(0.045 - 0.03)^3 + 271.5(0.03 - (-0.01))^3 + 11672((-0.01) - (-0.045))^3 \right] = \\
= 0.04643 \text{ MNm}^2/m = 46.43 \text{kNm}
\]

\[
D_{12} = \frac{1}{12} \left[ 181.1(0.045 - 0.03)^3 + 108.6(0.03 - (-0.01))^3 + 181.1((-0.01) - (-0.045))^3 \right] = \\
= 0.001277 \text{ MNm}^2/m = 1.277 \text{kNm}
\]
\[ D_{22} = \frac{1}{12} \left[ 452.8(0.045-0.03)^3 + 8043(0.03-(-0.01))^3 + 452.8((-0.01)-(-0.045))^3 \right] = \\
= 0.04464 \frac{\text{MNm}^2}{\text{m}} = 44.64 \text{kNm} \]

\[ D_{66} = \frac{1}{12} \left[ 690(0.045-0.03)^3 + 500(0.03-(-0.01))^3 + 690((-0.01)-(-0.045))^3 \right] = \\
= 0.005325 \frac{\text{MNm}^2}{\text{m}} = 5.325 \text{kNm} \]

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:
\[ D_{16} = D_{26} = 0. \]

The membrane part:
\[ A_{ij} = \sum_{k=1}^{N} (\overline{Q}_{y})_k (z_k - z_{k+1}) \quad (i,j = 1,2,6) \]

\[ A_{11} = 11672(0.045-0.03) + 271.5(0.03-(-0.01)) + 11672((-0.01)-(-0.045)) = \\
= 594.46 \frac{\text{MN}}{\text{m}} = 594460 \text{kN/m} \]

\[ A_{12} = 181.1(0.045-0.03) + 108.6(0.03-(-0.01)) + 181.1((-0.01)-(-0.045)) = \\
= 13.40 \frac{\text{MN}}{\text{m}} = 13400 \text{kN/m} \]

\[ A_{22} = 452.8(0.045-0.03) + 8043(0.03-(-0.01)) + 452.8((-0.01)-(-0.045)) = \\
= 344.4 \frac{\text{MN}}{\text{m}} = 344400 \text{kN/m} \]

\[ A_{66} = 690(0.045-0.03) + 500(0.03-(-0.01)) + 690((-0.01)-(-0.045)) = \\
= 54.5 \frac{\text{MN}}{\text{m}} = 54500 \text{kN/m} \]

In this case because the layers are orthogonal and the directions of orthotropy layer-by-layer coincide with the local system of the shell:
\[ A_{16} = A_{26} = 0. \]

Eccentricity part:
\[ B_{ij} = 0 \quad (i,j = 1,2,6) \]

Shear part:
In this case the calculation of the shear stiffnesses are similar than the calculation of the shear stiffnesses by a homogeneous isotropic slab:
Verification Examples

\[ S_{ij} = \frac{5}{6} \sum_{k=1}^{N} (Q_{ij})_k (z_k - z_{k+1}) \quad (i, j = 4, 5) \]

\[ S_{55} = \frac{5}{6} \left[ 690(0.045 - 0.03) + 50(0.03 - (-0.010)) + 690((-0.01) - (-0.045)) \right] = 30417 \text{ kN/m} \]

\[ S_{44} = \frac{5}{6} \left[ 100(0.045 - 0.03) + 500(0.03 - (-0.010)) + 100((-0.01) - (-0.045)) \right] = 20833 \text{ kN/m} \]

Thus the final homogenized laminated shell stiffnesses (in kNm and kN/m):

\[
\begin{bmatrix}
46.43 & 1.277 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.277 & 44.64 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5.325 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 594460 & 13400 & 0 & 0 & 0 \\
0 & 0 & 0 & 13400 & 344400 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 54500 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 30417 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 20833 & 0
\end{bmatrix}
\]

Fig. 21 shows the values based on FEM-Design. The values are the same.

![Stiffness matrix of timber panel](image)

*Figure 21 – The relevant stiffness values based on FEM-Design*
9.4.2.1.2.2 No glue at narrow side

The material stiffness properties are the same layer-by-layer in the shell local system just like in Chapter 6.1.1.2.

Layer No. 1 and 3:

\[
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_{1\&3} =
\begin{bmatrix}
11600 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 690
\end{bmatrix}_{\text{MPa}}
\]

\[
\begin{bmatrix}
\bar{Q}_{45} \\
\bar{Q}_{45}
\end{bmatrix}_{1\&3} =
\begin{bmatrix}
690 \\
0
\end{bmatrix}_{\text{MPa}}
\]

Layer No. 2:

\[
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_{2} =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 8000 & 0 \\
0 & 0 & 500
\end{bmatrix}_{\text{MPa}}
\]

\[
\begin{bmatrix}
\bar{Q}_{45} \\
\bar{Q}_{45}
\end{bmatrix}_{2} =
\begin{bmatrix}
50 \\
0
\end{bmatrix}_{\text{MPa}}
\]

Based on these information the ABD and shear part of the stiffness matrix can be calculated.

The bending part:

\[
D_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_{k} \frac{(z_{k} - z_{k+1})^{3}}{12}
\]

\[
D_{11} = \frac{1}{12} \left[ 11600(0.045-0.03)^{3} + 0(0.03-(0.01))^{3} + 11600((-0.01)-(-0.045))^{3} \right] = 0.04471 \frac{\text{MNm}^{2}}{\text{m}} = 44.71 \text{kNm}
\]

\[
D_{22} = \frac{1}{12} \left[ 0(0.045-0.03)^{3} + 8000(0.03-(0.01))^{3} + 0((-0.01)-(-0.045))^{3} \right] = 0.04267 \frac{\text{MNm}^{2}}{\text{m}} = 42.67 \text{kNm}
\]

\[
D_{66} = \frac{1}{12} \left[ 690(0.045-0.03)^{3} + 500(0.03-(0.01))^{3} + 690((-0.01)-(-0.045))^{3} \right] = 0.005325 \frac{\text{MNm}^{2}}{\text{m}} = 5.325 \text{kNm}
\]

\[
D_{12} = D_{16} = D_{26} = 0
\]
The membrane part:

\[ A_{ij} = \sum_{k=1}^{N} (\mathbf{Q}_j)_k (z_k - z_{k+1}) \quad (i,j = 1,2,6) \]

\[ A_{11} = 11600(0.045-0.03) + 0(0.03-(0.01)) + 11600((-0.01)-(-0.045)) = 580 \text{ MN/m} = 580000 \text{ kN/m} \]

\[ A_{22} = 0(0.045-0.03) + 8000(0.03-(0.01)) + 0((-0.01)-(-0.045)) = 320 \text{ MN/m} = 320000 \text{ kN/m} \]

\[ A_{66} = 690(0.045-0.03) + 500(0.03-(0.01)) + 690((-0.01)-(-0.045)) = 54.5 \text{ MN/m} = 54500 \text{ kN/m} \]

\[ A_{12} = A_{16} = A_{26} = 0. \]

Eccentricity part:

\[ B_{ij} = 0 \quad (i,j = 1,2,6) \]

Shear part:

In this case the calculation of the shear stiffnesses are similar than the calculation of the shear stiffnesses by a homogeneous isotropic slab:

\[ S_{ij} = \frac{5}{6} \sum_{k=1}^{N} (\overline{Q}_j)_k (z_k - z_{k+1}) \quad (i,j = 4,5) \]

\[ S_{55} = \frac{5}{6} \left[ 690(0.045-0.03) + 500(0.03-(0.01)) + 690((-0.01)-(-0.045)) \right] = 30417 \text{ kN/m} \]

\[ S_{44} = \frac{5}{6} \left[ 100(0.045-0.03) + 500(0.03-(0.01)) + 100((-0.01)-(-0.045)) \right] = 20833 \text{ kN/m} \]

Thus the final homogenized laminated shell stiffnesses (in kNm and kN/m):
Verification Examples

Fig. 22 shows the values based on FEM-Design. The values are the same.

![Stiffness matrix of timber panel](image)

*Figure 22 – The relevant stiffness values based on FEM-Design*

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.1 Calculation of the homogenized shell material stiffness matrix.str
9.4.2.2 Deflection and stresses of a CLT panel supported on two opposite edges

In this example we analyze a one-way CLT panel with beam-a-like behaviour and material properties to verify the calculation results compared with hand calculation. The geometry (10 m x 2.45 m) and the total distributed load can be seen with the supports in Fig. 23.

The material properties are reduced to consider beam-a-like behaviour (see Fig. 24) and to better comparison between the FEM-Design calculation and hand calculation, but in reality the Poisson's ratio and the $E_y$ modulus by a CLT panel is not equal to zero. The grain direction of the top layer is in the span direction.

The specific surface load in ULS:

$$q_u = 1.35 \cdot 0.9888 + 1.5 \cdot 2 = 4.335 \text{kN/m}^2$$
The maximum specific shear force at the support (calculated as a simply supported beam):

\[ V_{\text{max}} = q \frac{L}{2} = \frac{4.335 \times 10^2}{2} = 21.68 \text{kN/m} \]

The FEM-Design result according to Fig. 20:

\[ V_{\text{max}}^{\text{FEM-Design}} = 21.67 \text{kN/m} \]

The maximum specific bending moment at mid-span:

\[ M_{\text{max}} = q \frac{L^2}{8} = 4.335 \times 10^2 \frac{10^2}{8} = 54.19 \text{kNm/m} \]

The FEM-Design result according to Fig. 20:

\[ M_{\text{max}}^{\text{FEM-Design}} = 54.25 \text{kNm/m} \]

The relevant internal forces from FEM-Design can be seen in Fig. 25.

First we would like to calculate the relevant normal stress in the mid-span at the extreme fibers:

The relevant specific inertia (considering the 1st, 3rd, 5th and 7th layer), neglecting the lateral layers (2nd, 4th and 6th):

\[ I = 2 \left( \frac{2 \times 0.03^3}{12} + 0.03 \times 0.105^2 + 0.03 \times 0.035^2 \right) = 0.000744 \text{m}^4/\text{m} \]

The maximum normal stress in grain direction at the extreme fibers:

\[ \sigma_{c,t,0,d}^{\text{max}} = \frac{M_{\text{max}}}{I} \left( \frac{t}{2} \right) = \frac{54.19}{0.000744} \left( \frac{0.24}{2} \right) = 8740 \text{kPa} = 8.74 \text{MPa} \]
The FEM-Design result according to Fig. 26:

\[
\sigma_{\text{max}}^{\text{FEM - Design}} = 8.75 \text{ MPa}
\]

Second we would like to calculate the relevant transverse shear stress at the mid-plane (center of the 4\textsuperscript{th} layer) next to the support:

The specific maximum value of statical moment:

\[
S_{\text{max}} = 0.03 \cdot 0.105 + 0.03 \cdot 0.035 = 0.0042 \text{ m}^3/\text{m}
\]

The maximum rolling shear at the mid-plane next to the support:

\[
\tau_{\text{center \, yz, d}} = \tau_{\text{max \, Rolling, d}} = \frac{V_{\text{max}} S_{\text{max}}}{I b} = \frac{21.68 \cdot 0.0042}{0.000744 \cdot 1} = 122.4 \text{ kPa} = 0.1224 \text{ MPa}
\]

The FEM-Design result according to Fig. 26:

\[
\tau_{\text{max}}^{\text{FEM - Design}} = 0.1224 \text{ MPa}
\]

Third we would like to calculate the deflection in SLS considering creep.

The specific surface load in SLS quasi-permanent combination:

\[
q_{\text{Sq}} = 0.9888 + 0.3 \cdot 2 = 1.589 \text{ kN/m}^2
\]

The final deflection at mid-span considering creep and neglecting shear deformation:

\[
w_{\text{fin}} = \frac{5}{384} \frac{q_{\text{Sq}} L^4}{E I} \left(1 + k_{\text{def Sq}}\right) = \frac{5}{384} \frac{1.589 \cdot 10^4}{11000000 \cdot 0.000744} \left(1 + 0.8\right) = 0.0455 \text{ mm} = 45.5 \text{ mm}
\]

The FEM-Design result according to Fig. 27:

\[
w_{\text{fin}}^{\text{FEM - Design}} = 47.00 \text{ mm}
\]

The results based on the hand calculation and FEM-Design are identical except the deflection because in FEM-Design the shear deformations are also considered, therefore the maximum deflection at the mid-span is a bit larger compared to the hand calculation.
Download link to the example file:

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.2 Deflection and stresses of a CLT panel supported on two opposite edges.str
9.4.2.3 Deflection and stresses of a CLT simply supported two-way slab

By this verification example we calculated a simply supported two-way CLT slab (7m x 5m) with uniformly distributed load (see Fig. 28).

The layer composition with the mechanical properties can be seen in Fig. 29. The grain direction of the top layer is in the longer direction of the slab (see also Fig. 28). The total thickness is $t = 240$ mm. During the calculation we considered shear coupling between layers and glue at narrow sides.
Verification Examples

The considered combination is a ULS combination, thus $k_{a,q,U} = 0$. The design value of the considered value of the uniformly distributed surface load contains the self-weight of the CLT panel and a variable load:

$$q_U = 1.35 \left( \frac{420}{1000} \cdot 9.81 \cdot 0.24 \right) + 1.5 \cdot 2 = 4.335 \text{ kN/m}^2$$

We compared different FEM-Design results with ANSYS shell281 multilayer element results. Fig. 30 shows the applied finite element mesh size, the average element size was 0.36m.

![Finite element mesh in FEM-Design and ANSYS](image)

Fig 31. shows the deflection colour palette results. The maximum deflection from the calculations:

$$w_{\text{max, FEM-Design}} = 5.785 \text{ mm} ; \quad w_{\text{max, ANSYS}} = 5.786 \text{ mm}$$

Practically the results are the same.

![Deflections in FEM-Design and ANSYS](image)
The normal stress at the bottom of the lowermost layer in the $x'$ direction in shell local system:

\[ \sigma_{x' \text{bottom}}^{\text{FEM-Design}} = 1.302 \text{ MPa} ; \quad \sigma_{x' \text{bottom}}^{\text{ANSYS}} = 1.303 \text{ MPa} \]

The results are the same, see Fig. 32.

The normal stress at the bottom of the lowermost layer in the $y'$ direction in shell local system:

\[ \sigma_{y' \text{bottom}}^{\text{FEM-Design}} = 0.09774 \text{ MPa} ; \quad \sigma_{y' \text{bottom}}^{\text{ANSYS}} = 0.09771 \text{ MPa} \]

The results are the same, see Fig. 33.
The in-plane shear stress at the bottom of the lowermost layer in shell local system:

\[ \tau_{x'y'}^{\text{bottom}}^{\text{FEM-Design}} = 0.2380 \text{ MPa} \quad ; \quad \tau_{x'y'}^{\text{bottom}}^{\text{ANSYS}} = 0.2442 \text{ MPa} \]

The difference between the results is about 2.5%, see Fig. 34.

The normal stress at the top of the second layer from above in the \( y' \) direction in shell local system:

\[ \sigma_{y'}^{\text{top}}^{\text{FEM-Design}} = -1.992 \text{ MPa} \quad ; \quad \sigma_{y'}^{\text{top}}^{\text{ANSYS}} = -1.993 \text{ MPa} \]

The results are the same, see Fig. 35.
The transverse shear stresses at the middle of the slab (center of the 4th layer) in $x'z'$ direction in shell local system:

\[ \tau_{x'z'\text{middle}}^{\text{ANSYS}} = 0.05865 \text{ MPa} \quad ; \quad \tau_{x'z'\text{middle}}^{\text{ANSYS}} = 0.05864 \text{ MPa} \]

The results are the same, see Fig. 36.

The transverse shear stresses at the middle of the slab (center of the 4th layer) in $y'z'$ direction in shell local system:

\[ \tau_{y'z'\text{middle}}^{\text{ANSYS}} = 0.07181 \text{ MPa} \quad ; \quad \tau_{y'z'\text{middle}}^{\text{ANSYS}} = 0.07199 \text{ MPa} \]

The difference between the results is about 2.5%, see Fig. 37.
Based on the comparisons of the mentioned results we can say that FEM-Design results considering the laminated composite shell mechanical model by a two-way slab gives adequate results.

Download link to the example file:

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.3 Deflection and stresses of a CLT simply supported two-way slab.str
9.4.2.4 In-plane loaded CLT design check to shear failure at glued contact surface

In this example we will verify the FEM-Design design formulas in case of an in-plane loaded CLT panel according to Chapter 5.8.4 and compare the FEM-Design results with analytical/experimental data, see Ref. [40-41].

The calculated static layout can be seen in Fig. 38. The applied force is \( F = 237 \text{ kN} \) (thus \( F/2 = 118.5 \text{ kN} \), see Fig. 38. and Ref. [40]). The CLT here is an in-plane loaded panel with the following layer composition and material parameters, see Fig. 33-34. You can see in Fig. 38 that the applied plank (board) width is \( a = 150 \text{ mm} \) and the number of the layers \( N = 3 \). In FEM-Design calculation we use the No glue at narrow side option to reach this shear failure formula by the detailed results.

Figure 38 – The geometry, the loads and the analyzed RVE in case of in-plane loaded panel

Figure 39 – The layer composition with the thicknesses and the applied material parameters
The applied material and strength parameters are according to Ref. [40] to get more comparable results in the end. Fig. 40 shows a complete model about the problem, but according to the point supports and forces in this model there is going to be singularities which prevent the comparison between the FEM-Design calculation result with the benchmark reference values, therefore only a small section including section A-A (indicated with red box in Fig. 40) was modeled and loaded with the relevant internal force system assumed as an equilibrium force system with statically determinant supports (see the downloadable sample file below).

Thus the verification of failure of the glued contact surface will be performed in Section A-A, see Fig. 40.

First of all in FEM-Design the additional torsional stress at the glued surface in the mentioned RVE:

\[
\tau_{\text{tor},d} = \frac{3n_{xy}}{a(N-1)} = \frac{3 \times 264.65}{0.15(3-1)} = 2646.5 \text{kPa} = 2.647 \text{MPa},
\]

where the \( n_{xy} \) value is the in-plane specific shear force in the model, in the middle of the analyzed RVE, see Fig. 41 also.

According to Chapter 5.8.4 the relevant failure in this example will be at the surface of the vertical middle planks. At the RVE element the \( n_{x'} \) specific normal force in the shell local system (see. Fig. 42) will be the \( n_y \) value of the middle vertical board specific normal force perpendicular to the grain (see Fig. 38-39).
Verification Examples

The additional rolling shear at the glued surface in the middle vertical layer will be the following (see Fig. 42):

\[
\tau_{yz, d} = \frac{\Delta N_{x,y} - 1}{N - 1} = \frac{0.15}{3 - 1} = 329.2 \text{kPa} = 0.3292 \text{MPa}
\]

According to the mentioned design formula in Chapter 5.8.4 the following interaction should be checked, Fig. 42 also shows this interaction formula based on FEM-Design detailed result:

\[
\left| \tau_{tor, d} \right| + \left| \tau_{yz, inplane} \right| = 2.647 \frac{3.5}{3.5} + 0.3292 \frac{1.5}{1.5} = 0.9758 \leq 1.0 \text{ - OK}
\]

Based on Ref [40] results and the analytical theory in Ref. [41-42] the reference value of this utilization:

\[
\left| \tau_{tor, d} \right| + \left| \tau_{yz, inplane} \right| = 2.686 \frac{3.5}{3.5} + 0.25 \frac{1.5}{1.5} = 0.9341 \leq 1.0
\]

The literature data and the FEM-Design results are very close to each other, the difference in the interaction formula is smaller than 5%.
Download link to the example file:

http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.4 In-plane loaded CLT design check to shear failure at glued contact surface.str
9.4.2.5 Buckling check of a CLT wall

Let's consider the following layer composition in Fig. 43, with the given elastic and strength parameters.

The load-duration class is permanent thus the applied $k_{mod}=0.6$ and $\gamma_M=1.25$.

With this CLT panel we built a wall with $L$ height and 4 m width, applied a constant distributed load on the top edge of the wall and a surface distributed load perpendicular to the panel as you can see in Fig. 44-45. The boundary condition was simply supported. The planks are vertical at the outer layers.

In the first part (case a) ) of this verification example we analyzed different height walls ($L = 1, 2, 3, 5$ and $7.5$ m) with only theoretically centric load ($F = 543.73$ kN/m; $417$ kN/m; $274.84$ kN/m; $123$ kN/m; and $60$ kN/m respectively; $q = 0$) on the top edge. Here the FEM-Design calculation method will be shown firstly with the mentioned buckling calculation method (Chapter 5.8.5, effective length method with reduction factor) in details. After the buckling utilizations what we get from FEM-Design calculation we show a comparison with GMNIA (geometric and material non-linear analysis with imperfections) with the aim of a structural multphysics program. Nowadays the GMNIA is the most precise engineering method to get the load bearing capacity of a structure with numerical simulation. By the verification we used in this other program solid finite elements to model more realistic structural behaviour with GMNIA.
In the second part (case b) we show the behaviour in case of $L=3\text{m}$ height with different vertical and horizontal loads ($F = 274.84 \text{kN/m}; 252 \text{kN/m}; 234.99 \text{kN/m}; 208.33 \text{kN/m}; 171.53 \text{kN/m};$ and $130.5 \text{kN/m}$; $q = 0; 0.84 \text{kN/m}^2; 1.567 \text{kN/m}^2; 2.778 \text{kN/m}^2; 4.574 \text{kN/m}^2; 6.96 \text{kN/m}^2$ respectively). First the FEM-Design calculation method will be shown in details. After the buckling utilizations what we get from FEM-Design calculation we show a comparison with GMNIA with the same way what was introduced in case a).

Figure 44 – The CLT wall model in FEM-Design

Figure 45 – The simplified static system to the effective length reduction factor method (left) and to the GMNIA (right)
Case a)

In case of centric loading (see Fig. 44-45, $q=0$) the analyzed height and centric loads of the walls:

<table>
<thead>
<tr>
<th>$L$ [m]</th>
<th>$F$ [kN/m]</th>
<th>$q$ [kN/m²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>543.7</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>417.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>274.8</td>
<td>0.0</td>
</tr>
<tr>
<td>5</td>
<td>123.0</td>
<td>0.0</td>
</tr>
<tr>
<td>7.5</td>
<td>60.00</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Based on the layers material elastic properties in Fig. 43 and with the buckling length we would like to calculate the relative slenderness and the reduction factor. The $r$ reduction factor what was mentioned in Chapter 5.8.5 will help us to provide the characteristic values of the elastic properties to get the relative slenderness with the aim of the composite shell homogenization. The adjusted reduction factor on the elastic properties here was: $r = 0.8333$ (which is usually accepted by glue laminated or CLT products).

\[
\begin{align*}
E_{x,05} &= r E_{x,\text{mean}} = 0.8333 \cdot 11000 = 9170 \text{ MPa}, \\
E_{y,05} &= r E_{y,\text{mean}} = 0.8333 \cdot 370 = 308 \text{ MPa}, \\
G_{xy,05} &= r G_{xy,\text{mean}} = 0.8333 \cdot 690 = 575 \text{ MPa}, \\
G_{xz,05} &= r G_{xz,\text{mean}} = 0.8333 \cdot 690 = 575 \text{ MPa}, \\
G_{yz,05} &= r G_{yz,\text{mean}} = 0.8333 \cdot 69 = 57.5 \text{ MPa}.
\end{align*}
\]

From the FEM-Design laminated composite homogenization method in the background with the characteristic elastic modulus values based on Chapter 3.2 we get the following stiffness value to get the elastic critical force of the CLT panel:

\[
D_{11}' = 614 \frac{\text{kNm}^2}{\text{m}} \quad \text{as the relevant specific bending stiffness and}
\]

\[
S_{55}' = 8943 \text{ kN/m} \quad \text{as the relevant specific shear stiffness}.
\]

Based on these stiffnesses we can get the elastic critical forces which depends on the buckling length of the panel. In the first part of this example we will analyze theoretically centrically loaded panels with different buckling lengths. The $\beta$ factors are 1.0 because the wall was simply supported but the height of the walls are different. Thus the elastic critical forces are as follows:
Verification Examples

\[ n_{cr}^{1m} = \frac{1}{1} + \frac{1}{\left( \frac{D_{11} \pi^2}{(\beta L)^2} \right)} \left( \frac{S_{55}}{614 \pi^2 \left(\frac{1.0}{1}\right)^2} \right) + \frac{1}{8943} = 3612 \frac{kN}{m} \]

\[ n_{cr}^{2m} = \frac{1}{1} + \frac{1}{\left( \frac{D_{11} \pi^2}{(\beta L)^2} \right)} \left( \frac{S_{55}}{614 \pi^2 \left(\frac{1.0}{2}\right)^2} \right) + \frac{1}{8943} = 1296 \frac{kN}{m} \]

\[ n_{cr}^{3m} = \frac{1}{1} + \frac{1}{\left( \frac{D_{11} \pi^2}{(\beta L)^2} \right)} \left( \frac{S_{55}}{614 \pi^2 \left(\frac{1.0}{3}\right)^2} \right) + \frac{1}{8943} = 626.2 \frac{kN}{m} \]

\[ n_{cr}^{5m} = \frac{1}{1} + \frac{1}{\left( \frac{D_{11} \pi^2}{(\beta L)^2} \right)} \left( \frac{S_{55}}{614 \pi^2 \left(\frac{1.0}{5}\right)^2} \right) + \frac{1}{8943} = 236 \frac{kN}{m} \]

\[ n_{cr}^{7.5m} = \frac{1}{1} + \frac{1}{\left( \frac{D_{11} \pi^2}{(\beta L)^2} \right)} \left( \frac{S_{55}}{614 \pi^2 \left(\frac{1.0}{7.5}\right)^2} \right) + \frac{1}{8943} = 106.5 \frac{kN}{m} \]

Based on these elastic critical force values the relative slenderness could be calculated with the aim of the given data:

The thickness of the different layers:
\[ t_1 = t_2 = t_3 = t_4 = t_5 = 20 \text{ mm} = 0.020 \text{ m} \]

The characteristic compression strength in the grain and perpendicular to grain direction:
\[ f_{c,0,k,1} = f_{c,0,k,2} = f_{c,0,k,3} = 21000 \text{ kPa} \]
\[ f_{c,90,k,2} = f_{c,90,k,4} = 2500 \text{ kPa} \]

The generalized relative slenderness:
\[ \lambda_{rel} = \frac{\sum_{i=1}^{N} t_i f_{c,a,k,i}}{n_{cr}} \]

according to this equation the relative slenderness for the different height walls are as follows.
Verification Examples

\[ \lambda_{rel}^{1m} = \sqrt{\frac{\sum_{i=1}^{N} t_i f_{c, \alpha, k, i}}{n_{cr}^{1m}}} = \sqrt{\frac{3 \cdot 0.02 \cdot 21000 + 2 \cdot 0.02 \cdot 2500}{3612}} = 0.6136 \]

Similarly with the other lengths:

\[ \lambda_{rel}^{2m} = 1.024 \quad ; \quad \lambda_{rel}^{3m} = 1.474 \quad ; \quad \lambda_{rel}^{5m} = 2.401 \quad ; \quad \lambda_{rel}^{7.5m} = 3.574 \]

According to the used Ayrton-Perry formula in EC5 the reduction factor calculation:

\[ k = 0.5(1 + \beta \cdot (\lambda_{rel} - 0.3) + \lambda_{rel}^2) \]

\[ k_c = \frac{1}{k + \sqrt{k^2 - \lambda_{rel}}} \]

In these examples we used here \( \beta = 0.1 \) straightness factor, but it is adjustable in FEM-Design.

The reduction factors are as follows according to the previous equations and data:

\[ k_c^{1m} = 0.9534 \quad ; \quad k_c^{2m} = 0.7483 \quad ; \quad k_c^{3m} = 0.4210 \quad ; \quad k_c^{5m} = 0.1662 \quad ; \quad k_c^{7.5m} = 0.07617 \]

The table below summarizes the calculated values by the different cases.

<table>
<thead>
<tr>
<th>( \beta_c ) = 0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L ) [m]</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>7.5</td>
</tr>
</tbody>
</table>

To calculate the buckling utilizations it is necessary to define the design values of the compressive and bending strengths. The design values of the compressive and bending strength of the layers:

\[ f_{c, 0, d} = k_{mod} \cdot f_{c, 0, k} = 0.6 \cdot \frac{21}{1.25} = 10.08 \text{ MPa} \]

\[ f_{m, 0, d} = k_{mod} \cdot f_{m, 0, k} = 0.6 \cdot \frac{24}{1.25} = 11.52 \text{ MPa} \]
Fig. 46. shows the relevant stress results layer-by-layer from FEM-Design in the CLT panels with the given heights and loads.

With the calculated reduction factors the buckling utilization of the different height walls with the different centric loads (based on Fig. 46) in the grain directions of the relevant layers:

\[
\text{Util}_{1m} = \frac{\sigma_{\text{buckling}, c,0,d}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}, m,0,d}}{f_{m,0,d}} = \frac{-8.866}{0.9534 \cdot 10.08} + \frac{0}{11.52} = 0.9226
\]

\[
\text{Util}_{2m} = \frac{\sigma_{\text{buckling}, c,0,d}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}, m,0,d}}{f_{m,0,d}} = \frac{-6.799}{0.7483 \cdot 10.08} + \frac{0}{11.52} = 0.9014
\]

\[
\text{Util}_{3m} = \frac{\sigma_{\text{buckling}, c,0,d}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}, m,0,d}}{f_{m,0,d}} = \frac{-4.481}{0.4210 \cdot 10.08} + \frac{0}{11.52} = 1.056
\]

\[
\text{Util}_{5m} = \frac{\sigma_{\text{buckling}, c,0,d}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}, m,0,d}}{f_{m,0,d}} = \frac{-2.006}{0.1662 \cdot 10.08} + \frac{0}{11.52} = 1.197
\]

\[
\text{Util}_{7.5m} = \frac{\sigma_{\text{buckling}, c,0,d}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}, m,0,d}}{f_{m,0,d}} = \frac{-0.9783}{0.07617 \cdot 10.08} + \frac{0}{11.52} = 1.274
\]
After we have these utilization values we have made GMNIA analysis with an other software to make a comparison between the proposed practically usable effective length reduction factor method what was implemented in FEM-Design and one of the most precise numerical simulation method.

In the GMNIA the used finite elements were solid elements and the material model was orthotropic with the given nonlinear stress-strain models in Fig. 47 layer-by-layer.

![The considered non-linear stress-strain model by GMNIA](image)

In the GMNIA the adjusted elastic moduli were the following according to Ref. [41]:

\[
E_{0,d} = 0.8 \frac{E_{x,d}}{Y_M} = 0.8 \frac{0.84 \cdot 11000}{1.25} = 5914 \text{ MPa}
\]

\[
E_{90,d} = 0.8 \frac{E_{y,d}}{Y_M} = 0.8 \frac{0.84 \cdot 370}{1.25} = 198.9 \text{ MPa}
\]

\[
G_{xy,d} = 0.8 \frac{G_{xy,d}}{Y_M} = 0.8 \frac{0.84 \cdot 690}{1.25} = 370.9 \text{ MPa}
\]

\[
G_{xz,d} = 0.8 \frac{G_{xz,d}}{Y_M} = 0.8 \frac{0.84 \cdot 690}{1.25} = 370.9 \text{ MPa}
\]

\[
G_{yz,d} = 0.8 \frac{G_{yz,d}}{Y_M} = 0.8 \frac{0.84 \cdot 69}{1.25} = 37.09 \text{ MPa}
\]

The plastic limit values can be seen in Fig. 47, despite of the behaviour is symmetric in tension and compression the calculation is adequate because the analyzed cases are mainly under compression and on the tensioned side the values were not above the tension plastic limit at the failure load of GMNIA. It is important to note because it is very well-known that the timber material shows brittle fracture under pure tension, see Ref. [41].
By GMNIA another important question is the right consideration of the imperfections. First of all a global initial sway was considered, see Fig. 45 right side. The initial sway according to the Eurocode was:

\[ \phi = \alpha_h \frac{1}{200} \text{ ; where } \alpha_h = \frac{2}{\sqrt{L}} \text{ but } \frac{2}{3} \leq \alpha_h \leq 1.0 \text{ where } L \text{ is the wall height.} \]

Secondly it is necessary to consider an initial bow imperfection with its design value. This value is very important because it should contain not only the geometrical bow imperfection of the member but also other imperfections such as the material and load (eccentricity). According to the EC5 and Ref. [46-47] an adequate design value of the initial bow imperfection is:

\[ e_{imp} = \frac{L}{300}, \text{ where } L \text{ is the wall height.} \]

With these given values we performed the GMNIA with an independent software based on solid finite elements.

The given centric specific loads in case a) were the load-bearing capacity of the different height walls according to the GMNIA. Fig. 48 shows the back-calculated reduction factor values based on GMNIA.

Based on the fact that the reduction factor curves in Eurocode comes from stochastic and empiric calculation we can say that the results of the proposed FEM-Design buckling check method are very close to the GMNIA. The results are closer to each other if we consider that the practically relevant relative slenderness is up to around 2.0 here.
Fig. 49 shows the load-displacement curves from GMNIA. It is worth noting that by the first two short wall height (1 and 2 m) the load-displacement curves are almost straight, which means that the failure is close to the strength failure as expected. By the $L = 3$ m according to the results the failure is a mixture between the stability and strength failure. By the last two cases (5 and 7.5 m) the failure mode is stability failure which is obvious from Fig. 49.

**Figure 49 – The results of GMNIA – The applied vertical force versus the lateral displacement of the mid-section of the wall**
Case b)

In this case the height of the wall was constant, but we increased the lateral loads on the wall as well.

The analyzed cases can be seen in the table below where we indicated the applied height of the wall and furthermore the applied vertical and lateral loads (see Fig. 44-45).

<table>
<thead>
<tr>
<th>$L$ [m]</th>
<th>$F$ [kN/m]</th>
<th>$q$ [kN/m$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>274.8</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>252.0</td>
<td>0.840</td>
</tr>
<tr>
<td>3</td>
<td>235.0</td>
<td>1.567</td>
</tr>
<tr>
<td>3</td>
<td>208.3</td>
<td>2.778</td>
</tr>
<tr>
<td>3</td>
<td>171.5</td>
<td>4.574</td>
</tr>
<tr>
<td>3</td>
<td>130.5</td>
<td>6.960</td>
</tr>
</tbody>
</table>

Fig. 50 shows the relevant stress distribution in the walls under the different load conditions at the mid-section of the walls.

In case a) we calculated based on the proposed FEM-Design method the reduction factor to the 3 m height wall, thus we used that value by these calculations as well. The buckling utilizations of the walls with the different vertical and lateral loads by the six different load situations are as follows based on Fig. 45 and the mentioned stress separation in Chapter 5.8.5.4:
Verification Examples

\[
\text{Util}_1 = \frac{\sigma_{\text{buckling}}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}}}{f_{m,0,d}} = \frac{-4.481}{0.4210 \cdot 10.08} + \frac{0}{11.52} = 1.056
\]

\[
\text{Util}_2 = \frac{\sigma_{\text{buckling}}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}}}{f_{m,0,d}} = \frac{-4.109}{0.4210 \cdot 10.08} + \frac{0.707}{11.52} = 1.030
\]

\[
\text{Util}_3 = \frac{\sigma_{\text{buckling}}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}}}{f_{m,0,d}} = \frac{-3.832}{0.4210 \cdot 10.08} + \frac{1.319}{11.52} = 1.017
\]

\[
\text{Util}_4 = \frac{\sigma_{\text{buckling}}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}}}{f_{m,0,d}} = \frac{-3.397}{0.4210 \cdot 10.08} + \frac{2.339}{11.52} = 1.004
\]

\[
\text{Util}_5 = \frac{\sigma_{\text{buckling}}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}}}{f_{m,0,d}} = \frac{-2.797}{0.4210 \cdot 10.08} + \frac{3.853}{11.52} = 0.9936
\]

\[
\text{Util}_6 = \frac{\sigma_{\text{buckling}}}{k_c f_{c,0,d}} + \frac{\sigma_{\text{buckling}}}{f_{m,0,d}} = \frac{-2.128}{0.4210 \cdot 10.08} + \frac{5.861}{11.52} = 1.010
\]

To verify these utilizations we performed GMNIA with the same input parameters what were used in case a) but we modified the loads. The given load situations in case b) were the load-bearing capacities of the analyzed walls based on GMNIA. Fig. 51 shows the load-displacement curves of the analyzed walls. Do not forget that by these vertical loads there were simultaneously increasing lateral loads as well on the walls during the calculation.

According to these information we can say that the utilizations of FEM-Design proposed calculation are very close to GMNIA results. We can conclude that FEM-Design provides an adequate stability calculation method in its new CLT modul design calculation.
Download link to the example file:

Case a):
http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.5 Buckling check of a CLT wall case a.str

Case b):
http://download.strusoft.com/FEM-Design/inst190x/models/9.4.2.5 Buckling check of a CLT wall case b.str
9.5 Automatic calculation of flexural buckling length

9.5.1 Concrete frame building

In this example we will calculate the buckling lengths of the indicated isolated columns (C.1; C.4, see Fig. 9.5.1.1) according to EN 1992-1-1:2004 Chapter 5.8.3.2. After the hand calculation we will compare the results with the FEM-Design automatic buckling length calculation results.

The geometry is shown in Fig. 9.5.1.1. The material is C25/30 the columns have 300/300 mm, the beams have 300/500 mm cross-sections. We will calculate the buckling lengths of the middle isolated columns at the ground floor and at the first floor. The supports are fixed at the bottom of the ground floor columns.

![Figure 9.5.1.1 – The concrete planar frame geometry](image)

9.5.1.1 Non-sway case

If the frame is a non-sway frame the method according to EN 1992-1-1:2004 Chapter 5.8.3.2. is the following:

The bending stiffness of the columns:

\[ EI_c = 31000000 \times \frac{0.3^4}{12} = 20925 \text{ kNm}^2 \]

The bending stiffness of the beams in the relevant direction:

\[ EI_c = 31000000 \times \frac{0.3 \times 0.5^3}{12} = 96875 \text{ kNm}^2 \]
C.1 column (see Fig. 9.5.1.1):
The distribution factors:
At bottom:
\[ k_1 = 0 \quad \text{(fixed support)}; \]
At top:
\[
k_2 = \left( \frac{E I_c}{L_c} \right)_{\text{above}} + \left( \frac{E I_c}{L_c} \right)_{\text{below}} = \frac{20925}{3} + \frac{20925}{4} = 0.189
\]
By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

The beta factor of the buckling length:
\[
\beta_1 = \frac{L_{cr1}}{L_c} = 0.5 \cdot \sqrt{\left( 1 + \frac{k_1}{0.45 + k_1} \right) \left( 1 + \frac{k_2}{0.45 + k_2} \right)} = 0.5 \cdot \sqrt{\left( 1 + 0 \right) \left( 1 + \frac{0.1896}{0.45 + 0.189} \right)} = 0.569
\]

C.4 column (see Fig. 9.5.1.1):
The distribution factors:
At bottom:
\[
k_1 = \left( \frac{E I_c}{L_c} \right)_{\text{above}} + \left( \frac{E I_c}{L_c} \right)_{\text{below}} = \frac{20925}{3} + \frac{20925}{4} = 0.189
\]
By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

At top:
\[
k_2 = \left( \frac{E I_c}{L_c} \right)_{\text{above}} + \left( \frac{E I_c}{L_c} \right)_{\text{below}} = \frac{20925}{3} + \frac{20925}{3} = 0.216
\]
By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

The beta factor of the buckling length:
\[
\beta_4 = \frac{L_{cr4}}{L_{c4}} = 0.5 \cdot \sqrt{\left( 1 + \frac{0.189}{0.45 + 0.189} \right) \left( 1 + \frac{0.216}{0.45 + 0.216} \right)} = 0.655
\]
Based on FEM-Design auto buckling length calculation method the results are:

\[ \beta_{1FEM} = 0.576 \]
\[ \beta_{4FEM} = 0.663 \]

The difference between the calculations is less than 1.5%. Fig. 9.5.1.2 shows the results based on FEM-Design.

*Figure 9.5.1.2 – The buckling lengths of the columns in non-sway case*
9.5.1.2 Sway case

If the frame is a sway frame the method according to EN 1992-1-1:2004 Chapter 5.8.3.2. is the following:

C.1 column (see Fig. 9.5.1.1):

The distribution factors:

At bottom:
\[ k_1 = 0 \] (fixed support);

At top:
\[
\frac{(EI_c/L_{cd})_{\text{above}} + (EI_c/L_{cd})_{\text{below}}}{\sum c EI_b/L_b} = \frac{20925}{3} + \frac{20925}{4} = 0.063
\]

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

The beta factor of the buckling length:

\[
\beta_1 = \max \left[ \sqrt{1 + 10 \frac{k_1 k_2}{k_1 + k_2}} \right] = \max \left[ \sqrt{1 + 10 \frac{0.063}{0 + 0.063}} \right] = 1.06
\]

C.4 column (see Fig. 9.5.1.1):

The distribution factors:

At bottom:
\[
\frac{(EI_c/L_{cd})_{\text{above}} + (EI_c/L_{cd})_{\text{below}}}{\sum c EI_b/L_b} = \frac{20925}{3} + \frac{20925}{4} = 0.063
\]

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

At top:
\[
\frac{(EI_c/L_{cd})_{\text{above}} + (EI_c/L_{cd})_{\text{below}}}{\sum c EI_b/L_b} = \frac{20925}{3} + \frac{20925}{3} = 0.072
\]
By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

The beta factor of the buckling length:

\[ \beta_4 = \frac{L_{crd}}{L_4} = \max \left[ \sqrt{\frac{1+10\cdot0.063\cdot0.072}{0.063+0.072}}, \frac{0.063}{1+0.063}, \frac{0.072}{1+0.072} \right] = 1.156 \]

Based on FEM-Design auto buckling length calculation method the results are:

\[ \beta_{IFEM} = 1.07 \]
\[ \beta_{4FEM} = 1.15 \]

The difference between the calculations is less than 1%. Fig. 9.5.1.3 shows the results based on FEM-Design.

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.5.1 Auto Buckling length concrete building.str
9.5.2 Steel frame building

In this example we will calculate the buckling lengths of the indicated isolated columns (C.2; C.6, see Fig. 9.5.2.1) according to the method in Ref. [17] which is basically identical with the given method in the former ENV 1993-1-1:1992 Annex E. After the hand calculation we will compare the results with FEM-Design automatic buckling length calculation results.

The geometry is shown in Fig. 9.5.2.1. The material is S235, the outer columns have HEB220, the inner columns have HEB260, the beams have IPE450 and the beams at the roof have IPE360 cross-sections. We will calculate the buckling lengths of the middle isolated columns at the ground floor and at the first floor. The supports are hinged at the bottom of the ground floor columns.

9.5.2.1 Non-sway case

If the frame is a non-sway frame the method according to Ref. [17] is the following:

The bending stiffness of the columns (HEB260):

\[ EI_c = 210000000 \cdot 0.0001492 = 31332 \text{ kNm}^2 \]

The bending stiffness of the beams (IPE450):

\[ EI_b = 210000000 \cdot 0.0003374 = 70854 \text{ kNm}^2 \]

The rotational stiffness coefficient of the columns being analyzed (C.2 and C.6):

\[ K_c = 4 \frac{EI_c}{L_c} = 4 \frac{31332}{3.5} = 35808 \text{ kNm} \]
C.2 column (see Fig. 9.5.2.1):
The distribution factors:
At bottom:
\[
\eta_1 = \frac{K_c}{K_c} = 1.0 \quad \text{(hinged support)}
\]
At top:
\[
\eta_2 = \frac{K_c + K_2}{K_c + K_2 + K_{21} + K_{22}} = \frac{4 \frac{EI_c}{L_c} + 4 \frac{EI_c}{L_c}}{35808 + 35808 + 2 \frac{70854}{6.5} + 2 \frac{70854}{6.5}} = 0.622
\]
By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

The beta factor of the buckling length:
\[
\beta_2 = \frac{1 + 0.145(\eta_1 + \eta_2) - 0.265 \eta_1 \eta_2}{2 - 0.364(\eta_1 + \eta_2) - 0.247 \eta_1 \eta_2} = \frac{1 + 0.145(1.0 + 0.622) - 0.265 \cdot 1.0 \cdot 0.622}{2 - 0.364(1.0 + 0.622) - 0.247 \cdot 1.0 \cdot 0.622} = 0.852
\]

C.6 column (see Fig. 9.5.2.1):
The distribution factors:
At bottom:
\[
\eta_1 = \frac{K_c + K_1}{K_c + K_1 + K_{11} + K_{12}} = \frac{4 \frac{EI_c}{L_c} + 4 \frac{EI_c}{L_c}}{35808 + 35808 + 2 \frac{70854}{6.5} + 2 \frac{70854}{6.5}} = 0.622
\]
By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

At top:
\[
\eta_2 = \frac{K_c + K_2}{K_c + K_2 + K_{21} + K_{22}} = \frac{4 \frac{EI_c}{L_c} + 4 \frac{EI_c}{L_c}}{35808 + 35808 + 2 \frac{70854}{6.5} + 2 \frac{70854}{6.5}} = 0.622
\]
By the beams rotational stiffnesses we assumed a single curvature due to the non-sway situation.

The beta factor of the buckling length:

$$\beta_6 = \frac{1 + 0.145 (\eta_1 + \eta_2) - 0.265 \eta_1 \eta_2}{2 - 0.364 (\eta_1 + \eta_2) - 0.247 \eta_1 \eta_2} = \frac{1 + 0.145 (0.622 + 0.622) - 0.265 \cdot 0.622 \cdot 0.622}{2 - 0.364 (0.622 + 0.622) - 0.247 \cdot 0.622 \cdot 0.622}$$

$$\beta_6 = 0.743$$

Based on FEM-Design auto buckling length calculation method the results are:

$$\beta_{2FEM} = 0.852$$
$$\beta_{6FEM} = 0.742$$

The calculations are identical to each other (see Fig. 9.5.2.2).

9.5.2.2 Sway case

If the frame is a sway frame the method according to Ref. [17] is the following:

C.2 column (see Fig. 9.5.2.1):

The distribution factors:

At bottom:

$$\eta_1 = \frac{K_c}{K_c} = 1.0 \quad \text{(hinged support)}$$

Figure 9.5.2.2 – The buckling lengths of the columns in non-sway case
At top:

\[ \eta_2 = \frac{K_c + K_2}{K_c + K_2 + K_{21} + K_{22}} = \frac{4 EI_c + 4 EI_c}{4 EI_c + 4 EI_c + 6 EI_b + 6 EI_b} = \frac{35808 + 35808}{35808 + 35808 + 6 \cdot \frac{70854}{6.5} + 6 \cdot \frac{70854}{6.5}} \]

\[ \eta_2 = 0.354 \]

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

The beta factor of the buckling length:

\[ \beta_2 = \sqrt{\frac{1 - 0.2(1.0 + 0.354) - 0.12 \cdot 1.0 \cdot 0.354}{1 - 0.8(1.0 + 0.354) + 0.6 \cdot 1.0 \cdot 0.354}} \]

\[ \beta_2 = 2.305 \]

C.6 column (see Fig. 9.5.2.1):

The distribution factors:

At bottom:

\[ \eta_1 = \frac{K_c + K_1}{K_c + K_1 + K_{11} + K_{12}} = \frac{4 EI_c + 4 EI_c}{4 EI_c + 4 EI_c + 6 EI_b + 6 EI_b} = \frac{35808 + 35808}{35808 + 35808 + 6 \cdot \frac{70854}{6.5} + 6 \cdot \frac{70854}{6.5}} \]

\[ \eta_1 = 0.354 \]

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

At top:

\[ \eta_2 = \frac{K_c + K_2}{K_c + K_2 + K_{21} + K_{22}} = \frac{4 EI_c + 4 EI_c}{4 EI_c + 4 EI_c + 6 EI_b + 6 EI_b} = \frac{35808 + 35808}{35808 + 35808 + 6 \cdot \frac{70854}{6.5} + 6 \cdot \frac{70854}{6.5}} \]

\[ \eta_2 = 0.354 \]

By the beams rotational stiffnesses we assumed double curvature due to the sway situation.

The beta factor of the buckling length:

\[ \beta_6 = \sqrt{\frac{1 - 0.2(0.354 + 0.354) - 0.12 \cdot 0.354 \cdot 0.354}{1 - 0.8(0.354 + 0.354) + 0.6 \cdot 0.354 \cdot 0.354}} \]

\[ \beta_6 = 1.287 \]
Based on FEM-Design auto buckling length calculation method the results are:

\[
\beta_{2FEM} = 2.31 \\
\beta_{6FEM} = 1.29
\]

The calculations are identical to each other (see Fig. 9.5.2.3).

Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.5.2 Auto Buckling length steel building.str
9.5.3 A column and a supporting beam with various angles

Fig. 9.5.3.1 shows the analyzed problem. The vertical column is HEB260 and the bottom is fixed. The horizontal supporting beam is IPE360 and connected to the upper end of the column (the other end of the beam is simply supported). The angle of the connecting beam is varied between $0^\circ$-90°. The plain of the various angle beams is perpendicular to the column (see Fig. 9.5.3.1) thus the supporting beam is always horizontal.

The connecting beam rigidity has effect on the stiff and the weak buckling lengths of the column. After the calculation of the buckling lengths of the column based on the solution of the stability eigenvalue problem (stability calculation) we compared the beta factors with the FEM-Design automatic flexural buckling calculation results.

Around the stiff direction the buckling length is increasing because the supporting effect of the connecting beam is decreasing. Around the weak direction the buckling length is decreasing because the supporting effect of the connecting beam is increasing (see Fig. 9.5.3.1, Table 9.5.3.1-2).

The critical forces of the hinged-hinged column (so-called Euler force) based on the stability calculation in FEM-Design are:

\[ F_{cr1} = 11672 \text{kN} \quad \text{around stiff direction}, \]
\[ F_{cr2} = 4230 \text{kN} \quad \text{around weak direction}. \]

Be careful, these values contain the shear deformations and not only the deformation from bending because in FEM-Design the beam model is the Timoshenko model.
For example the beta factor around stiff direction based on the solution of the stability eigenvalue problem when the $\alpha$ angle is equal to 76°:

$$\beta_{76}^{\text{stiff}} = \sqrt{\frac{F_{\text{cre}}}{F_{\text{cr}}^{\text{stiff}}}} = \sqrt{\frac{11672}{22877}} = 0.714$$

<table>
<thead>
<tr>
<th>Angle [degree]</th>
<th>Critical load [kN]</th>
<th>Eigenvalue</th>
<th>Beta factor</th>
<th>AutoBucklingLength</th>
<th>Difference [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>30373</td>
<td>0.620</td>
<td>0.589</td>
<td>-0.0499</td>
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</tr>
<tr>
<td>14</td>
<td>29948</td>
<td>0.624</td>
<td>0.592</td>
<td>-0.0517</td>
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</tr>
<tr>
<td>27</td>
<td>28899</td>
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<td>0.600</td>
<td>-0.0559</td>
<td></td>
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<tr>
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<td>0.650</td>
<td>0.611</td>
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<td></td>
</tr>
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<td>45</td>
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<tr>
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<td>24076</td>
<td>0.696</td>
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</tr>
<tr>
<td>76</td>
<td>22877</td>
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<td>0.686</td>
<td>-0.0396</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>22372</td>
<td>0.722</td>
<td>0.700</td>
<td>-0.0309</td>
<td></td>
</tr>
</tbody>
</table>

*Table 9.5.3.1 – The beta factor around the stiff direction in the function of the given angle*

<table>
<thead>
<tr>
<th>Angle [degree]</th>
<th>Critical load [kN]</th>
<th>Eigenvalue</th>
<th>Beta factor</th>
<th>AutoBucklingLength</th>
<th>Difference [-]</th>
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<tbody>
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<td>8963</td>
<td>0.687</td>
<td>0.665</td>
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<td></td>
</tr>
<tr>
<td>27</td>
<td>9811</td>
<td>0.657</td>
<td>0.616</td>
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<td></td>
</tr>
<tr>
<td>37</td>
<td>10737</td>
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<td>-0.0648</td>
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</tr>
<tr>
<td>45</td>
<td>11523</td>
<td>0.606</td>
<td>0.571</td>
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<td></td>
</tr>
<tr>
<td>53</td>
<td>12295</td>
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</tr>
<tr>
<td>63</td>
<td>13171</td>
<td>0.567</td>
<td>0.551</td>
<td>-0.0277</td>
<td></td>
</tr>
<tr>
<td>76</td>
<td>13952</td>
<td>0.551</td>
<td>0.545</td>
<td>-0.0102</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>14284</td>
<td>0.544</td>
<td>0.543</td>
<td>-0.0022</td>
<td></td>
</tr>
</tbody>
</table>

*Table 9.5.3.2 – The beta factor around the weak direction in the function of the given angle*

The differences between the two calculation methods are less than 6% (see Table 9.5.3.1-2). In FEM-Design by the automatic beta factor calculation the column was assumed as a non-sway column according to the original supporting condition (see Fig. 9.5.3.1).

Fig. 9.5.3.2 shows the tendency of the beta factors in function of the supporting beam angle.
Download link to the example file:
http://download.strusoft.com/FEM-Design/inst180x/models/9.5.3 A column and a supporting beam with various angles.str

Figure 9.5.3.2 – The tendency of the beta factors (stiff 1, weak 2) in the function of the various angle
10 Cross section editor

10.1 Calculation of a compound cross section

An example for compound cross section is taken from [7] where the authors calculated the cross sectional properties with the assumption of thin-walled simplifications. The welded cross section is consisting of U300 and L160x80x12 (DIN) profiles. In the Section Editor the exact cold rolled geometry was analyzed as it is seen in Figure 8.1.1.

![Figure 8.1.1 – The analyzed cross section](image)

The following table contains the results of the two independent calculations with several cross sectional properties.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Ref. [1]</th>
<th>Section Editor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ [cm$^2$]</td>
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<td>86.30</td>
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<tr>
<td>$y_G$ [cm]</td>
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<td>1.442</td>
</tr>
<tr>
<td>$z_G$ [cm]</td>
<td>19.20</td>
<td>19.22</td>
</tr>
<tr>
<td>$y'_S$ [cm]</td>
<td>1.39</td>
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<tr>
<td>$z'_S$ [cm]</td>
<td>10.06</td>
<td>10.36</td>
</tr>
<tr>
<td>$I_y$ [cm$^4$]</td>
<td>11379.9</td>
<td>11431.2</td>
</tr>
<tr>
<td>$I_z$ [cm$^4$]</td>
<td>4513.3</td>
<td>4372.9</td>
</tr>
<tr>
<td>$I_{yz}$ [cm$^4$]</td>
<td>3013.2</td>
<td>3053.5</td>
</tr>
<tr>
<td>$I_x$ [cm$^4$]</td>
<td>48.83</td>
<td>52.11</td>
</tr>
<tr>
<td>$I_w$ [cm$^6$]</td>
<td>-</td>
<td>203082.0</td>
</tr>
</tbody>
</table>

Table 8.1.1 – The results of the example
Download link to the example file:

http://download.strusoft.com/FEM-Design/inst170x/models/8.1 Calculation of a compound cross section.sec
References


Notes