StruSoft FEM-Design

FEM-Design

Verification Examples

version 1.1
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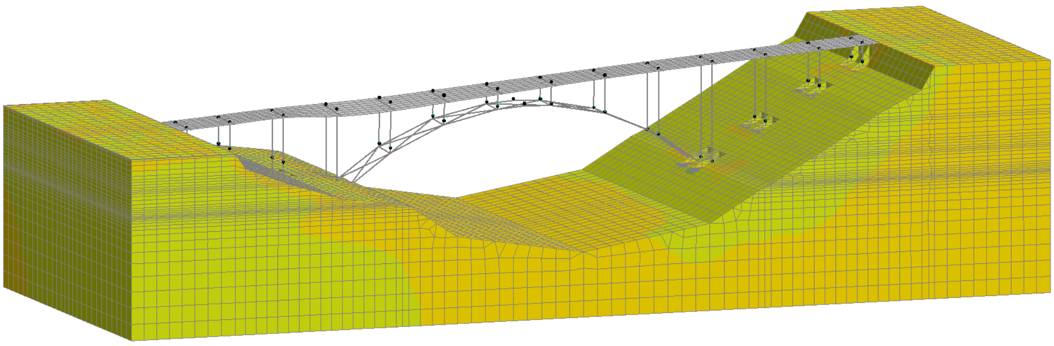

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Verification Examples

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Contents

In this verification handbook we highlighted the analytical results with green and the finite element results with blue background for better comparison. The analytical closed formulas are highlighted with a black frame.

If the finite element mesh is not mentioned during the example it means that the automatically generated mesh was used.

1. Linear static calculations

1.1 Beam with two point loading at one-third of its span

Fig. 1.1.1 left side shows the simple supported problem. The loads, the geometric and material properties are as follows:

Force	F = 150 kN
Length	L = 6 m
Cross section	Steel I beam HEA 300
The second moment of inertia in the relevant direction	$I_1 = 1.8264 \cdot 10^{-4}\ m^4$
The shear correction factor in the relevant direction	$\rho_2 = 0.21597$
The area of the cross section	$A = 112.53\ cm^2$
Young's modulus	E = 210 GPa
Shear modulus	G = 80.769 GPa

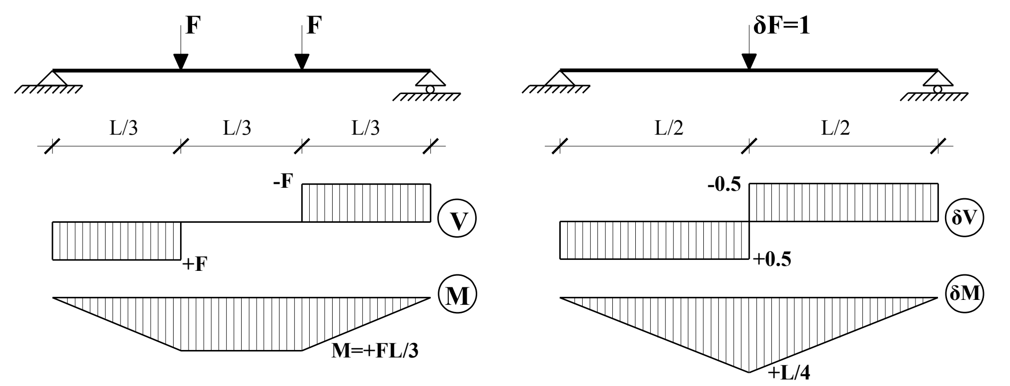


Figure 1.1.1 – The beam theory and the application of a virtual force

The deflection of the mid-span based on the hand calculation (based on virtaul force theorem [1], see Fig. 1.1.1 right side also):

$$e=\frac{2M}{EI}\left[\frac{L}{3}\frac{2}{3}\frac{2}{3}\frac{L}{4}\frac{1}{2}+\frac{L}{6}\frac{2}{3}\frac{L}{4}+\frac{L}{6}\frac{1}{3}\frac{L}{4}\frac{1}{2}\right]+\frac{2F}{\rho GA}\left[0.5\frac{L}{3}\right]=\frac{23}{648}\frac{FL^3}{EI}+\frac{FL}{3\rho GA}$$

$$e=\frac{23}{648}\frac{150\cdot 6^3}{210000000\cdot 1.8264\cdot 10^{-4}}+\frac{150\cdot 6}{3\cdot 0.21597\cdot 80769000\cdot 0.011253}=0.03151\ m=31.51\ mm$$

The first part of this equation comes from the bending deformation and the second part comes from the consideration of the shear deformation as well, because FEM-Design uses Timoshenko beam theory (see the Scientific Manual).

The deflection and the bending moment at the mid-span based on the linear static calculation with three 2-noded beam elements (Fig. 1.1.2 and Fig. 1.1.3):

$e_{FEM} = 31.51 \text{ mm}$ and the bending moment $M_{FEM} = 300 \text{ kNm}$

The theoretical solution in this case (three 2-noded beam elements) must be equal to the finite element solution because with three beam elements the shape functions order coincides with the order of the theoretical function of the deflection (the solution of the differtial equation).

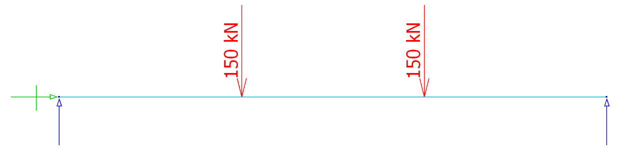


Figure 1.1.2 – The finite element model

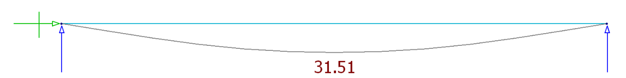


Figure 1.1.3 – The mid-span deflection [mm]

Therefore the difference between the results of the two calculations is zero.

1.2 Calculation of a circular plate with concentrated force at its center

In this chapter a circular steel plate with a concentrated force at its center will be analyzed. First of all the maximum deflection (translation) of the plate will be calculated at its center and then the bending moments in the plate will be presented.

Two different boundary conditions will be applied at the edge of the plate. In the first case the edge is clamped (Case I.) and in the second case is simply supported (Case II.), see Fig. 1.2.1.

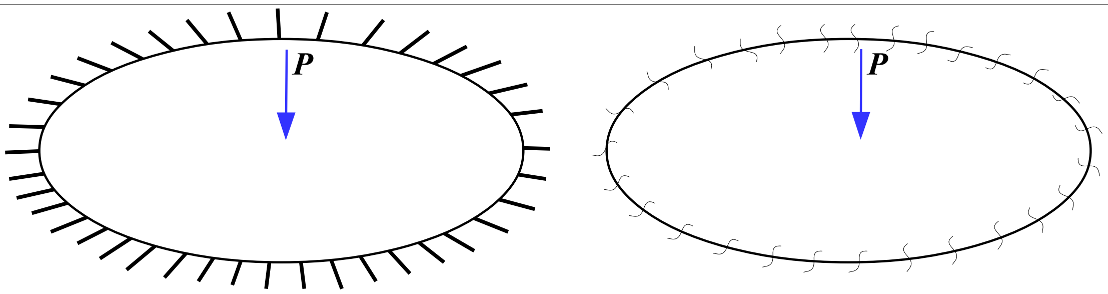


Figure 1.2.1 – Clamped (Case I.) and simply supported (Case II.) circular plate with concentrated force

The input parameters are as follows:

The concentrated force	P = 10 kN
The thickness of the plate	h = 0.05 m
The radius of the circular plate	R = 5 m
The elastic modulus	E = 210 GPa
The Poisson's ratio	ν = 0.3

The ratio between the diameter and the thickness is 2R/h = 200. It means that based on the geometry the shear deformation only have negligible effects on the maximum deflections. It is important because FEM-Design uses the Mindlin plate theory (considering the shear deformation, see Scientific Manual for more details), but in this case the solution of Kirchhoff's plate theory and the finite element result must be close to each other based on the mentioned ratio.

The analytical solution of Kirchhoff's plate theory is given in a closed form [2][3].

Case I.:

For the clamped case the maximum deflection at the center is:

$$w_{cl}=\frac{P\,R^2}{16\,\pi\left(\frac{E\,h^3}{12\left(1-\nu^2\right)}\right)}$$

The reaction force at the edge:

$$Q_r=\frac{P}{2\,\pi\,R}$$

And the bending moment in the radial direction at the edge:

$$M_{cl}=\frac{P}{4\,\pi}$$

With the given input parameters the results based on the analytical and the finite element solutions (with the default finite element mesh size, see Fig. 1.2.2) are:

$$w_{cl}=\frac{10\cdot 5^2}{16\,\pi\left(\frac{210000000\cdot 0.05^3}{12\left(1-0.3^2\right)}\right)}=0.002069\,\mathrm{m}=2.069\,\mathrm{mm}$$

$$w_{clFEM}=2.04\,\mathrm{mm}$$

$$Q_r=\frac{10}{2\,\pi\,5}=0.318\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$$Q_{rFEM}=0.318\,\frac{\mathrm{kN}}{\mathrm{m}}$$

$$M_{cl}=\frac{P}{4\,\pi}=\frac{10}{4\,\pi}=0.796\,\frac{\mathrm{kNm}}{\mathrm{m}}$$

$$M_{clFEM}=0.796\,\frac{\mathrm{kNm}}{\mathrm{m}}$$

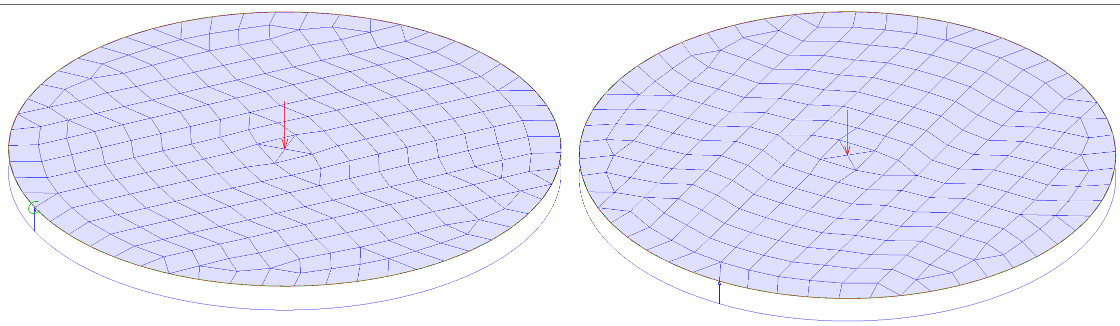


Figure 1.2.2 – The clamped (Case I.) and the simply supported (Case II.) plate with the default mesh

Case II.:

For the simply supported case the maximum deflection in the center is:

$$w_{ss}=\frac{P R^2}{16\pi\left(\frac{E h^3}{12(1-\nu^2)}\right)}\left(\frac{3+\nu}{1+\nu}\right)$$

The reaction force at the edge:

$$Q_r=\frac{P}{2\pi R}$$

With the given input parameters the results based on the analytical and the finite element solutions (with the default finite element mesh size, see Fig. 1.2.2) are:

$$w_{ss}=\frac{10\cdot 5^2}{16\pi\left(\frac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}\left(\frac{3+0.3}{1+0.3}\right)=0.005252\,\mathrm{m}=5.252\,\mathrm{mm}$$

$$w_{ssFEM}=5.00\,\mathrm{mm}$$

$$Q_r=\frac{10}{2\pi 5}=0.318\,\mathrm{kN/m}$$

$$Q_{rFEM}=0.318\,\frac{\mathrm{kN}}{\mathrm{m}}$$

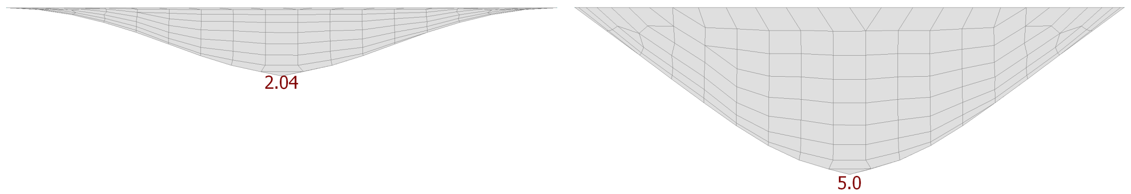


Figure 1.2.3 – The deflected shape of Case I. (clamped) and Case II. (simply supported) with the default mesh

Fig. 1.2.3 shows the two deflected shape in side view. The different boundary conditions are obvious based on the two different displacement shape. The differences between the analytical solutions and finite element solutions are less than 5% but the results could be more accurate if the applied mesh is more dense than the default size.

Based on the analytical solution the bending moments in plates under concentrated loads are infinite. It means that if more and more dense mesh will be applied the bending moment under the concentrated load will be greater and greater. Thus the following diagram and table (Fig. 1.2.4 and Table 1.2.1) shows the convergence analysis of Case I. respect to the deflection and bending moment. The deflection converges to the analytical solution (w_{cl} = 2.07 mm) and the bending moment converges to infinite.

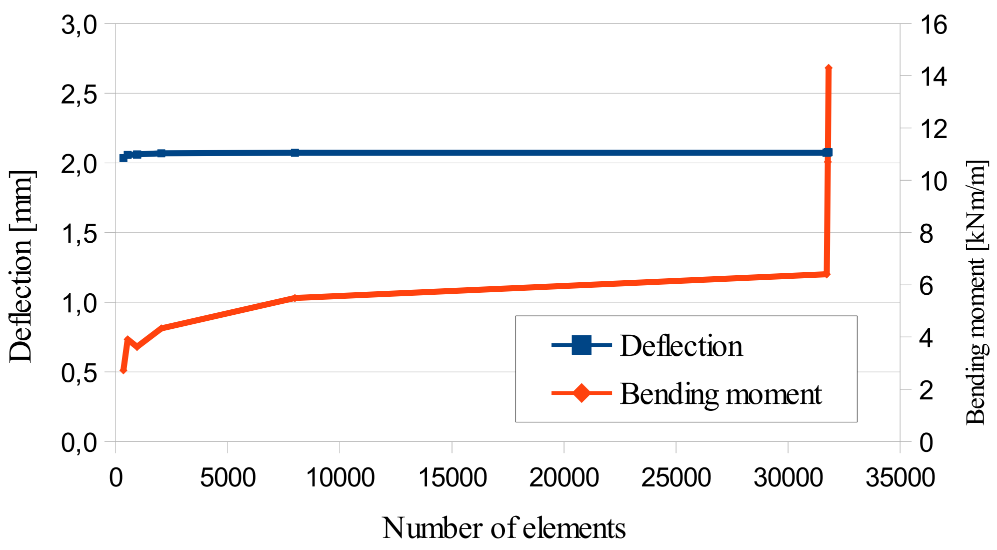


Figure 1.2.4 – Convergence analysis regarding to deflection and bending moment

Number of elements [pcs.]	Deflection [mm]	Bending moment [kNm/m]	Average element size [m]
341	2,034	2,73	0,5
533	2,057	3,91	0,4
957	2,060	3,62	0,3
2035	2,068	4,33	0,2
7994	2,072	5,49	0,1
31719	2,073	6,40	0,05
31772	2,075	10,70	Local refinement 1
31812	2,076	14,30	Local refinement 2

Table 1.2.1 – The convergence analysis

1.3 A simply supported square plate with uniform load

In this example a simply supported concrete square plate will be analyzed. The external load is a uniform distributed load (see Fig. 1.3.1). We compare the maximum displacements and maximum bending moments of the analytical solution of Kirchhoff's plate theory and finite element results.

The input parameters are in this table:

The intensity of the uniform load	$p = 40\ kN/m^2$
The thickness of the plate	$h = 0.25\ m$
The edge of the square plate	$a = 5\ m$
The elastic modulus	$E = 30\ GPa$
Poisson's ratio	$\nu = 0.2$

The ratio between the span and the thickness is a/h = 20. It means that based on the geometry the shear deformation may have effects on the maximum deflection. It is important because FEM-Design uses the Mindlin plate theory (considering the shear deformation, see Scientific Manual for more details), therefore in this case the results of Kirchhoff's theory and the finite element result could be different from each other due to the effect of shear deformations.

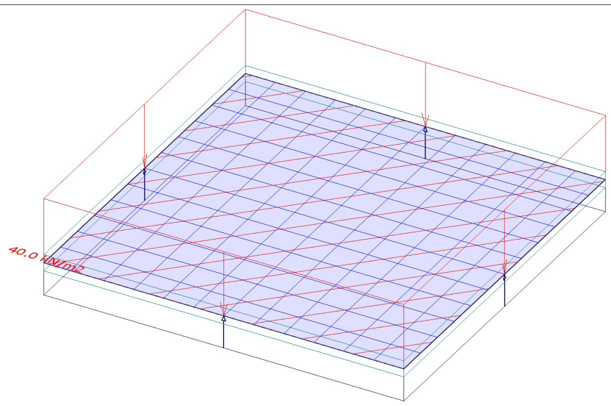


Figure 1.3.1 – The square plate with simply supported edges, uniform load and default mesh size

Based on Kirchhoff's plate theory [2][3] the maximum deflection is in the center of the simply supported square plate and its intensity can be given with the following closed form:

$$w_{max} = 0.00416 \frac{p\,a^4}{\left(\dfrac{E\,h^3}{12(1-\nu^2)}\right)}$$

The maximum bending moment in the plate if the Poisson's ratio ν = 0.2:

$$M_{max} = 0.0469\, p\, a^2$$

According to the input parameters and the analytical solutions the results of this problem are the following:

The deflection at the center of the plate:

$$w_{max} = 0.00416 \frac{40 \cdot 5^4}{\left(\frac{30000000 \cdot 0.25^3}{12(1-0.2^2)}\right)} = 0.002556\,\text{m} = 2.556\,\text{mm}$$

$$w_{maxFEM} = 2.632\,\text{mm}$$

The bending moment at the center of the plate:

$$M_{max} = 0.0469 \cdot 40 \cdot 5^2 = 46.9 \frac{\text{kNm}}{\text{m}}$$

$$M_{maxFEM} = 45.97 \frac{\text{kNm}}{\text{m}}$$

Next to the analytical solutions the results of the FE calculations are also indicated (see Fig, 1.3.2 and 1.3.3). The difference is less than 3% and it also comes from the fact that FEM-Design considers the shear deformation also (Mindlin plate theory).

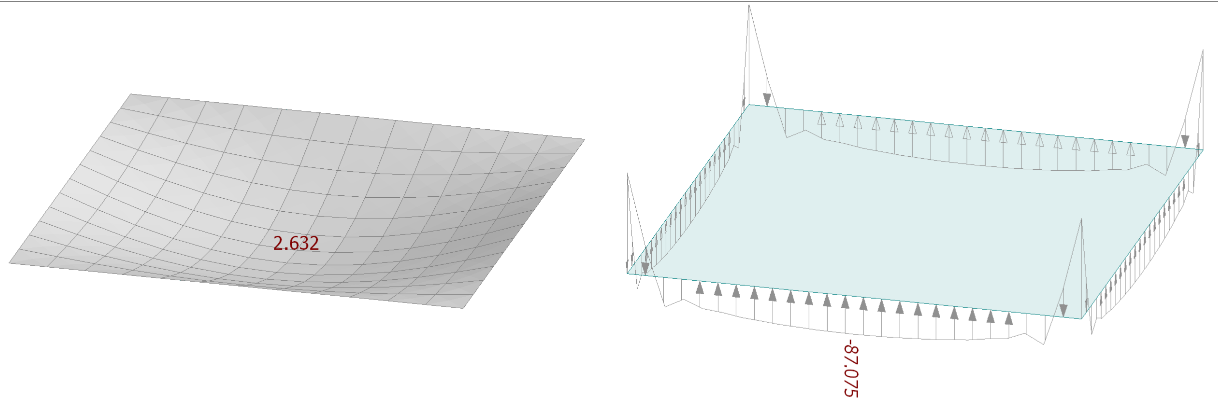


Figure 1.3.2 – The deflected shape [mm] *and the reaction forces* [kN/m] *with the default mesh*

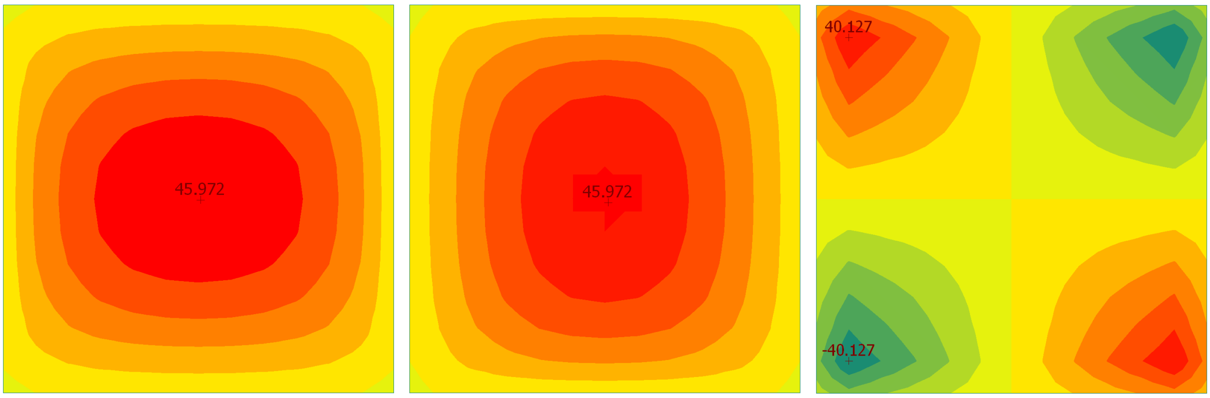


Figure 1.3.3 – The internal forces; M_x – M_y – M_{xy} [kNm/m]

2. Second order analysis

2.1 A column with vertical and horizontal loads

We would like to analyze the following column (see Fig. 2.1.1) with second order theory. First of all we make a hand calculation with third order theory according to Ref. [6] and [8] with stability functions. After this step we compare the results with FEM-Design. In this moment we need to consider that in FEM-Design second order analysis is implemented and the hand calulation will be based on third order theory therefore the final results won't be exactly the same. By the FEM-Design calulation we split the column into 3 bar elements thus the finite element number of the bars was 3 for more precise results.

The input parameters:

Elastic modulus	E = 30 GPa
Normal force	P = 2468 kN
Horizontal load	q = 10 kN/m
Cross section	0.2 m x 0.4 m (rectangle)
Second moment of inertia in the relevant direction	I = 0.0002667 m^4
Column length	L = 4 m

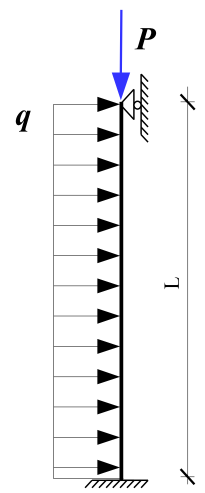


Figure 2.1.1 – The column with vertical and horizontal loads

According to Ref. [6] and [8] first of all we need to calculate the following assistant quantities:

$$\rho = \frac{P}{P_E} = \frac{P}{\left(\frac{\pi^2 EI}{L^2}\right)} = \frac{2468}{\left(\frac{\pi^2 30000000 \cdot 0.0002667}{4^2}\right)} = 0.500$$

The constants based on this value for the appropriate stability functions:

$$s=3.294 \quad ; \quad c=0.666 \quad ; \quad f=1.104$$

With these values the bending moments and the shear forces based on third order theory and FEM-Design calculation:

$$M_{clamped}=f(1+c)\frac{qL^2}{12}=1.104(1+0.666)\frac{10\cdot 4^2}{12}=24.52\,\text{kNm}$$

$$M_{2ndFEMclamped}=25.55\,\text{kNm}$$

$$M_{roller}=0.0\,\text{kNm}$$

$$M_{2ndFEMroller}=0.0\,\text{kNm}$$

$$V_{clamped}=\left[1+\frac{f(1+c)}{6}\right]\left(\frac{qL}{2}\right)=\left[1+\frac{1.104(1+0.666)}{6}\right]\left(\frac{10\cdot 4}{2}\right)=26.13\,\text{kN}$$

$$V_{2ndFEMclamped}=26.38\,\text{kN}$$

$$V_{roller}=\left[1-\frac{f(1+c)}{6}\right]\left(\frac{qL}{2}\right)=\left[1-\frac{1.104(1+0.666)}{6}\right]\left(\frac{10\cdot 4}{2}\right)=13.87\,\text{kN}$$

$$V_{2ndFEMroller}=13.61\,\text{kNm}$$

The differences are less than 5 %.

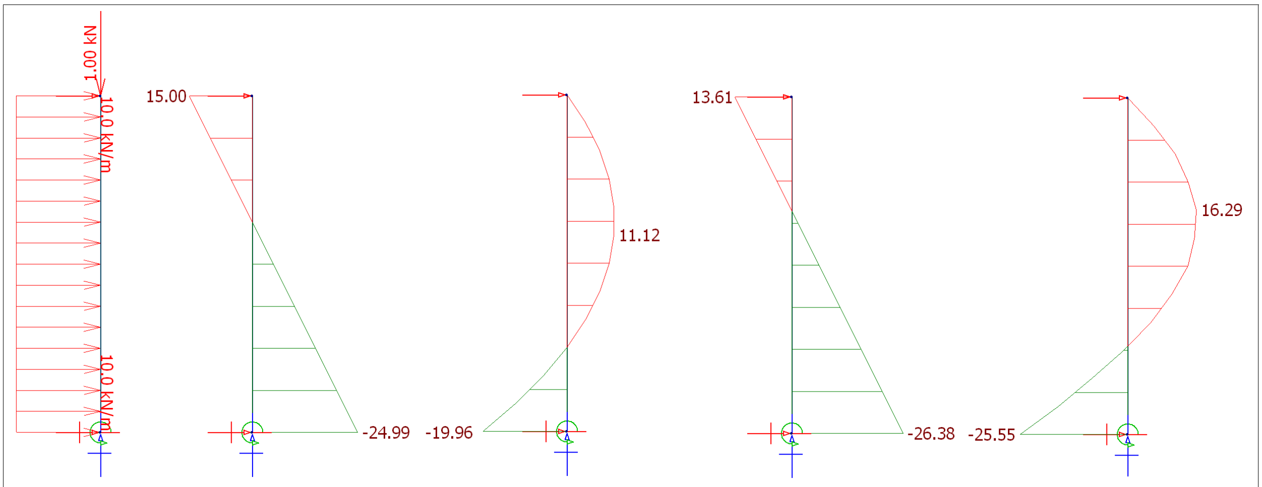


Figure 2.1.2 – The shear [kN] *and bending moment* [kNm] *diagram with 1st and 2nd order theory*

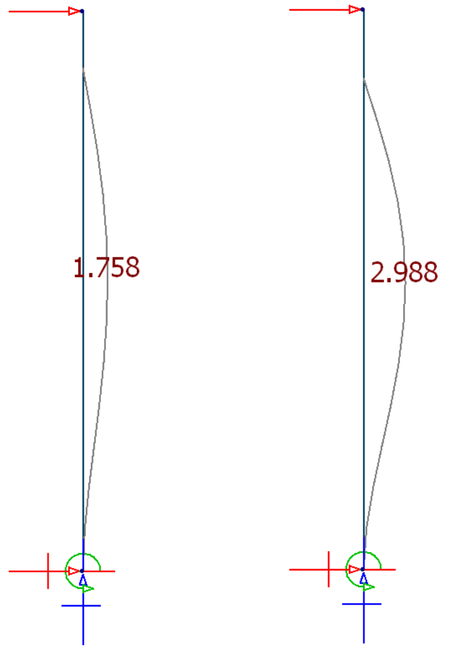


Figure 2.1.3 – The lateral translations with 1st and 2nd order theory

2.2 A plate with in-plane and out-of-plane loads

In this chapter we will analyze a rectangular plate with single supported four edges. The load is a specific normal force at the shorter edge and a lateral distributed total load perpendicular to the plate (see Fig. 2.2.1). The displacement and the bending moment are the question based on a 2nd order analysis. First of all we calculate the results with analytical solution and then we compare the results with FE calculations.

In this case the material and the geometric properties are the following:

The thickness of the plate	h = 0.05 m
The dimensions of the plate	a = 8 m; b = 6 m
The elastic modulus	E = 210 GPa
Poisson's ratio	ν = 0.3
The specific normal force	N_x = 1000 kN/m
The lateral distributed load	q_z = 10 kN/m^2

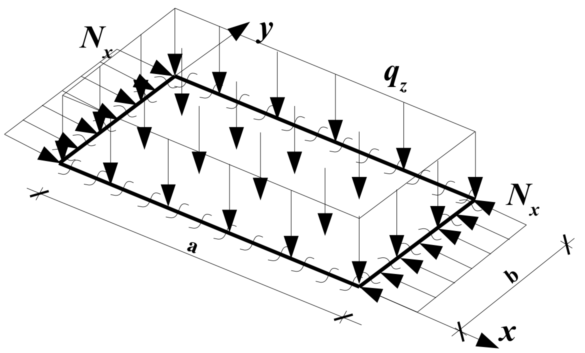


Figure 2.2.1 – The single supported edges, the lateral distributed load and the specific normal force

The maximum displacement and moments based on the 1st order linear calculation:

$$w_{max} = 35.38\,\text{mm} \quad , \quad m_{x,max} = 18.05\,\frac{\text{kNm}}{\text{m}} \quad , \quad m_{y,max} = 25.62\,\frac{\text{kNm}}{\text{m}} \quad , \quad m_{xy,max} = 13.68\,\frac{\text{kNm}}{\text{m}}$$

In Chapter 3.2 the critical specific normal force for this example is:

$$N_{cr} = 2860\,\frac{\text{kN}}{\text{m}}$$

If the applied specific normal force is not so close to the critical value (now it is lower than the half of the critical value) we can assume the second order displacements and internal forces based on the linear solutions with the following formulas (with blue highlight we indicated the results of the FE calculation):

$$w_{max,2nd} = w_{max} \frac{1}{\left(1-\frac{N_x}{N_{cr}}\right)} = 35.38 \frac{1}{\left(1-\frac{1000}{2860}\right)} = 54.41\,\text{mm} \quad , \quad w_{max,2nd,FEM} = 54.69\,\text{mm}$$

$$m_{x,max,2nd} = m_{x,max} \frac{1}{\left(1-\frac{N_x}{N_{cr}}\right)} = 18.05 \frac{1}{\left(1-\frac{1000}{2860}\right)} = 27.76 \frac{\text{kNm}}{\text{m}} \quad , \quad m_{x,max,2nd,FEM} = 28.50 \frac{\text{kNm}}{\text{m}}$$

$$m_{y,max,2nd} = m_{y,max} \frac{1}{\left(1-\frac{N_x}{N_{cr}}\right)} = 25.62 \frac{1}{\left(1-\frac{1000}{2860}\right)} = 39.40 \frac{\text{kNm}}{\text{m}} \quad , \quad m_{y,max,2nd,FEM} = 40.30 \frac{\text{kNm}}{\text{m}}$$

$$m_{xy,max,2nd} = m_{xy,max} \frac{1}{\left(1-\frac{N_x}{N_{cr}}\right)} = 13.68 \frac{1}{\left(1-\frac{1000}{2860}\right)} = 21.04 \frac{\text{kNm}}{\text{m}} \quad m_{xy,max,2nd,FEM} = 20.53 \frac{\text{kNm}}{\text{m}}$$

The differences are less than 3 %.

Figure 2.2.2 shows the problem in FEM-Design with the default mesh.

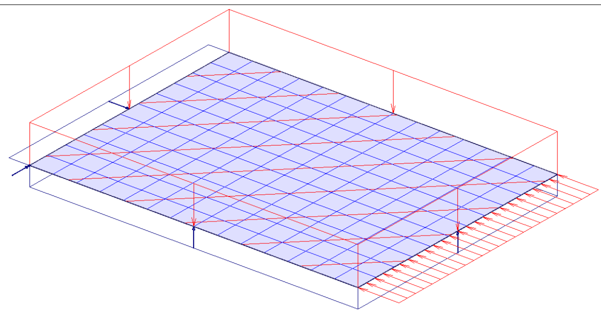


Figure 2.2.2 – The single supported slab with in-plane and out-of-plane loads

The following figures show the moment distribution in the plate and the displacements with FEM-Design according to 1st and 2nd order theory.

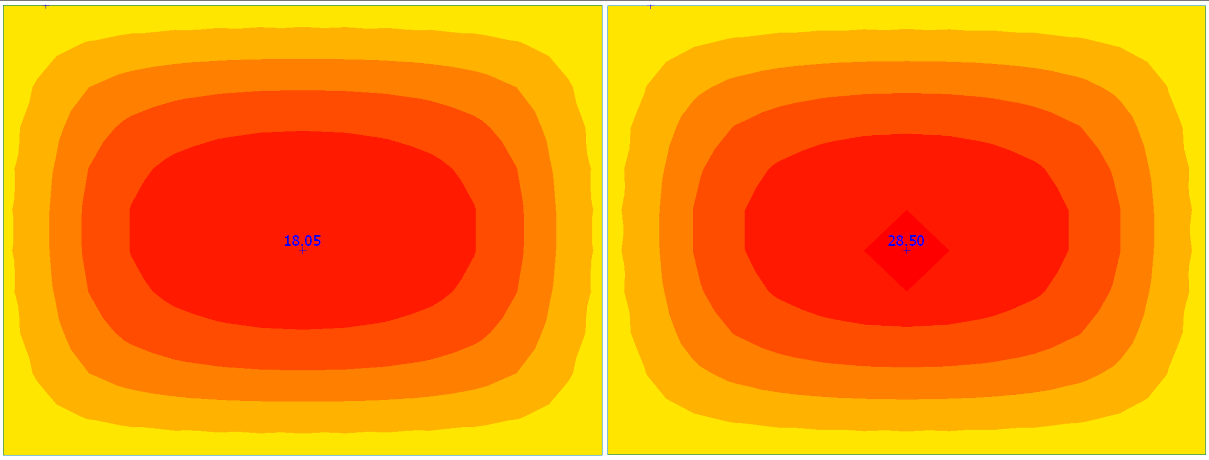


Figure 2.2.3 – The m_x [kNm/m] *moment with 1st and 2nd order analysis*

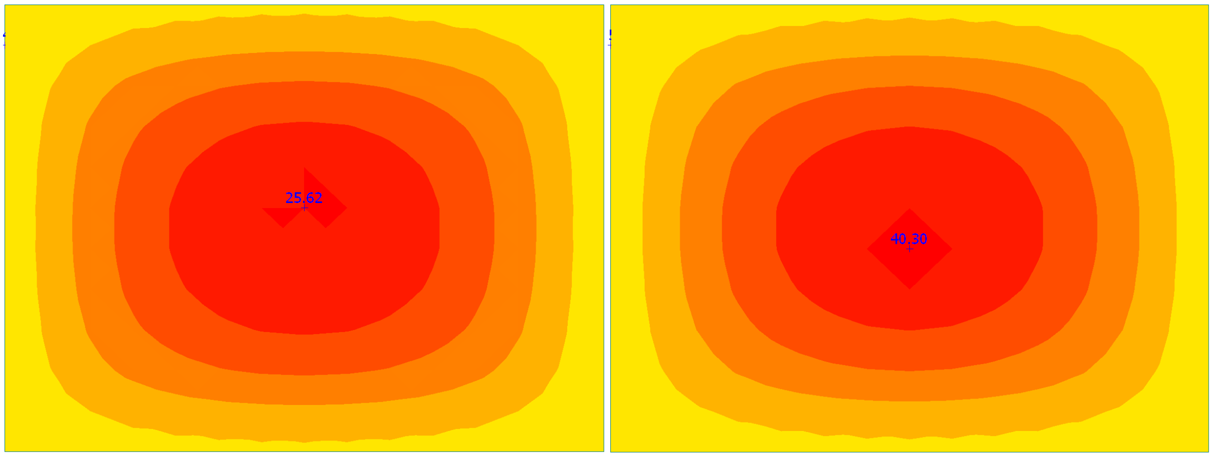


Figure 2.2.4 – The m_y [kNm/m] *moment with 1st and 2nd order analysis*

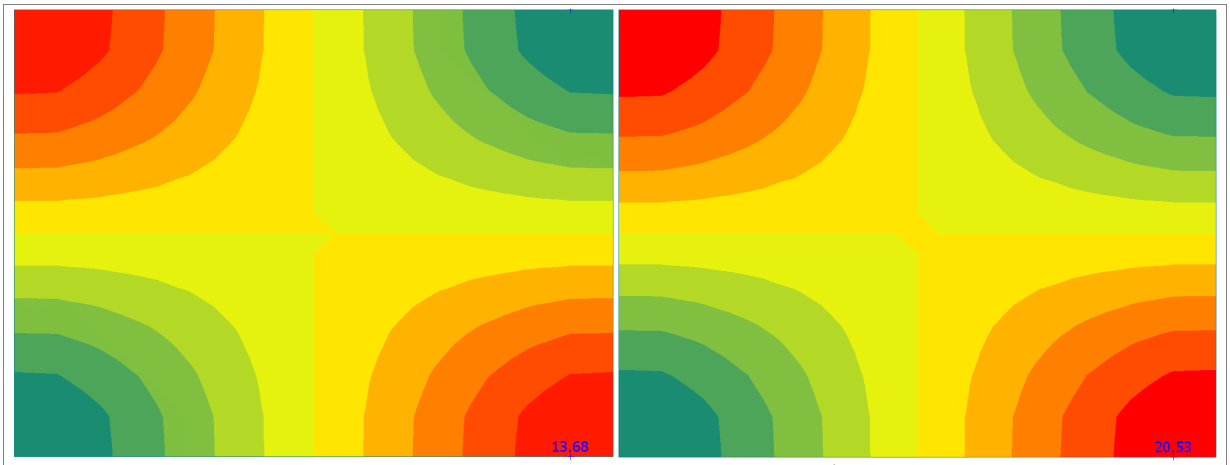
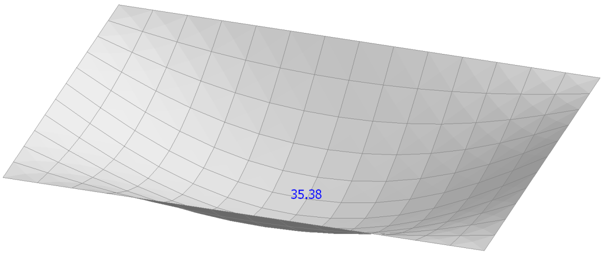


Figure 2.2.5 – The m_{xy} [kNm/m] *moment with 1st and 2nd order analysis*



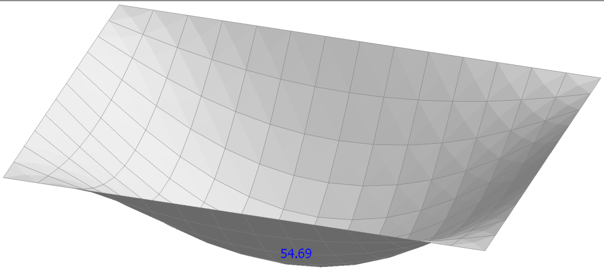


Figure 2.2.6 – The vertical translation [mm] *with 1st and 2nd order analysis*

3. Stability analysis

3.1 Flexural buckling analysis of a beam modell with different boundary conditions

The cross section is a rectangular section	see Fig. 3.1.1
The material	C20/25 concrete
The elastic modulus	E = 30000 GPa
The second moment of inertia about the weak axis	$I_2 = 2.667 \cdot 10^{-4}$ m^4
The length of the column	L = 4 m
The boundary conditions	see Fig. 3.1.1

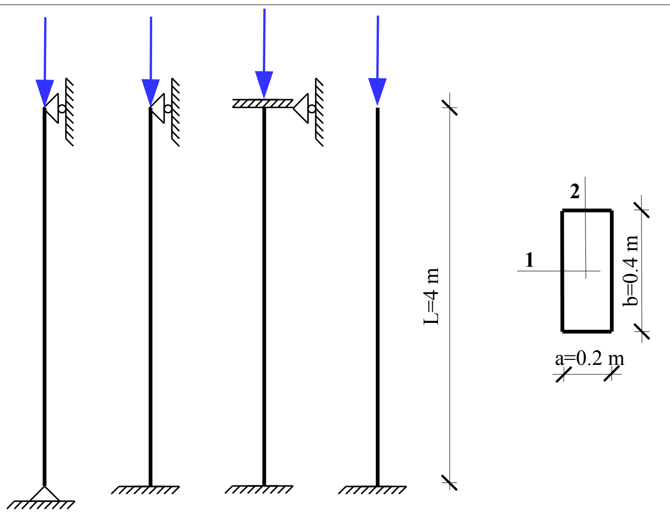


Figure 3.1.1 – The buckling problem with the different boundary conditions and the cross section

The critical load parameters according to the Euler's theory are as follows and next to the analytical solution [1] the relevant results of the FEM-Design calculation can be seen.

Pinned-pinned boundary condition:

$$F_{cr1} = \frac{\pi^2 E I_2}{L^2} = 4934.8\,\text{kN}$$

$$F_{crFEM1} = 4910.9\,\text{kN}$$

Fixed-pinned boundary condition:

$$F_{cr2} = \frac{\pi^2 E I_2}{(0.6992\,L)^2} = 10094.1\,\text{kN}$$

$$F_{crFEM2} = 9974.6\,\text{kN}$$

Fixed-fixed boundary condition:

$$F_{cr3}=\frac{\pi^2 E I_2}{(0.5L)^2}=19739.2\,\text{kN}$$

$$F_{crFEM3}=19318.7\,\text{kN}$$

Fixed-free boundary condition:

$$F_{cr4}=\frac{\pi^2 E I_2}{(2L)^2}=1233.7\,\text{kN}$$

$$F_{crFEM4}=1233.1\,\text{kN}$$

The difference between the two calculations are less than 3% but keep in mind that FEM-Design considers the shear deformation therefore we can be sure that the Euler's results give higher critical values. Fig. 3.1.2 shows the first mode shapes of the problem with the different boundary conditions.

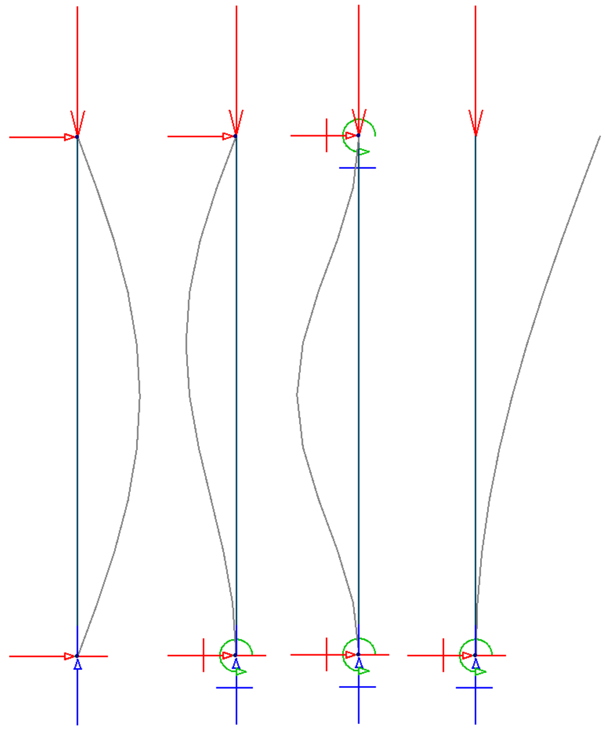


Figure 3.1.2 – The buckling mode shapes for different boundary conditions pinned-pinned; fixed-pinned; fixed-fixed; fixed-free

3.2 Buckling analysis of a plate with shell modell

In this chapter we will analyze a rectangular plate with single supported four edges. The load is a specific normal force at the shorter edge (see Fig. 3.2.1). The critical force parameters are the questions due to this edge load, therefore it is a stability problem of a plate.

In this case the material and the geometric properties are the following:

The thickness of the plate	h = 0.05 m
The dimensions of the plate	a = 8 m; b = 6 m
The elastic modulus	E = 210 GPa
Poisson's ratio	ν = 0.3

The solutions of the differential equation of the plate buckling problem are as follows [6]:

$$N_{cr}=\left(\frac{m\,b}{a}+\frac{n^2 a}{m\,b}\right)^2 \frac{\pi^2\left(\frac{E\,h^3}{12(1-\nu^2)}\right)}{b^2} \quad , \quad m=1,2,3\ldots \quad , \quad n=1,2,3\ldots$$

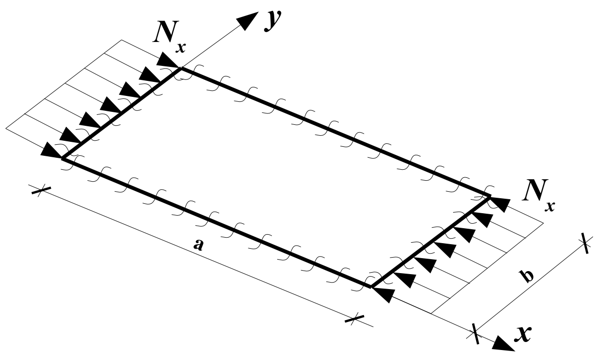


Figure 3.2.1 – The single supported edges and the specific normal force

Figure 3.2.2 shows the problem in FEM-Design with the default mesh.

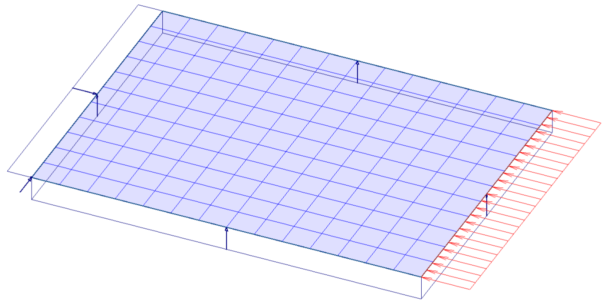


Figure 3.2.2 – The stability problem of a plate with single supported edges

According to the analytical solution the first five critical load parameters are:

$m=1 \quad , \quad n=1$

$$N_{cr1}=\left(\frac{1\cdot 6}{8}+\frac{1^2 8}{1\cdot 6}\right)^2 \frac{\pi^2\left(\dfrac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{6^2}=2860.36\,\frac{\text{kN}}{\text{m}}$$

$$N_{crFEM1}=2862.58\,\frac{\text{kN}}{\text{m}}$$

$m=2 \quad , \quad n=1$

$$N_{cr2}=\left(\frac{2\cdot 6}{8}+\frac{1^2 8}{2\cdot 6}\right)^2 \frac{\pi^2\left(\dfrac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{6^2}=3093.77\,\frac{\text{kN}}{\text{m}}$$

$$N_{crFEM2}=3109.96\,\frac{\text{kN}}{\text{m}}$$

$m=3 \quad , \quad n=1$

$$N_{cr3}=\left(\frac{3\cdot 6}{8}+\frac{1^2 8}{3\cdot 6}\right)^2 \frac{\pi^2\left(\dfrac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{6^2}=4784.56\,\frac{\text{kN}}{\text{m}}$$

$$N_{crFEM3}=4884.90\,\frac{\text{kN}}{\text{m}}$$

$m=4 \quad , \quad n=1$

$$N_{cr4}=\left(\frac{4\cdot 6}{8}+\frac{1^2 8}{4\cdot 6}\right)^2 \frac{\pi^2\left(\dfrac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{6^2}=7322.53\,\frac{\text{kN}}{\text{m}}$$

$$N_{crFEM4}=7655.58\,\frac{\text{kN}}{\text{m}}$$

$m=3 \quad , \quad n=2$

$$N_{cr5}=\left(\frac{3\cdot 6}{8}+\frac{2^2 8}{3\cdot 6}\right)^2 \frac{\pi^2\left(\dfrac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{6^2}=10691.41\,\frac{\text{kN}}{\text{m}}$$

$$N_{crFEM5}=10804.62\,\frac{\text{kN}}{\text{m}}$$

Next to these values we indicated the critical load parameters what were calculated with the FEM-Design.

The difference between the calculations less than 5 %.

Figure 3.2.3 shows the first five stability mode shapes of rectangular single supported plate.

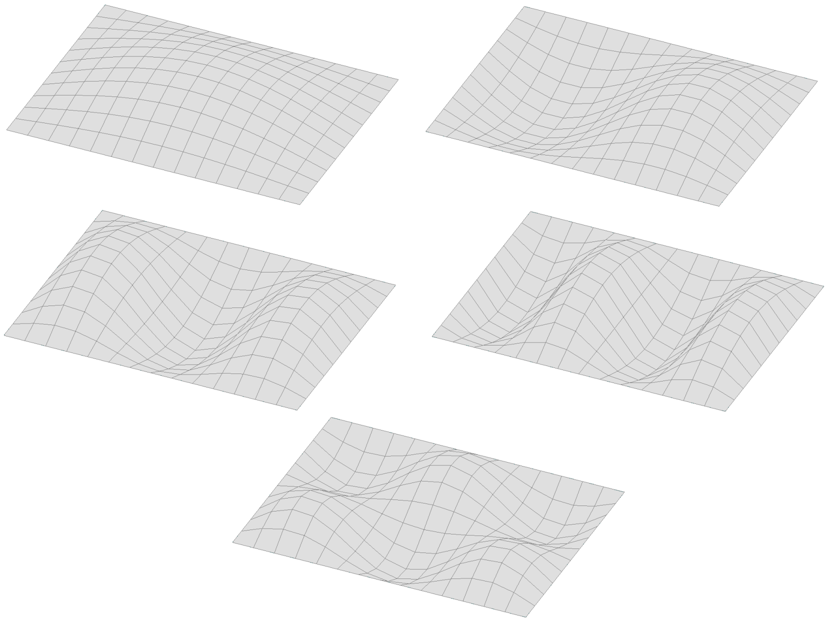


Figure 3.2.3 – The first five stability mode shape of the described problem

3.3 Lateral torsional buckling of an I section with shell modell

The purpose of this example is calculate the lateral torsional critical moment of the following simple supported beam (see Fig. 3.3.1).

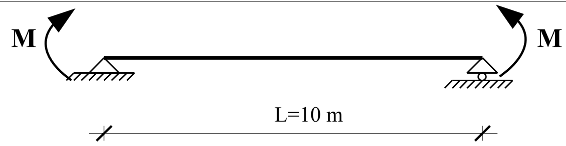


Figure 3.3.1 – The static frame of a simple supported beam loaded with bending moments of both ends

The length of the beam is	L = 10 m
The cross section	see Fig. 3.3.2
The warping constant of the section	I_w = 125841 cm^6
The St. Venant torsional inertia	I_t = 15.34 cm^4
The minor axis second moment of area	I_z = 602.7 cm^4
The elastic modulus	E = 210 GPa
The shear modulus	G = 80.77 GPa

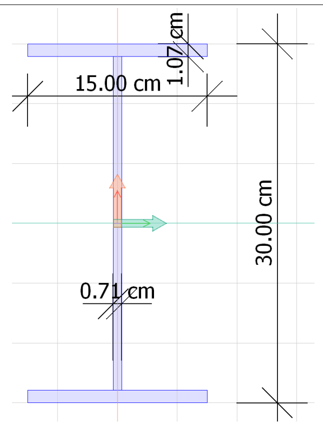


Figure 3.3.2 – The dimensions of the double symmetric cross section

In this case the critical moment can be calculated with the following formula based on the analytical solution [6]:

$$M_{cr}=\frac{\pi^2 E I_z}{L^2}\sqrt{\frac{I_w}{I_z}+\frac{L^2 G I_t}{\pi^2 E I_z}}$$

$$M_{cr}=\frac{\pi^2\cdot 21000\cdot 602.7}{1000^2}\sqrt{\frac{125841}{602.7}+\frac{1000^2\cdot 8077\cdot 15.34}{\pi^2\cdot 21000\cdot 602.7}}=4328\,\text{kNcm}$$

In FEM-Design a shell modell was built to analyze this problem. The bending moments in the shell model were considered with line loads at the end of the flanges (see Fig. 3.3.3).

The supports provides the simple supported beam effects with a fork support for the shell model (see Fig. 3.3.3).

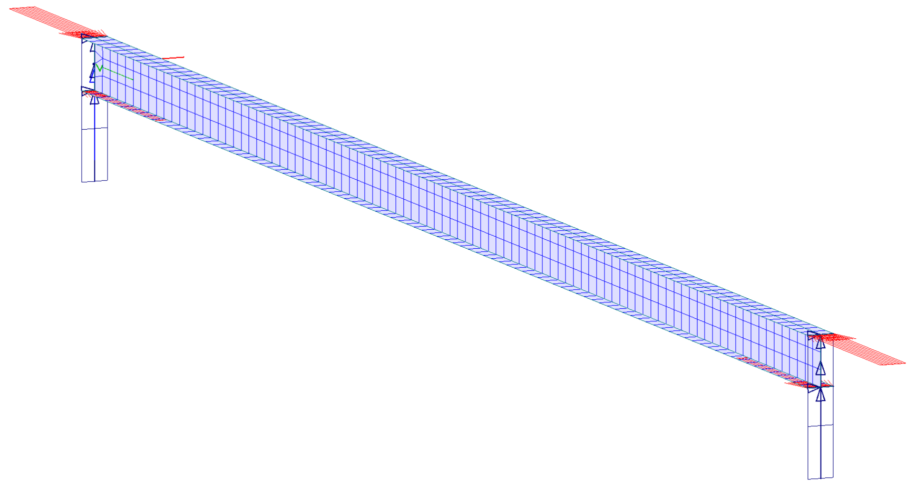


Figure 3.3.3 – The FEM model with the supports and the loads (moments) at the ends

From the FEM-Design stability calculation the critical moment value for this lateral torsional buckling problem is:

$$M_{cr\mathrm{FEM}} = 4363\,\mathrm{kNcm}$$

The critical shape is in Fig 3.3.4. The finite element mesh size was provided based on the automatic mesh generator of FEM-Design.

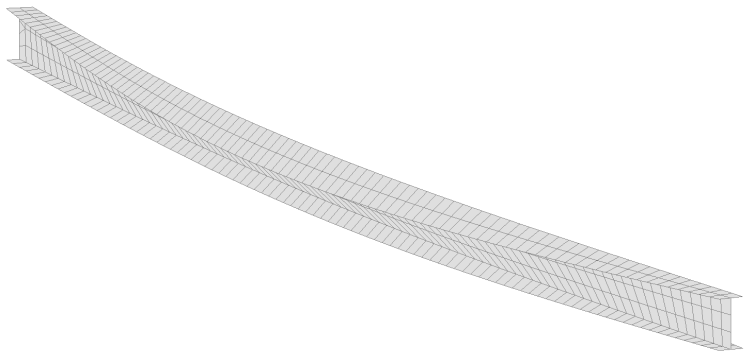


Figure 3.3.4 – The critical mode shape of the problem

The difference between the two calculated critical moments is less than 1%.

3.4 Lateral torsional buckling of a cantilever with elongated rectangle section

The purpose of this example is calculate the critical force at the end of a cantilever beam (see Fig. 3.4.1). If the load is increasing the state of the cantilever will be unstable due to lateral torsional buckling.

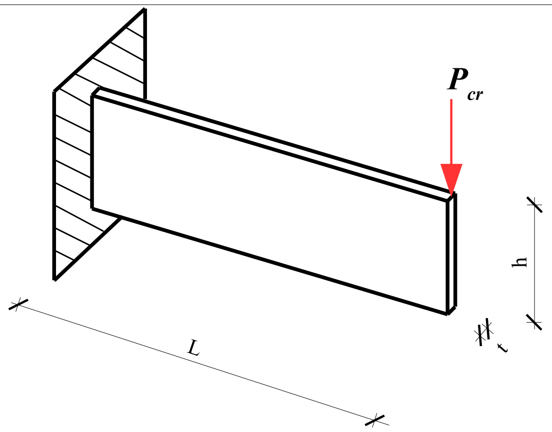


Figure 3.4.1 – The cantilever beam with concentrated load

The input parameters:

The length of the beam is	L = 10 m
The cross section	t = 40 mm; h = 438 mm; see Fig. 3.4.1
The St. Venant torsional inertia	I_t = 8806246 mm^4
The minor axis second moment of area	I_z = 2336000 mm^4
The elastic modulus	E = 210 GPa
The shear modulus	G = 80.77 GPa

In this case (elongated rectangle cross section with cantilever boundary condition) the critical concentrated force at the end can be calculated with the following formula based on analytical solution:

$$P_{cr}=\frac{4.01\,E\,I_z}{L^2}\sqrt{\frac{G\,I_t}{E\,I_z}}$$

$$P_{cr}=\frac{4.01\cdot 210000\cdot 2336000}{10000^2}\sqrt{\frac{80770\cdot 8806246}{210000\cdot 2336000}}=23687\,\mathrm{N}=23.69\,\mathrm{kN}$$

In FEM-Design a shell modell was built to analyze this problem. The concentrated load at the end of the cantilever was considered at the top of the beam (see Fig. 3.4.2).

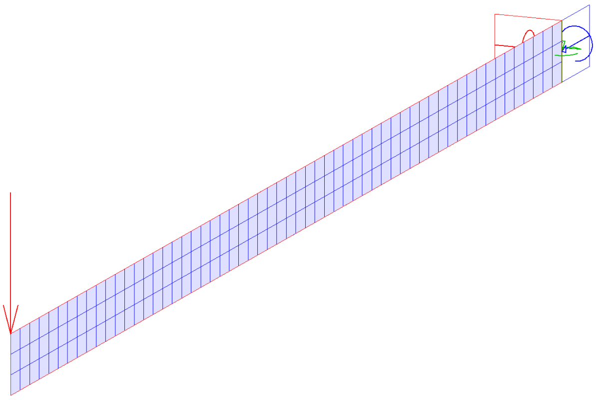


Figure 3.4.2 – The FE model of the cantilever beam with the default mesh

With the FEM-Design stability calculation the critical concentrated force value for this lateral torsional buckling problem is:

$P_{cr\mathrm{FEM}} = 24.00\,\mathrm{kN}$

The critical shape is in Fig 3.4.3. The finite element mesh size was provided based on the automatic mesh generator of FEM-Design.

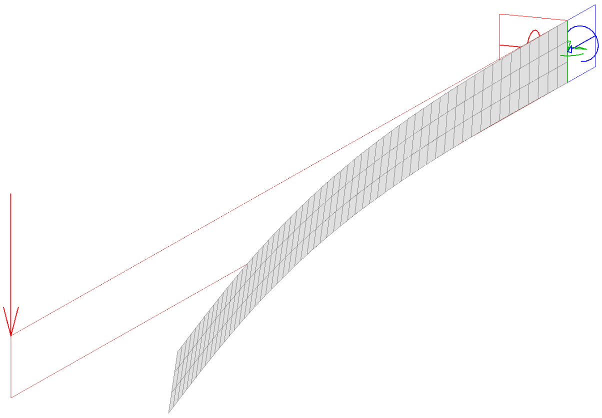


Figure 3.4.3 – The critical mode shape of the problem

The difference between the two calculated critical load parameters is less than 2%.

4. Calculation of eigenfrequencies with linear dynamic theory

4.1 Continuous mass distribution on a cantilever column

Column height	H = 4 m
The cross section	square with 0.4 m edge
The second moment of inertia	$I = 0.002133\ m^4$
The area of the cross section	$A = 0.16\ m^2$
The shear correction factor	ρ = 5/6 = 0.8333
The elastic modulus	E = 30 GPa
The shear modulus	G = 12.5 GPa
The specific self-weight of the column	$\gamma = 25\ kN/m^3$
The mass of the column	m = 1.631 t

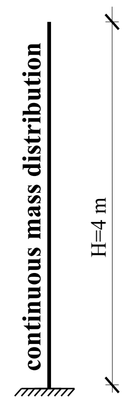


Figure 4.1.1 – The cantilever with continuous mass distribution

Based on the analytical solution [4] the angular frequencies for this case is:

$$\omega_B = \mu_{Bi}\sqrt{\frac{EI}{m H^3}} \quad ; \quad \mu_{B1} = 3.52;\ \mu_{B2} = 22.03;\ \mu_{B3} = 61.7$$

if only the bending deformations are considered.

The angular frequencies are [4]:

$$\omega_S = \mu_{Si}\sqrt{\frac{\rho GA}{mH}} \quad ; \quad \mu_{S1} = 0.5\pi\,;\, \mu_{S2} = 1.5\pi\,;\, \mu_{S3} = 2.5\pi$$

if only the shear deformations are considered.

Based on these two equations (considering bending and shear deformation) using the Föppl theorem the angular frequency for a continuous mass distribution column is:

$$\frac{1}{\omega_n^2} = \frac{1}{\omega_B^2} + \frac{1}{\omega_S^2}$$

Based on the given equations the first three angular frequencies separately for bending and shear deformations are:

$$\omega_{B1} = 3.52\sqrt{\frac{30000000\cdot 0.002133}{16\cdot 4^3}} = 87.16\frac{1}{\mathrm{s}}$$

$$\omega_{B2} = 22.03\sqrt{\frac{30000000\cdot 0.002133}{1.631\cdot 4^3}} = 545.4\frac{1}{\mathrm{s}}$$

$$\omega_{B3} = 61.7\sqrt{\frac{30000000\cdot 0.002133}{1.631\cdot 4^3}} = 1527.7\frac{1}{\mathrm{s}}$$

$$\omega_{S1} = 0.5\pi\sqrt{\frac{0.8333\cdot 12500000\cdot 0.16}{1.631\cdot 4}} = 793.9\frac{1}{\mathrm{s}}$$

$$\omega_{S2} = 1.5\pi\sqrt{\frac{0.8333\cdot 12500000\cdot 0.16}{1.631\cdot 4}} = 2381.8\frac{1}{\mathrm{s}}$$

$$\omega_{S3} = 2.5\pi\sqrt{\frac{0.8333\cdot 12500000\cdot 0.16}{1.631\cdot 4}} = 3969.6\frac{1}{\mathrm{s}}$$

According to the Föppl theorem the resultant first three angular frequencies of the problem are:

$$\omega_{n1} = 86.639\frac{1}{\mathrm{s}} \quad , \quad \omega_{n2} = 531.64\frac{1}{\mathrm{s}} \quad , \quad \omega_{n3} = 1425.8\frac{1}{\mathrm{s}}$$

And based on these results the first three eigenfrequencies are ($f = \omega/(2\pi)$):

$$f_{n1} = 13.789\frac{1}{\mathrm{s}} \quad , \quad f_{n2} = 84.613\frac{1}{\mathrm{s}} \quad , \quad f_{n3} = 226.923\frac{1}{\mathrm{s}}$$

In FEM-Design to consider the continuous mass distribution 200 beam elements were used for the cantilever column. The first three planar mode shapes are as follows according to the FE calculation:

$f_{FEM1}=13.780\frac{1}{s}$, $f_{FEM2}=83.636\frac{1}{s}$, $f_{FEM3}=223.326\frac{1}{s}$

The first three mode shapes can be seen in Fig. 4.1.2.

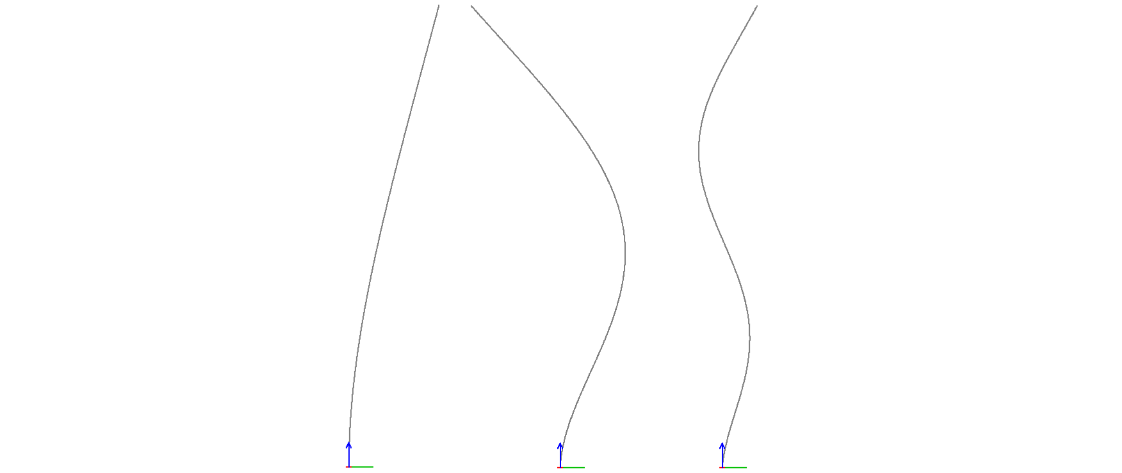


Figure 4.1.2 – The first three mode shapes for the cantilever with continuous mass distribution

The differences between the analytical and FE solutions are less than 2 %.

4.2 Free vibration shapes of a clamped circular plate due to its self-weight

In the next example we will analyze a circular clamped plate. The eigenfrequencies are the question due to the self-weight of the slab.

In this case the material and the geometric properties are the following:

The thickness of the plate	h = 0.05 m
The radius of the circular plate	R = 5 m
The elastic modulus	E = 210 GPa
Poisson's ratio	ν = 0.3
The density	ρ = 7.85 t/m³

The solution of the dynamic differential equation for the first two angular frequencies of a clamped circular plate are [5]:

$$\omega_{nm}=\frac{\pi^2}{R^2}\beta_{nm}{}^2\sqrt{\frac{\left(\frac{E\,h^3}{12(1-\nu^2)}\right)}{\rho\,h}} \quad , \quad \beta_{10}=1.015 \quad , \quad \beta_{11}=1.468$$

Figure 4.2.1 shows the problem in FEM-Design with the clamped edges and with the default mesh.

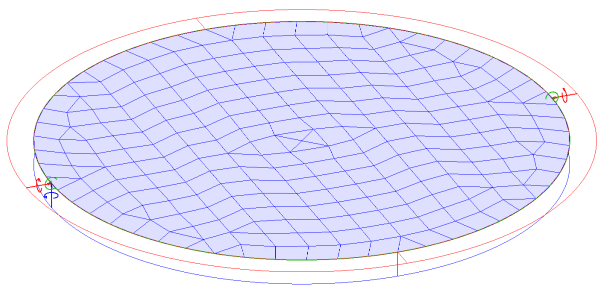


Figure 4.2.1 – The clamped circular plate and the default finite element mesh

According to the analytical solution the first two angular frequencies are:

$$\omega_{10}=\frac{\pi^2}{5^2}1.015^2\sqrt{\frac{\left(\frac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{7.85\cdot 0.05}}=31.83\frac{1}{s} \quad , \quad f_{10}=5.066\frac{1}{s} \quad , \quad f_{10FEM}=5.129\frac{1}{s}$$

$$\omega_{11}=\frac{\pi^2}{5^2}1.468^2\sqrt{\frac{\left(\frac{210000000\cdot 0.05^3}{12(1-0.3^2)}\right)}{7.85\cdot 0.05}}=66.58\frac{1}{s} \quad , \quad f_{11}=10.60\frac{1}{s} \quad , \quad f_{11FEM}=10.731\frac{1}{s}$$

Based on the angular frequencies we can calculate the eigenfrequencies in a very easy way. Next to these values we indicated the eigenfrequencies what were calculated with the FEM-Design.

The difference between the calculations less than 2 %.

Figure 4.2.2 shows the first two vibration mode shapes of the circular clamped plate.

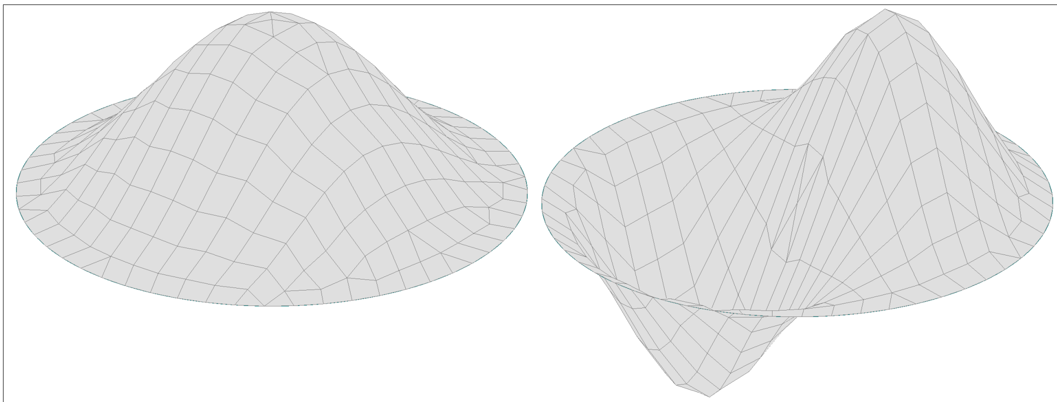


Figure 4.2.2 – The first two vibration shape mode of a clamped circular plate

5. Seismic calculation

5.1 Lateral force method with linear shape distribution on a cantilever

Inputs:

Column height	H = 10 m
The cross section	square with 0.4 m edge
The second moment of inertia	I = 0.002133 m^4
The elastic modulus	E = 31 GPa
The concentrated mass points	10 pieces of 1.0 t (see Fig. 5.1.1)
The total mass	m = 10.0 t

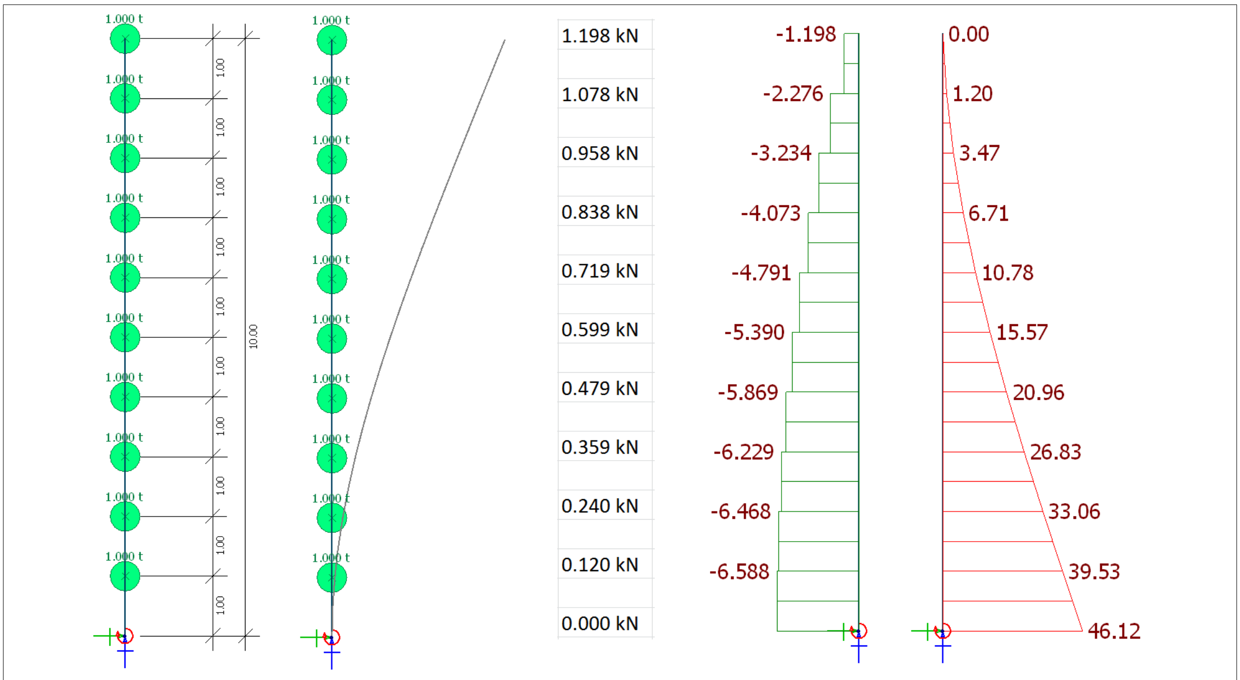


Figure 5.1.1 – The cantilever column with the concentrated mass points, the first vibration shape [T=0.765 s], *the equivalent forces* [kN], *the shear force diagram* [kN] *and the bending moment diagram* [kNm] *with FEM-Design*

First of all based on a hand calculation we determine the first fundamental period:

The first fundamental period of a cantilever column (length *H*) with a concentrated mass at the end (*m* mass) and *EI* bending stiffness [4]:

$$T_i = \frac{2\pi}{\sqrt{\frac{3\,EI}{m_i H_i^3}}}$$

The fundamental period separately for the mass points from bottom to top:

$$T_1=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 1^3}}}=0.01411\,\text{s} \quad ; \quad T_2=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 2^3}}}=0.03990\,\text{s} \quad ;$$

$$T_3=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 3^3}}}=0.07330\,\text{s} \quad ; \quad T_4=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 4^3}}}=0.1129\,\text{s} \quad ;$$

$$T_5=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 5^3}}}=0.1577\,\text{s} \quad ; \quad T_6=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 6^3}}}=0.2073\,\text{s} \quad ;$$

$$T_7=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 7^3}}}=0.2613\,\text{s} \quad ; \quad T_8=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 8^3}}}=0.3192\,\text{s} \quad ;$$

$$T_9=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 9^3}}}=0.3809\,\text{s} \quad ; \quad T_{10}=\frac{2\pi}{\sqrt{\frac{3\cdot 31000000\cdot 0.0021333}{1\cdot 10^3}}}=0.4461\,\text{s} \quad .$$

The approximated period based on these values according to the Dunkerley summary and the result of FE calculation:

$$T_{HC}=\sqrt{\sum_{i=1}^{10} T_i^2}=0.7758\,\text{s} \qquad T_{FEM}=0.765\,\text{s}$$

The difference between the hand calculation and FEM-Design calculation is less than 2%, for further information on the period calculation see Chapter 4.

The base shear force according to the fundamental period of vibration (see Fig. 5.1.1) and the response spectrum (see Fig. 5.1.2):

$$F_b=S_d(T_1)\,m\,\lambda=0.6588\cdot 10\cdot 1.0=6.588\,\text{kN}$$

We considered the response acceleration based on the period from FE calculation to get a more comparable results at the end. Thus the equivalent forces on the different point masses are:

$$F_i=F_b\frac{z_i m_i}{\sum z_j m_j}=6.588\frac{z_i m_i}{1\cdot 1+2\cdot 1+3\cdot 1+4\cdot 1+5\cdot 1+6\cdot 1+7\cdot 1+8\cdot 1+9\cdot 1+10\cdot 1}=6.588\frac{z_i m_i}{55}$$

The equivalent forces from the bottom to the top on each point mass:

$F_1 = 0.120\,kN$; $F_2 = 0.240\,kN$; $F_3 = 0.359\,kN$; $F_4 = 0.479\,kN$; $F_5 = 0.599\,kN$;
$F_6 = 0.719\,kN$; $F_7 = 0.838\,kN$; $F_8 = 0.958\,kN$; $F_9 = 1.078\,kN$; $F_{10} = 1.198\,kN$

These forces are identical with the FEM-Design calculation and the shear force and bending moment diagrams are also identical with the hand calculation.

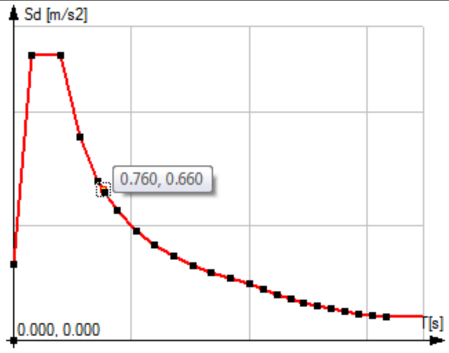


Figure 5.1.2 – The response spectrum [T = 0.765 s; S_d = 0.6588 m/s^2]

5.2 Lateral force method with fundamental mode shape distribution on a cantilever

Inputs:

Column height	H = 10 m
The cross section	square with 0.4 m edge
The second moment of inertia	I = 0.002133 m^4
The elastic modulus	E = 31 GPa
The concentrated mass points	10 pieces of 1.0 t (see Fig. 5.1.1)
The total mass	m = 10.0 t

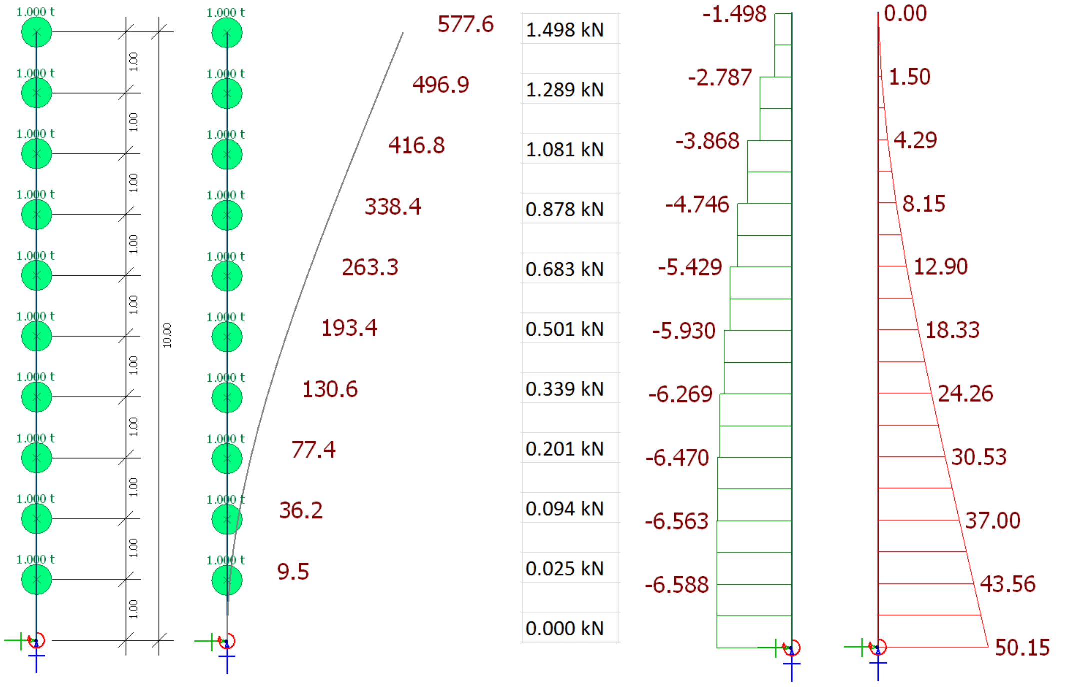


Figure 5.2.1 – The cantilever column with the concentrated mass points, the first vibration shape with the value of the eigenvector [T=0.765 s], *the equivalent forces* [kN], *the shear force diagram* [kN] *and the bending moment diagram* [kNm] *with FEM-Design*

The base shear force according to the fundamental period of vibration (see Fig. 5.2.1) and the response spectrum (see Fig. 5.2.2):

$$F_b = S_d(T_1)\, m\, \lambda = 0.6588 \cdot 10 \cdot 1.0 = 6.588\,\text{kN}$$

We considered the response acceleration based on the period from FE calculation to get a more comparable results at the end. Thus the equivalent forces on the different point masses are:

$$F_i = F_b \frac{s_i m_i}{\sum s_j m_j} =$$

$$= 6.588 \frac{s_i m_i}{9.5 \cdot 1 + 36.2 \cdot 1 + 77.4 \cdot 1 + 130.6 \cdot 1 + 193.4 \cdot 1 + 263.3 \cdot 1 + 338.4 \cdot 1 + 416.8 \cdot 1 + 496.9 \cdot 1 + 577.6 \cdot 1} = 6.588 \frac{s_i m_i}{2540.1}$$

The equivalent forces from the bottom to the top on each point mass:

$F_1 = 0.0246\,\text{kN}$; $F_2 = 0.0939\,\text{kN}$; $F_3 = 0.201\,\text{kN}$; $F_4 = 0.339\,\text{kN}$; $F_5 = 0.502\,\text{kN}$;
$F_6 = 0.683\,\text{kN}$; $F_7 = 0.878\,\text{kN}$; $F_8 = 1.081\,\text{kN}$; $F_9 = 1.289\,\text{kN}$; $F_{10} = 1.498\,\text{kN}$

These forces are identical with the FEM-Design calculation and the shear force and bending moment diagrams are also identical with the hand calculation.

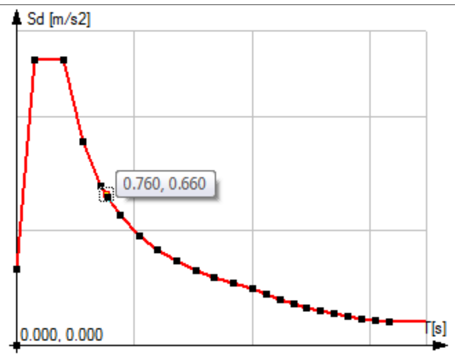


Figure 5.2.2 – The response spectrum [T = 0.765 s; S_d = 0.6588 m/s^2]

5.3 Modal analysis of a concrete frame building

In this chapter we show a worked example for modal analysis on a concrete frame building according to EN 1998-1:2008 with hand calculation and compare the results with FEM-Design. This example is partly based on [4]. The geometry, the dimensions, the material and the bracing system are in Fig. 5.3.1-3 and in the following table.

Inputs:

Column height/Total height	h = 3.2 m; H = 2·3.2=6.4 m
The cross sections	Columns: 30/30 cm; Beams: 30/50 cm
The second moment of inertia	$I_c = 0.000675\ m^4$; $I_b = 0.003125\ m^4$
The elastic modulus	E = 28.80 GPa
The concentrated mass points	12 pieces of 13.358 t on 1st storey and 12 pieces of 11.268 t on 2nd storey (see Fig. 5.3.2)
The total mass	1st storey: $m_1 = 160.3$ t 2nd storey: $m_2 = 135.2$ t total mass: M = 295.5 t
Reduction factor for elastic modulus considering the cracking according to EN 1998-1:2008	$\alpha = 0.5$
Behaviour factors	$q = 1.5$, $q_d = 1.5$
Accidental torsional effect do not considered	$\xi = 0.05$ (damping factor)

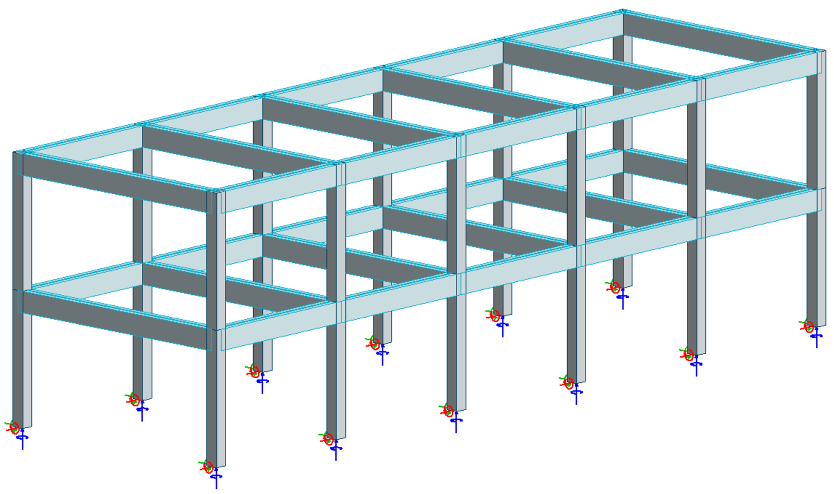


Figure 5.3.1 – The concrete frame building with the columns and beams

The first exercise is the determination of the fundamental periods and mode shapes. There are several hand calculation modes to get these values but in this chapter the details of the modal analysis are important therefore we considered the first two fundamental periods based on FEM-Design calculation (see Fig. 5.3.5). See the details and example on the eigenfrequency calculation in Chapter 4.

The dead loads and the live loads are considered in the mass points (see Fig. 5.3.2).

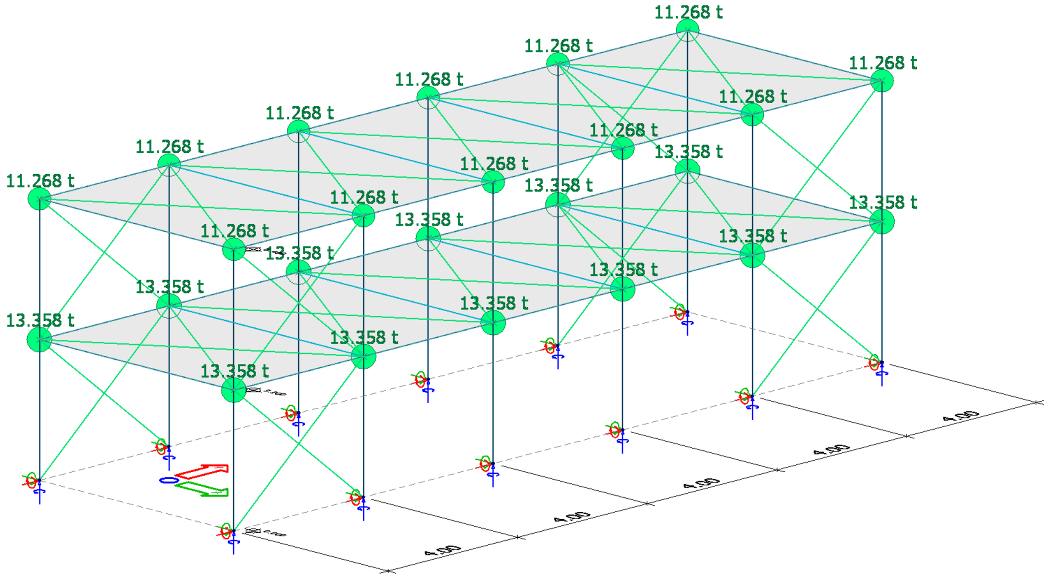


Figure 5.3.2 – The frame building with the masses and bracings

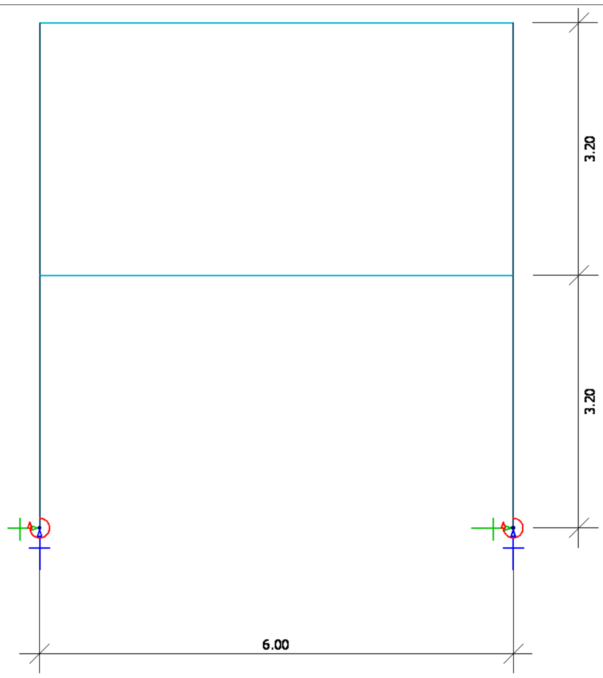


Figure 5.3.3 – The side view of the building

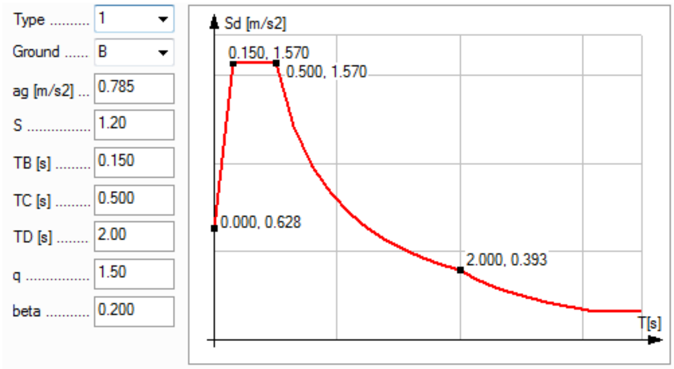


Figure 5.3.4 – The considered design response specra according to EN 1998-1:2008

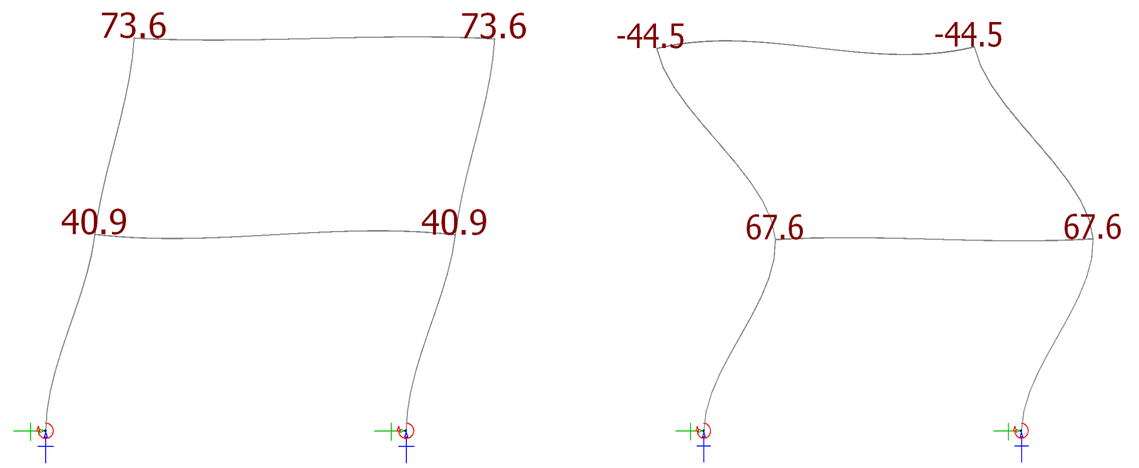


Figure 5.3.5 – The first two fundamental mode shapes [-], $T_1 = 0.704$ s; $T_2 = 0.252$ s

According to the fundamental periods in Fig. 5.3.5 the response accelerations from Fig. 5.3.4 are:

$$T_1 = 0.704\,\text{s} \qquad S_{d1} = 1.115\,\frac{\text{m}}{\text{s}^2} \quad ;$$

$$T_2 = 0.252\,\text{s} \qquad S_{d2} = 1.57\,\frac{\text{m}}{\text{s}^2} \quad .$$

The second step is to calculate the effective modal masses based on this formula:

$$m_i^* = \frac{\left(\mathbf{\Phi}_\mathbf{i}^\mathrm{T}\mathbf{m}\,\boldsymbol{\iota}\right)^2}{\mathbf{\Phi}_\mathbf{i}^\mathrm{T}\mathbf{m}\,\mathbf{\Phi}_\mathbf{i}}$$

During the hand calculation we assume that the structure is a two degrees of freedom system in the x direction with the two storeys, because the first two modal shapes are in the same plane see Fig. 5.3.5. Thus we only consider the seismic loads in one direction because in this way the hand calculation is more comprehensible.

$$m_1^* = \frac{\left(\begin{bmatrix} 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2}{\begin{bmatrix} 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 40.9 \\ 73.6 \end{bmatrix}} = 272.3\,\mathrm{t} \quad ; \quad \frac{m_1^*}{\mathrm{M}} = \frac{272.3}{295.5} = 92.1\,\%$$

$$m_2^* = \frac{\left(\begin{bmatrix} 67.6 & -44.5 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2}{\begin{bmatrix} 67.6 & -44.5 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 67.6 \\ -44.5 \end{bmatrix}} = 23.23\,\mathrm{t} \quad ; \quad \frac{m_2^*}{\mathrm{M}} = \frac{23.23}{295.5} = 7.9\,\%$$

According to the assumption of a two degrees of freedom system the sum of the effective modal masses is equal to the total mass:

$$\frac{m_1^*}{\mathrm{M}} + \frac{m_2^*}{\mathrm{M}} = \frac{272.3}{295.5} + \frac{23.23}{295.5} = 100.0\,\%$$

Calculation of the base shear forces:

$$F_{b1} = S_{d1}\, m_1^* = 1.115 \cdot 272.3 = 303.6\,\mathrm{kN} \quad ; \quad F_{b2} = S_{d2}\, m_1^* = 1.570 \cdot 23.23 = 36.5\,\mathrm{kN}$$

The equivalent forces come from this formula:

$$\mathbf{p}_i = \mathbf{m}\,\boldsymbol{\Phi}_i \frac{\boldsymbol{\Phi}_i^{\mathrm{T}}\,\mathbf{m}\,\boldsymbol{\iota}}{\boldsymbol{\Phi}_i^{\mathrm{T}}\,\mathbf{m}\,\boldsymbol{\Phi}_i} S_{di}$$

The equivalent forces at the storeys respect to the mode shapes considering the mentioned two degrees of freedom model:

$$\mathbf{p}_1 = \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 40.9 \\ 73.6 \end{bmatrix} \frac{\begin{bmatrix} 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 40.9 & 73.6 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 40.9 \\ 73.6 \end{bmatrix}} 1.115 = \begin{bmatrix} 120.6 \\ 183.0 \end{bmatrix} \mathrm{kN}$$

$$\mathbf{p_2} = \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 67.6 \\ -44.5 \end{bmatrix} \frac{\begin{bmatrix} 67.6 & -44.5 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 67.6 & -44.5 \end{bmatrix} \begin{bmatrix} 160.3 & 0 \\ 0 & 135.2 \end{bmatrix} \begin{bmatrix} 67.6 \\ -44.5 \end{bmatrix}} 1.570 = \begin{bmatrix} 81.98 \\ -44.52 \end{bmatrix} \text{kN}$$

The equivalent forces on one frame from the six (see Fig. 5.3.1):

$$\mathbf{p_{f1}} = \begin{bmatrix} 120.6/6 \\ 183.0/6 \end{bmatrix} = \begin{bmatrix} 20.10 \\ 30.50 \end{bmatrix} \text{kN}$$

$$\mathbf{p_{f2}} = \begin{bmatrix} 81.98/6 \\ -44.52/6 \end{bmatrix} = \begin{bmatrix} 13.66 \\ -7.420 \end{bmatrix} \text{kN}$$

The shear forces between the storeys respect to the two different mode shapes:

$$\mathbf{V}_1 = \begin{bmatrix} 20.1 + 30.5 \\ 30.5 \end{bmatrix} = \begin{bmatrix} 50.6 \\ 30.5 \end{bmatrix} \text{kN} \qquad \mathbf{V}_2 = \begin{bmatrix} 13.66 - 7.42 \\ -7.42 \end{bmatrix} = \begin{bmatrix} 6.24 \\ -7.42 \end{bmatrix} \text{kN}$$

The shear forces in the columns respect to the two different mode shapes:

$$\mathbf{V}_{c1} = \begin{bmatrix} 50.6/2 \\ 30.5/2 \end{bmatrix} = \begin{bmatrix} 25.30 \\ 15.25 \end{bmatrix} \text{kN} \qquad \mathbf{V}_{c2} = \begin{bmatrix} 6.24/2 \\ -7.42/2 \end{bmatrix} = \begin{bmatrix} 3.13 \\ -3.71 \end{bmatrix} \text{kN}$$

The bending moments in the columns respect to the two different mode shapes from the relevant shear forces (by the hand calculation we assumed zero bending moment points in the middle of the columns):

$$\mathbf{M}_{c1} = \begin{bmatrix} 25.30 \cdot 3.2/2 \\ 15.25 \cdot 3.2/2 \end{bmatrix} = \begin{bmatrix} 40.48 \\ 24.40 \end{bmatrix} \text{kNm} \qquad \mathbf{M}_{c2} = \begin{bmatrix} 3.13 \cdot 3.2/2 \\ -3.71 \cdot 3.2/2 \end{bmatrix} = \begin{bmatrix} 5.008 \\ -5.936 \end{bmatrix} \text{kNm}$$

The bending moments in the beams respect to the two different mode shapes:

$$\mathbf{M}_{b1} = \begin{bmatrix} 40.48 + 24.40 \\ 24.40 \end{bmatrix} = \begin{bmatrix} 64.88 \\ 24.40 \end{bmatrix} \text{kNm} \qquad \mathbf{M}_{b2} = \begin{bmatrix} 5.008 - 5.936 \\ -5.936 \end{bmatrix} = \begin{bmatrix} -0.928 \\ -5.936 \end{bmatrix} \text{kNm}$$

The SRSS summation on the internal forces:

$$\mathbf{V_c}=\begin{bmatrix}\sqrt{25.30^2+3.13^2}\\\sqrt{15.25^2+(-3.71)^2}\end{bmatrix}=\begin{bmatrix}25.49\\15.69\end{bmatrix}kN \qquad \mathbf{M_c}=\begin{bmatrix}\sqrt{40.48^2+5.008^2}\\\sqrt{24.40^2+5.936^2}\end{bmatrix}=\begin{bmatrix}40.79\\25.11\end{bmatrix}kNm$$

$$\mathbf{M_b}=\begin{bmatrix}\sqrt{64.88^2+(-0.928)^2}\\\sqrt{24.40^2+(-5.936)^2}\end{bmatrix}=\begin{bmatrix}64.89\\25.11\end{bmatrix}kNm$$

The CQC summation on the internal forces:

$$\alpha_{12}=\frac{T_2}{T_1}=\frac{0.252}{0.704}=0.358$$

$$r_{12}=\frac{8\xi^2(1+\alpha_{12})\alpha_{12}^{3/2}}{\left(1-\alpha_{12}^2\right)^2+4\xi^2\alpha_{12}(1+\alpha_{12})^2}=\frac{8\cdot0.05^2(1+0.358)0.358^{3/2}}{\left(1-0.358^2\right)^2+4\cdot0.05^2\cdot0.358(1+0.358)^2}=0.007588$$

$$\mathbf{r}=\begin{bmatrix}1 & 0.007588\\0.007588 & 1\end{bmatrix}$$

And based on these values the results of the CQC summation:

$$\mathbf{V_c}=\begin{bmatrix}\sqrt{25.30^2+3.13^2+2\cdot25.3\cdot3.13\cdot0.007588}\\\sqrt{15.25^2+(-3.71)^2+2\cdot15.25\cdot(-3.71)\cdot0.007588}\end{bmatrix}=\begin{bmatrix}25.52\\15.67\end{bmatrix}kN$$

$$\mathbf{M_c}=\begin{bmatrix}\sqrt{40.48^2+5.008^2+2\cdot40.48\cdot5.008\cdot0.007588}\\\sqrt{24.40^2+5.936^2+2\cdot24.40\cdot5.936\cdot0.007588}\end{bmatrix}=\begin{bmatrix}40.83\\25.16\end{bmatrix}kNm$$

$$\mathbf{M_b}=\begin{bmatrix}\sqrt{64.88^2+(-0.928)^2+2\cdot64.88\cdot(-0.928)\cdot0.007588}\\\sqrt{24.40^2+(-5.936)^2+2\cdot24.40\cdot(-5.936\cdot0.007588)}\end{bmatrix}=\begin{bmatrix}64.88\\25.07\end{bmatrix}kNm$$

The following displacements come from the FEM-Design calculation on the complete frame structure to ensure the comprehensible final results on the P-Δ effect.

The displacements at the storeys respect to the two different mode shapes considering the displacement behaviour factor:

$$\mathbf{u_1}=q_d\begin{bmatrix}9.54\\17.15\end{bmatrix}=1.5\begin{bmatrix}9.54\\17.15\end{bmatrix}=\begin{bmatrix}14.31\\25.73\end{bmatrix}mm \qquad \mathbf{u_2}=q_d\begin{bmatrix}0.818\\-0.540\end{bmatrix}=1.5\begin{bmatrix}0.818\\-0.540\end{bmatrix}=\begin{bmatrix}1.227\\-0.810\end{bmatrix}mm$$

Based on these values the storey drifting respect to the two different mode shapes:

$$\mathbf{\Delta_1}=\begin{bmatrix}14.31\\25.73-14.31\end{bmatrix}=\begin{bmatrix}14.31\\11.42\end{bmatrix}mm \qquad \mathbf{\Delta_2}=\begin{bmatrix}1.227\\-0.810-1.227\end{bmatrix}=\begin{bmatrix}1.227\\-2.037\end{bmatrix}mm$$

SRSS summation on the story drifting:

$$\Delta=\begin{bmatrix}\sqrt{14.31^2+1.227^2}\\ \sqrt{11.42^2+(-2.027)^2}\end{bmatrix}=\begin{bmatrix}14.36\\ 11.60\end{bmatrix}\text{mm}$$

P-Δ effect checking on the total building:

$$\mathbf{P}_{\mathbf{tot}}=\begin{bmatrix}(m_1+m_2)g\\ m_2 g\end{bmatrix}=\begin{bmatrix}(160.3+135.2)9.81\\ 135.2\cdot 9.81\end{bmatrix}=\begin{bmatrix}2899\\ 1326\end{bmatrix}\text{kN}$$

$$\mathbf{V}_{\mathbf{tot}}=\begin{bmatrix}6\cdot\sqrt{50.6^2+6.24^2}\\ 6\cdot\sqrt{30.5^2+(-7.42)^2}\end{bmatrix}=\begin{bmatrix}305.9\\ 188.3\end{bmatrix}\text{kN}$$

$$\theta_1=\frac{P_{tot1}\Delta_1}{V_{tot1}h}=\frac{2899\cdot 14.36}{305.9\cdot 3200}=0.0425$$

$$\theta_2=\frac{P_{tot2}\Delta_2}{V_{tot2}h}=\frac{1326\cdot 11.60}{188.3\cdot 3200}=0.0255$$

After the hand calculation let's see the results from the FEM-Design calculation and compare them to each other. Fig. 5.3.6 shows the effective modal masses from the FE calculation. Practically these values coincide with the hand calculation.

Shape no.	T	mx'	mx'
[-]	[s]	[%]	[t]
1	0.704	92.2	272.374
2	0.252	7.8	23.139

Figure 5.3.6 – The first two fundamental periods and the effective modal masses from FEM-Design

Fig. 5.3.7 and the following table shows the equivalent resultant shear forces and the base shear forces respect to the first two mode shapes. The differences between the two calculations are less than 2 %.

	Storey 1 equivalent resultant [kN]		Storey 2 equivalent resultant [kN]		Base shear force [kN]	
	Hand	FEM	Hand	FEM	Hand	FEM
Mode shape 1	120.6	121.9	81.98	81.80	303.6	306.9
Mode shape 2	183.0	185.0	– 44.52	– 45.47	36.50	36.33

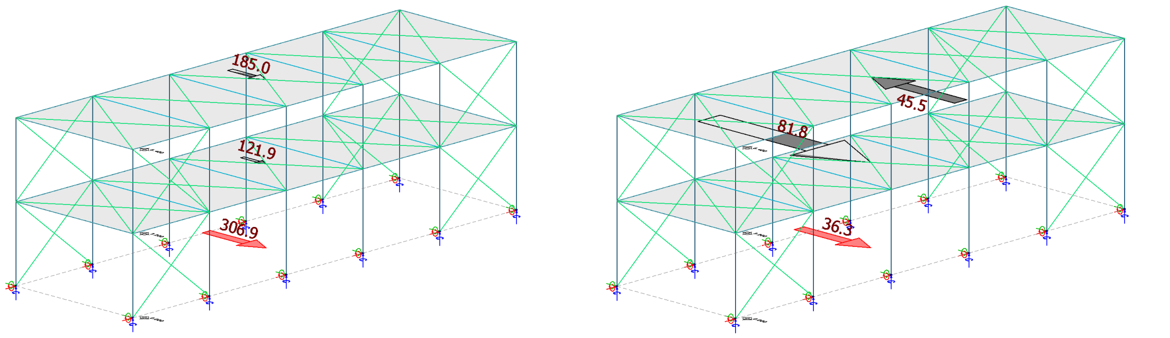


Figure 5.3.7 – The equivalent forces respect to the storeys and the base shear forces for the first two mode shapes [kN]

Fig. 5.3.8-9 and the following table shows the internal forces after the differenr summation modes (SRSS and CQC). The differences between the two calculations are less than 2 %.

	Column shear force [kN]		Column bending moment [kNm]		Beam bending moment [kNm]	
	Storey 1	Storey 2	Storey 1	Storey 2	Storey 1	Storey 2
SRSS Hand	25.49	15.69	40.79	25.11	64.89	25.11
SRSS FEM	25.78	15.89	(37.18+45.32)/2= 41.25	27.51	59.33	27.51
CQC Hand	25.50	15.67	40.83	25.16	64.88	25.07
CQC FEM	25.80	15.86	(37.21+45.36)/2= 41.29	27.46	59.32	27.46

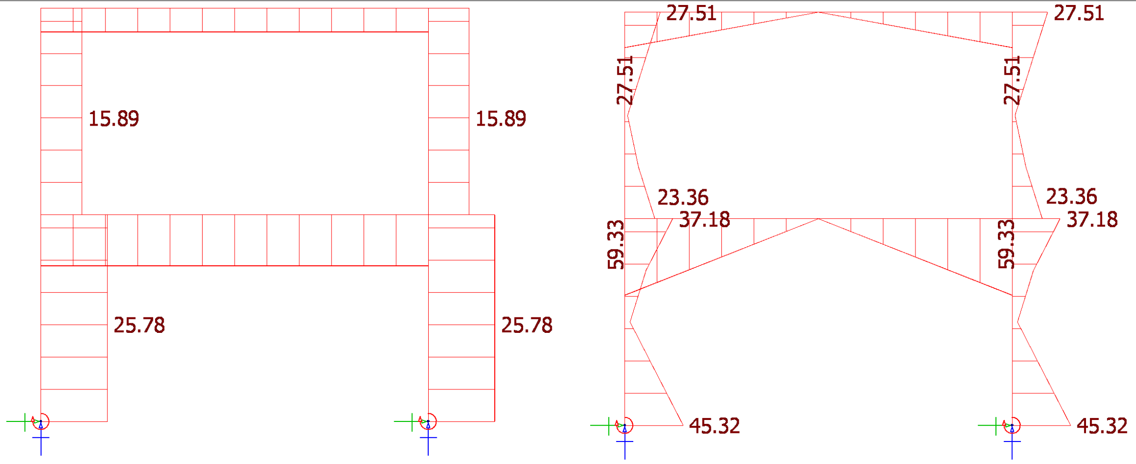


Figure 5.3.8 – The shear force [kN] *and bending moment diagram* [kNm] *after the SRSS summation rule*

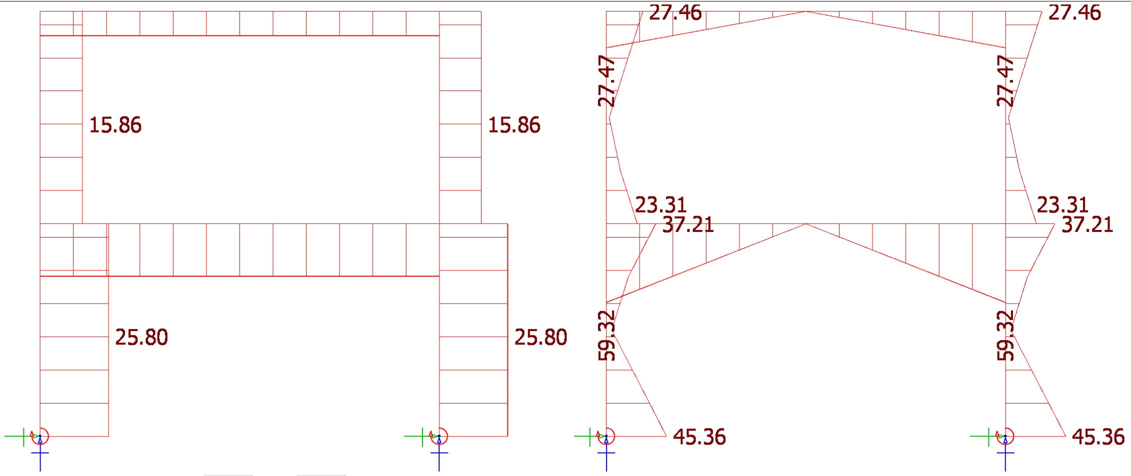


Figure 5.3.9 – The shear force [kN] *and bending moment diagram* [kNm] *after the CQC summation rule*

Fig. 5.3.10 shows the *Θ* values from FEM-Design. The differences between the hand calculation and FEM-Design are less than 3 %.

Storey	Theta x
1	0.0420
2	0.0253

Figure 5.3.10 – The θ values at the different storeys from FEM-Design

6. Calculation considering diaphragms

6.1. A simple calculation with diaphragms

If we apply two diaphragms on the two storeys of the building from Chapter 5.3 then the eigenfrequencies and the periods will be the same what we indicated in Chapter 5.3.

6.2. The calculation of the shear center

In this example we show that how can we calculate the shear center of a storey based on the FEM-Design calculation. We analyzed a bottom fixed cantilever structure made of three concrete shear walls which are connected to each other at the edges (see Fig. 6.2.1). The diaphragm is applied at the top plane of the structure (see also Fig. 6.2.1 right side). If the height of the structure is high enough then the shear center will be on the same geometry point where it should be when we consider the complete cross section of the shear walls as a "thin-walled" "C" cross section (see Fig. 6.2.1 left side). Therefore we calculate by hand the shear center of the "C" profile assumed to be a thin-walled cross section then compare the solution what we can get from FEM-Design calculation with diaphragms.

Secondly we calculate the idealized bending stiffnesses in the principal rigidity directions by hand and compare the results what we can calculate with FEM-Design results.

Inputs:

Height of the walls	H = 63 m
The thickness of the walls	t = 20 cm
The width of wall number 1 and 3	$w_1 = w_3 = 4.0$ m
The width of wall number 2	$w_2 = 6.0$ m
The applied Young's modulus of concrete	E = 9.396 GPa

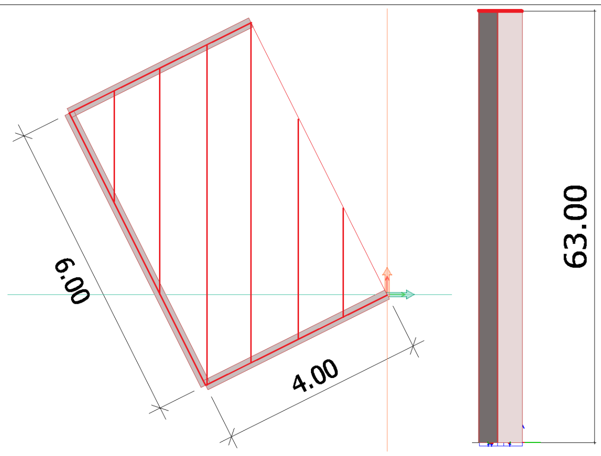


Figure 6.2.1 – The geometry of the bracing core and the height of the bottom fixed structure (the diaphragm is lying on the top plane, see the red line and hatch)

First of all lets see Fig. 6.2.2. The applied cross section is a symmetric cross section. In the web the shear stress distribution comes from the shear formula regarding bending (see Fig. 6.2.2). Therefore it is a second order polynom. In the flanges the shear stress distribution is linear according to the thin walled theory. With the resultant of these shear stress ditribution (see Fig. 6.2.2, V_1, V_2 and V_3) the position of the shear center can be calculated based on the statical (equilibrium) equations.

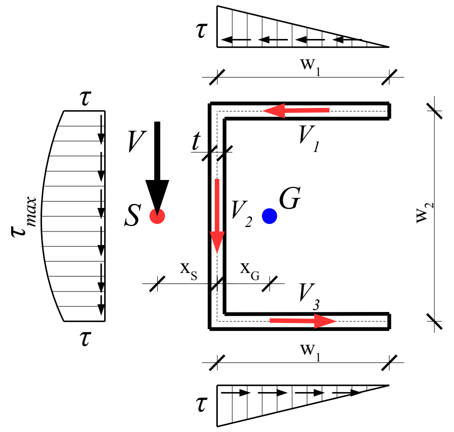


Figure 6.2.2 – The shear stress distribution in a thin-walled cross section if the shear force acting on the shear center

The shear stress values (see Fig. 6.2.2):

$$\tau = \frac{V\,S}{I\,t} = \frac{1\cdot(0.2\cdot 4\cdot 3)}{\left(\frac{0.2\cdot 6^3}{12}+\frac{4\cdot 6.2^3}{12}-\frac{4\cdot 5.8^3}{12}\right)\cdot 0.2} = 0.6665\,\frac{\text{kN}}{\text{m}^2}$$

$$\tau_{max} = \frac{V\,S_{max}}{I\,t} = \frac{1\cdot(0.2\cdot 4\cdot 3+0.2\cdot 3\cdot 1.5)}{\left(\frac{0.2\cdot 6^3}{12}+\frac{4\cdot 6.2^3}{12}-\frac{4\cdot 5.8^3}{12}\right)\cdot 0.2} = 0.9164\,\frac{\text{kN}}{\text{m}^2}$$

Based on these stresses the resultant in the flanges and in the web:

$$V_1 = V_3 = \frac{\tau\,t\,w_1}{2} = \frac{0.6665\cdot 0.2\cdot 4}{2} = 0.2667\,\text{kN}$$

$$V_2 = \frac{2}{3}(\tau_{max}-\tau)\,w_2 t + \tau\,w_2 t = \frac{2}{3}(0.9164-0.6665)\,6\cdot 0.2 + 0.6665\cdot 6\cdot 0.2 = 0.9997\,\text{kN}$$

Respect to the equilibrium (sum of the forces):

$$V = 1\,\text{kN} \approx V_2 = 0.9997\,\text{kN}$$

And also respect to the equilibrium (if the external load is acting on the shear center, see Fig. 6.2.2) the sum of the moments:

$$V_1 w_2 = V_2 x_S$$

$$x_S = \frac{V_1 w_2}{V_2} = \frac{0.2667 \cdot 6}{0.9997} = 1.601\,\text{m}$$

Thus the shear center is lying on the symmetry axis and it is x_S=1.601 m from the web (see Fig. 6.2.2). In FEM-Design the global coordinate system does not coincide with the symmetry axis of the structure (see Fig. 6.2.1). Therefore we need no transform the results.

Lets be a selected key node at the diaphragm in the global coordinate system (see Fig. 6.2.1):

$$x_m = 0\,\text{m} \quad ; \quad y_m = 0\,\text{m}$$

Based on the unit forces (1 kN) and moment (1 kNm) on the key node the displacements of the key node are as follows based on the FEM-Design calculation (see the theory manual Calculation considering diaphragm chapter also):

According to unit force on key node in X direction:

$$u_{xx} = 1.5852\,\text{mm} \qquad u_{yx} = 0.72166\,\text{mm} \qquad \varphi_{zx} = 0.29744 \cdot 10^{-4}\,\text{rad}$$

According to unit force on key node in Y direction:

$$u_{xy} = 0.72166\,\text{mm} \qquad u_{yy} = 7.3314\,\text{mm} \qquad \varphi_{zy} = 0.10328 \cdot 10^{-2}\,\text{rad}$$

According to unit moment on key node around Z direction:

$$\varphi_{zz} = 0.16283 \cdot 10^{-3}\,\text{rad}$$

Based on these finite element results the global coordinates of the shear center of the diaphragm are:

$$x_S = x_m - \frac{\varphi_{zy}}{\varphi_{zz}} = 0 - \frac{0.10328 \cdot 10^{-2}}{0.16283 \cdot 10^{-3}} = -6.343\,\text{m}$$

$$y_S = y_m + \frac{\varphi_{zx}}{\varphi_{zz}} = 0 + \frac{0.29744 \cdot 10^{-4}}{0.16283 \cdot 10^{-3}} = +0.1827\,\text{m}$$

In FEM-Design the coordinates of the middle point of the web are (see Fig. 6.2.1):

$$x_{mid} = -4.919\,\text{m} \quad ; \quad y_{mid} = +0.894\,\text{m}$$

With the distance between these two points we get a comparable solution with the hand calculation.

$$x_{SFEM}=\sqrt{(x_S-x_{mid})^2+(y_S-y_{mid})^2}=\sqrt{(-6.343-(-4.919))^2+(0.1827-0.894)^2}=1.592\,\mathrm{m}$$

The difference between FEM and hand calculation is less than 1%.

The gravity center of the cross section (Fig. 6.2.2) can be calculated based on the statical moments. And of course the gravity center lying on the symmetry axis. The distance of the gravity center from the web is:

$$x'_G=\frac{S_y'}{A}=\frac{2(0.2\cdot4\cdot2)}{0.2(4+6+4)}=1.143\,\mathrm{m}$$

With the input Young's modulus and with the second moments of inertia the idealized bending stiffnesses in the principal directions can be calculated by hand.

$$E\,I_1=9396\cdot10^3\cdot\left(\frac{0.2\cdot6^3}{12}+\frac{4\cdot6.2^3}{12}-\frac{4\cdot5.8^3}{12}\right)=1.692\cdot10^8\,\mathrm{kNm^2}$$

$$E\,I_2=9396\cdot10^3\cdot\left(2\frac{0.2\cdot4^3}{12}+2(0.2\cdot4(2-1.143)^2)+\frac{6\cdot0.2^3}{12}+0.2\cdot6(1.143)^2\right)=4.585\cdot10^7\,\mathrm{kNm^2}$$

With the finite element results we can calculate the translations of the shear center according to the unit forces and moment on the key node (see the former calculation method).

The distances between the shear center and the selected key node are:

$$\Delta x=x_S-x_m=-6.343-0=-6.343\,\mathrm{m}=-6343\,\mathrm{mm}$$

$$\Delta y=y_S-y_m=+0.1827-0=+0.1827\,\mathrm{m}=182.7\,\mathrm{mm}$$

The translations of the shear center are as follows:

$$u_{Sxx}=u_{xx}-\varphi_{zx}\,\Delta y=1.5852-0.29744\cdot10^{-4}\cdot182.7=1.5798\,\mathrm{mm}$$

$$u_{Syx}=u_{yx}+\varphi_{zx}\,\Delta x=0.72166+0.29744\cdot10^{-4}\cdot(-6343)=0.5330\,\mathrm{mm}$$

$$u_{Sxy}=u_{xy}-\varphi_{zy}\,\Delta y=0.72166-0.10328\cdot10^{-2}\cdot182.7=0.5330\,\mathrm{mm}$$

$$u_{Syy}=u_{yy}+\varphi_{zy}\,\Delta x=7.3314+0.10328\cdot10^{-2}\cdot(-6343)=0.7803\,\mathrm{mm}$$

Based on these values the translations of the shear center in the principal directions:

$$u_1 = \frac{u_{Sxx}+u_{Syy}}{2} + \sqrt{\left(\frac{u_{Sxx}-u_{Syy}}{2}\right)^2 + u_{Sxy}{}^2} = \frac{1.5798+0.7803}{2} + \sqrt{\left(\frac{1.5798-0.7803}{2}\right)^2 + 0.5330^2} =$$
$$= 1.8463 \text{mm}$$

$$u_2 = \frac{u_{Sxx}+u_{Syy}}{2} - \sqrt{\left(\frac{u_{Sxx}-u_{Syy}}{2}\right)^2 + u_{Sxy}{}^2} = \frac{1.5798+0.7803}{2} - \sqrt{\left(\frac{1.5798-0.7803}{2}\right)^2 + 0.5330^2} =$$
$$= 0.5138 \text{mm}$$

According to these values the angles of the principal rigidity directions:

$$\alpha_{1FEM} = \arctan\frac{u_1 - u_{Sxx}}{u_{Sxy}} = \arctan\frac{1.8463-1.5798}{0.533} = 26.57^\circ$$

$$\alpha_{2FEM} = \arctan\frac{u_2 - u_{Sxx}}{u_{Sxy}} = \arctan\frac{0.5138-1.5798}{0.533} = -63.43^\circ$$

The directions coincide with the axes of symmetries (see Fig. 6.2.1-2) which is one of the principal rigidity direction in this case.

Then with FEM-Design results we can calculate the idealized bending stiffnesses of the structure:

$$EI_{1FEM} = \frac{H^3}{3u_2} = \frac{63^3}{3\cdot(0.5138/1000)} \qquad EI_{1FEM} = 1.622\cdot10^8 \text{kNm}^2$$

$$EI_{2FEM} = \frac{H^3}{3u_1} = \frac{63^3}{3\cdot(1.8463/1000)} \qquad EI_{2FEM} = 4.514\cdot10^7 \text{kNm}^2$$

The difference between FEM and hand calculation is less than 4%.

7. Calculations considering nonlinear effects

7.1 Uplift calculation

7.1.1 A trusses with limited compression members

In this example a truss will be analyzed. First of all we calculate the normal forces in the truss members and the maximum deflection for the gived concentrated loads. After this step we calculate the load multiplier when the vertical truss members reaches its limit compression bearing capacity what we set. See the inputs in the following table. After the hand calulation we compare the results with the FEM-Design nonlinear calculation results.

Inputs:

Column height/Span	H = 2.0 m; L = 8.0 m
The cross sections	KKR 80x80x6
The area of the cross sections	$A = 1652\ mm^2$
The elastic modulus	E = 210 GPa, steel
The concentrated loads	F = 40 kN
Limited compression of the vertical truss members	$P_{cr} = 700$ kN

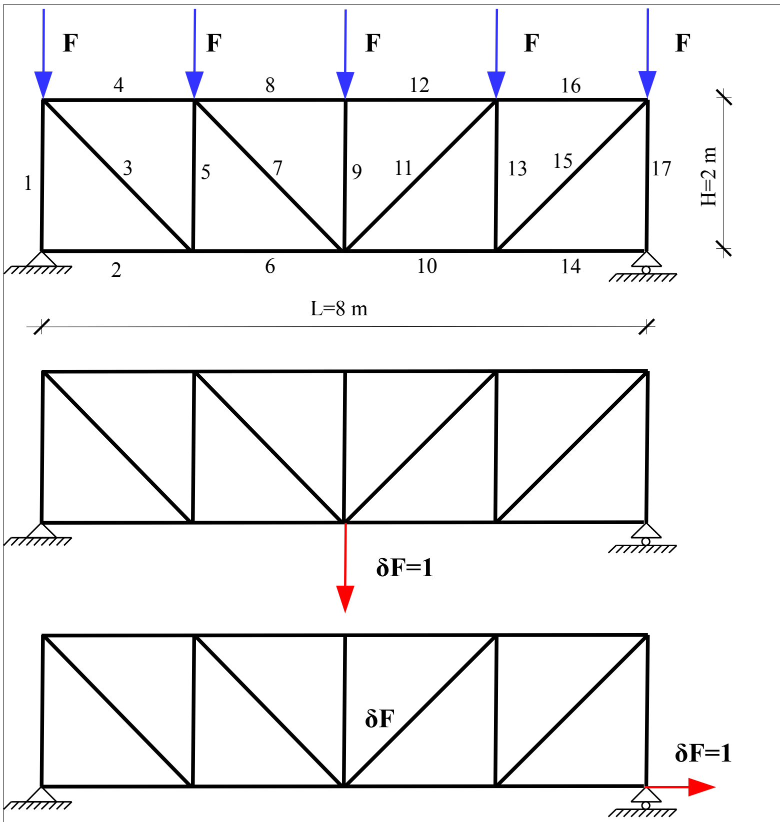


Figure 7.1.1.1 – The truss with the concentrated loads and with the virtual loads for the translation claculation

The normal forces in the truss members based on the hand calculation (without further details) are:

$N_1 = N_{17} = -100\,\mathrm{kN}$; $N_2 = N_{14} = 0\,\mathrm{kN}$; $N_3 = N_{15} = +84.85\,\mathrm{kN}$;

$N_4 = N_5 = N_{13} = N_{16} = -60.00\,\mathrm{kN}$; $N_6 = N_{10} = +60.00\,\mathrm{kN}$;

$N_7 = N_{11} = +28.28\,\mathrm{kN}$; $N_8 = N_{12} = -80.00\,\mathrm{kN}$; $N_9 = -40.00\,\mathrm{kN}$.

The normal forces in the truss members according to the vertical virtual force (see Fig. 7.1.1.1):

$N_{1,1} = N_{1,17} = -0.5\,\mathrm{kN}$; $N_{1,2} = N_{1,14} = \mathrm{N}_{1,9} = 0\,\mathrm{kN}$;

$N_{1,3} = N_{1,15} = N_{1,7} = N_{1,11} = +0.7071\,\mathrm{kN}$;

$N_{1,4} = N_{1,5} = N_{1,13} = N_{1,16} = -0.5\,\mathrm{kN}$; $N_{1,6} = N_{1,10} = +0.5\,\mathrm{kN}$;

$N_{1,8} = N_{1,12} = -1.0\,\mathrm{kN}$.

The normal forces in the truss members according to the horizontal virtual force (see Fig. 7.1.1.1):

$N_{2,2} = N_{2,6} = N_{2,10} = N_{2,14} = +1.0\,\mathrm{kN}$;

$N_{2,1} = N_{2,3} = N_{2,4} = N_{2,5} = N_{2,7} = N_{2,8} = N_{2,9} = N_{2,11} = N_{2,12} = N_{2,13} = N_{2,15} = N_{2,16} = N_{2,17} = 0\,\mathrm{kN}$.

The hand calculation of the vertical translation at the mid-span with the virtual force method:

$$e_z = \frac{1}{EA} \sum_{i=1}^{17} N_i \,\delta N_{1,i}\, l_i = 0.003841\,\mathrm{m} = 3.841\,\mathrm{mm}$$

The hand calculation of the horizontal translation at right roller with the virtual force method:

$$e_x = \frac{1}{EA} \sum_{i=1}^{17} N_i \,\delta N_{2,i}\, l_i = 0.0006918\,\mathrm{m} = 0.6918\,\mathrm{mm}$$

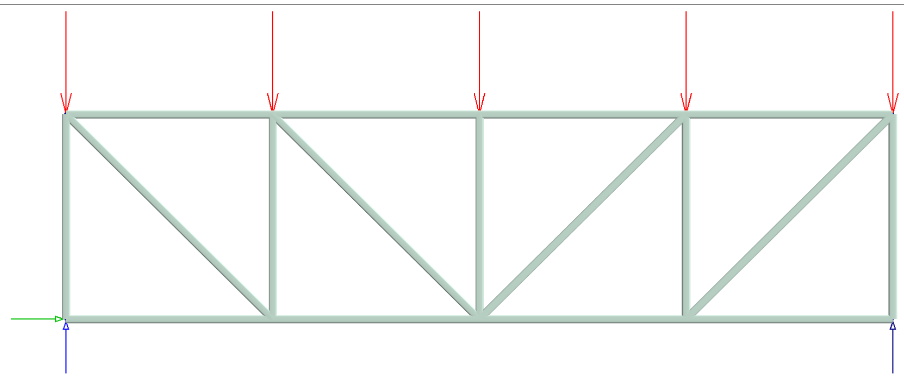


Figure 7.1.1.2 – The truss with the concentrated loads in FEM-Design

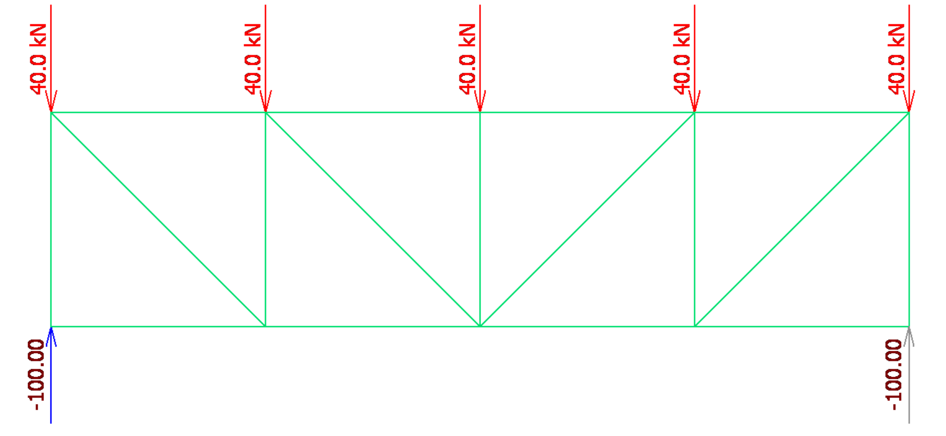


Figure 7.1.1.3 – The reaction forces

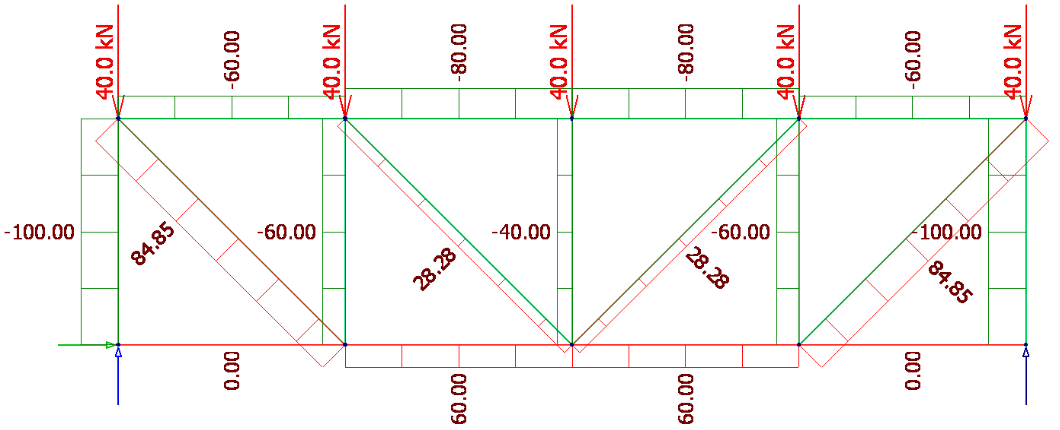


Figure 7.1.1.4 – The normal forces in the truss members

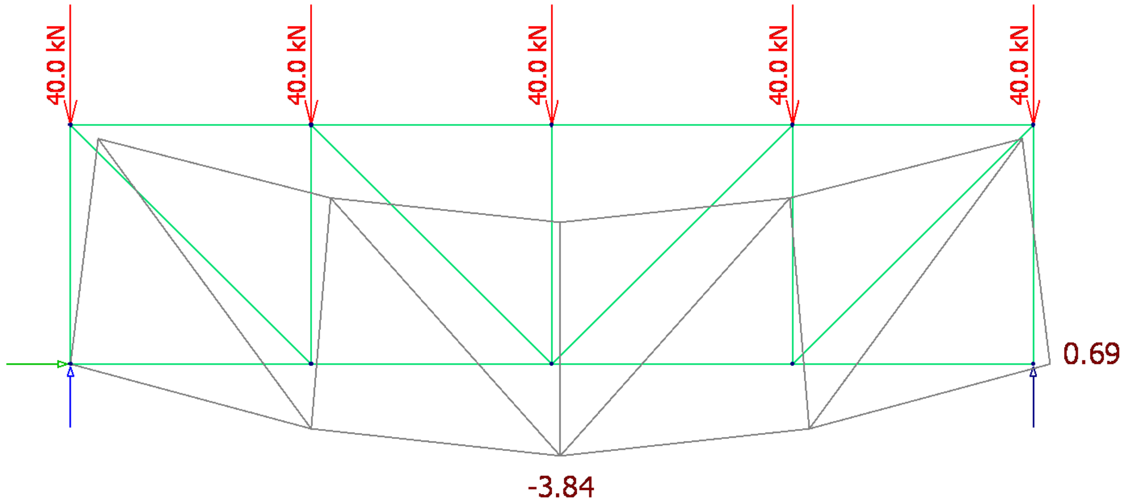


Figure 7.1.1.5 – The vertical translation at the mid-span and the horizontal translation at the right roller [mm]

The translations and the normal forces in the truss members based on the hand calculation are identical with the FEM-Design calculation, see Fig. 7.1.1.2-5.

After this step we would like to know the maximum load multiplier when the vertical truss members reaches its limit compression bearing capacity what we set, P_{cr} = 700 kN. The maximum compression force arises in the side columns, see the hand calculation, N_1 = (–)100 kN. Therefore the load multiplier based on the hand calcualtion is $\lambda = 7.0$.

Let's see the FEM-Design uplift calculation considering the limit compression in the vertical members.

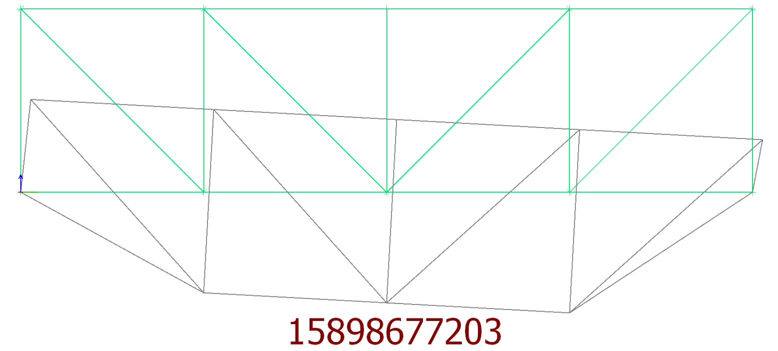


Figure 7.1.1.6 – Large nodal displacements when the side truss members reached the limit compression value [mm]

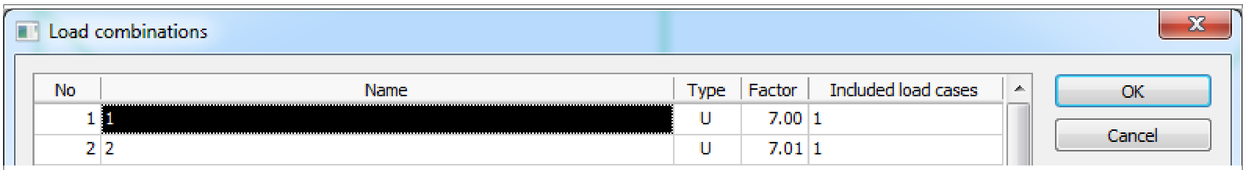


Figure 7.1.1.7 – The two different analyzed load multiplier in FEM-Design

With λ_{FEM} = 7.00 multiplier the FEM-Design analysis gives the accurate result but with λ_{FEM} = 7.01 (see Fig. 7.1.1.7) large nodal displacements occurred, see Fig. 7.1.1.6. Thus by this structure if we neglect the effect of the side members the complete truss became a statically over-determinated structure. FEM-Design solve this problem with iterative solver due to the fact that these kind of problems are nonlinear.

7.1.2 A continuous beam with three supports

In this example we analyse non-linear supports of a beam. Let's consider a continuous beam with three supports with the following parameters:

Inputs:

Span length	L = 2 m, total length = 2x2 = 4 m
The cross sections	Rectangle: 120x150 mm
The elastic modulus	E = 30 GPa, concrete C20/25
Intensity of distributed load (total, partial)	p = 10 kN/m

In Case I. the distribution of the external load and the nonlinearity of the supports differ from Case II. See the further details below (Fig. 7.1.2.1 and Fig. 7.1.2.8).

a) Case I.

In this case the distributed load is a total load (Fig. 7.1.2.1). In the first part all of the three supports behave the same way for compression and tension. In the second part of this case the middle support only bears tension. We calculate in both cases the deflections, shear forces and bending moments by hand and compared the results with FEM-Design uplift (nonlinear) calculations.

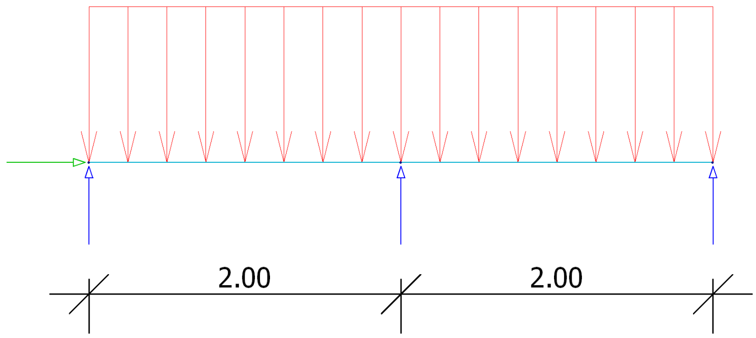


Figure 7.1.2.1 – The beam with three supports and uniform distributed load

In first part of this case the maximum deflection comes from the following formula considering only the bending deformations in the beam:

$$e_{max}=\frac{2.1}{384}\frac{pL^4}{EI}=\frac{2.1}{384}\frac{10\cdot 2^4}{30000000\cdot 0.12\cdot 0.15^3/12}=0.0008642\,\mathrm{m}=0.8642\,\mathrm{mm}$$

The relevant results with FEM-Design:

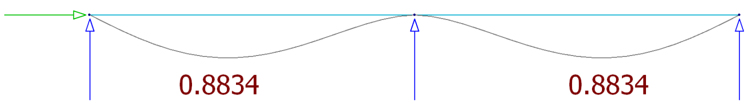


Figure 7.1.2.2 – The deflection of the beam with three supports (total load)

The extremums of the shear force without signs:

$$V_1 = \frac{3}{8} p L = \frac{3}{8} 10 \cdot 2 = 7.5\,\text{kN} \quad ; \quad V_2 = \frac{5}{8} p L = \frac{5}{8} 10 \cdot 2 = 12.5\,\text{kN}$$

The relevant results with FEM-Design:

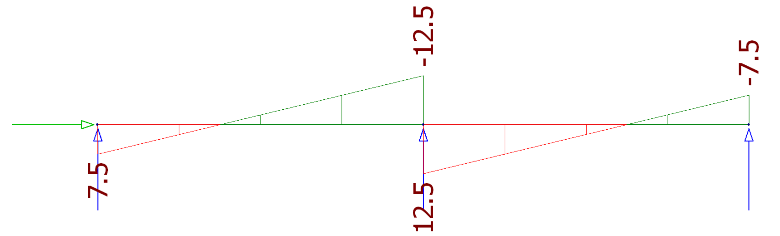


Figure 7.1.2.3 – The shear diagram of the beam with three supports (total load)

The extremums of the bending moment without signs:

$$M_{midspan} = \frac{9}{128} p L^2 = \frac{9}{128} 10 \cdot 2^2 = 2.812\,\text{kNm} \quad ; \quad M_{middle} = \frac{1}{8} p L^2 = \frac{1}{8} 10 \cdot 2^2 = 5.0\,\text{kNm}$$

The relevant results with FEM-Design:

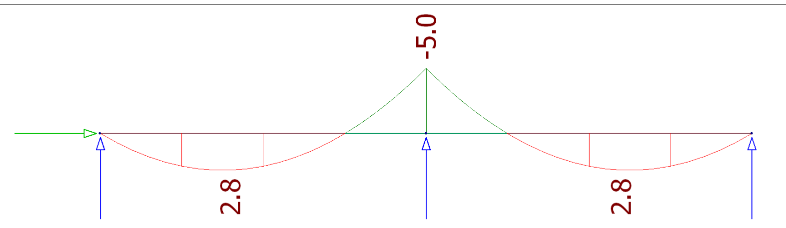


Figure 7.1.2.4 – The bending moment diagram of the beam with three supports (total load)

When the middle support only bear tension (second part of this case) basically under the total vertical load (Fig. 7.1.2.1) the middle support is not active (support nonlinearity). Therefore it works as a simply supported beam with two supports. The deflection, the shear forces and the bending moments are the following:

The maximum deflection comes from the following formula considering only the bending deformations in the beam:

$$e_{max} = \frac{5}{384} \frac{p (L+L)^4}{EI} = \frac{5}{384} \frac{10 \cdot (2+2)^4}{30000000 \cdot 0.12 \cdot 0.15^3 / 12} = 0.03292\,\text{m} = 32.92\,\text{mm}$$

The relevant results with FEM-Design:

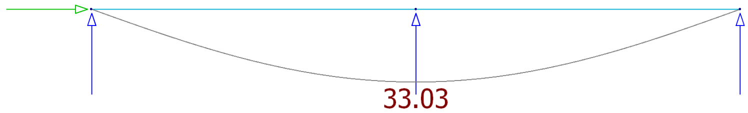


Figure 7.1.2.5 – The deflection of the beam when the middle support only bear tension (total load)

The maximum of the shear force without sign:

$$V=\frac{1}{2}p(L+L)=\frac{1}{2}10(2+2)=20\,\text{kN}$$

The relevant results with FEM-Design:

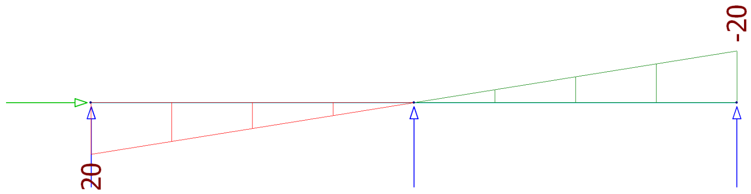


Figure 7.1.2.6 – The shear diagram of the beam when the middle support only bear tension (total load)

The extremum of the bending moment without sign:

$$M_{max}=\frac{1}{8}p(L+L)^2=\frac{1}{8}10\cdot(2+2)^2=20\,\text{kNm}$$

The relevant results with FEM-Design:

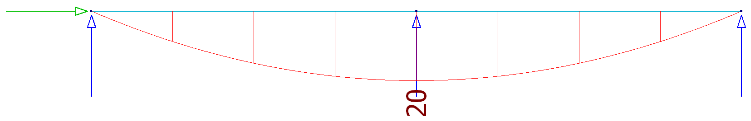


Figure 7.1.2.7 – The bending moment diagram of the beam when the middle support only bear tension (total load)

The differences between the calculated results by hand and by FEM-Design are less than 2%.

b) Case II.

In this case the distributed load is a partial load (Fig. 7.1.2.8). In the first part all of the three supports behave the same way for compression and tension. In the second part of this case the right side support only bears compression. We calculate in both cases the deflections, shear forces and bending moments by hand and compared the results with FEM-Design calculations.

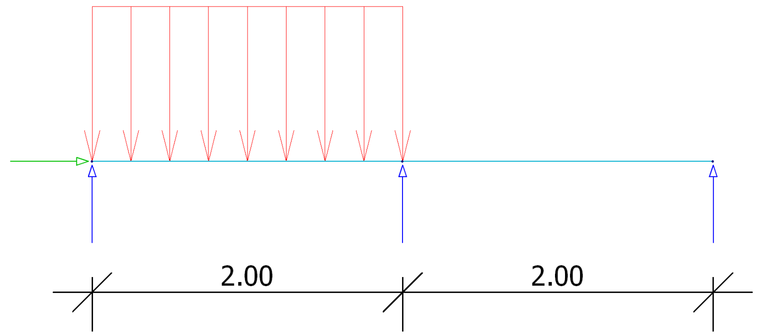


Figure 7.1.2.8 – The beam with three supports and uniform partial load

The extremums of the deflection come from the following formulas considering only the bending deformations in the beam (without signs):

$$e_{max} \approx \frac{2.1}{384}\frac{(p/2)L^4}{EI} + \frac{5}{384}\frac{(p/2)L^4}{EI} = \frac{2.1}{384}\frac{10/2 \cdot 2^4}{EI} + \frac{5}{384}\frac{10/2 \cdot 2^4}{EI} = 0.001461\,\mathrm{m} = 1.461\,\mathrm{mm}$$

$$e_{min} \approx \frac{5}{384}\frac{(p/2)L^4}{EI} - \frac{2}{384}\frac{(p/2)L^4}{EI} = \frac{5}{384}\frac{10/2 \cdot 2^4}{EI} - \frac{2}{384}\frac{10/2 \cdot 2^4}{EI} = 0.0006173\,\mathrm{m} = 0.6173\,\mathrm{mm}$$

The relevant results with FEM-Design:

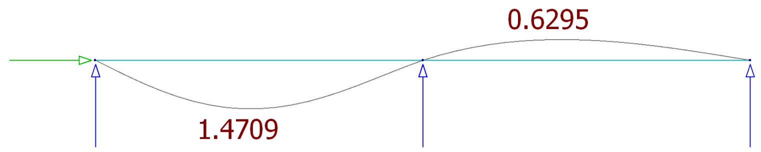


Figure 7.1.2.9 – The deflection of the beam with three supports (partial load)

The extremums of the shear force without signs:

$$V_1 = \frac{7}{16} p\,L = \frac{7}{16} 10 \cdot 2 = 8.75\,\mathrm{kN} \quad ; \quad V_2 = \frac{9}{16} p\,L = \frac{9}{16} 10 \cdot 2 = 11.25\,\mathrm{kN} \quad ;$$

$$V_3 = \frac{1}{16} p\,L = \frac{1}{16} 10 \cdot 2 = 1.25\,\mathrm{kN}$$

The relevant results with FEM-Design:

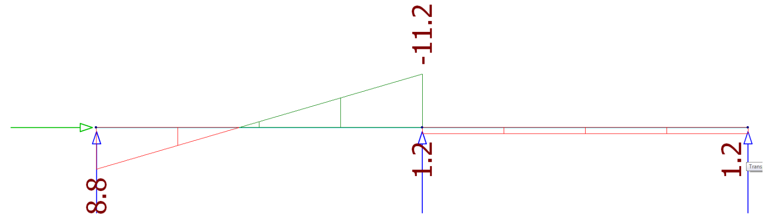


Figure 7.1.2.10 – The shear diagram of the beam with three supports (partial load)

The extremums of the bending moment without signs:

$$M_{midspan} = \frac{\left(\frac{7}{16} p L\right)^2}{2p} = \frac{\left(\frac{7}{16} 10 \cdot 2\right)^2}{2 \cdot 10} = 3.828\,\text{kNm} \quad ; \quad M_{middle} = \frac{1}{16} p L^2 = \frac{1}{16} 10 \cdot 2^2 = 2.5\,\text{kNm}$$

The relevant results with FEM-Design:

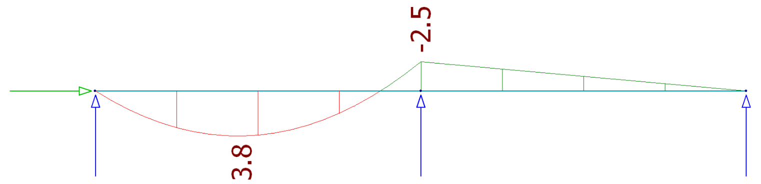


Figure 7.1.2.11 – The bending moment diagram of the beam with three supports (partial load)

When the right side support only bear compression (second part of this case) basically under the partial vertical load (Fig. 7.1.2.8) the right side support is not active (support nonlinearity). Therefore it works as a simply supported beam with two supports. The deflection, the shear forces and the bending moments are the following:

The maximum deflection comes from the following formula considering only the bending deformations in the beam:

$$e_{midspan} = \frac{5}{384} \frac{p L^4}{EI} = \frac{5}{384} \frac{10 \cdot 2^4}{EI} = 0.002058\,\text{m} = 2.058\,\text{mm}$$

$$e_{right} = \frac{1}{24} \frac{p L^4}{EI} = \frac{1}{24} \frac{10 \cdot 2^4}{EI} = 0.006584\,\text{m} = 6.584\,\text{mm}$$

The relevant results with FEM-Design:

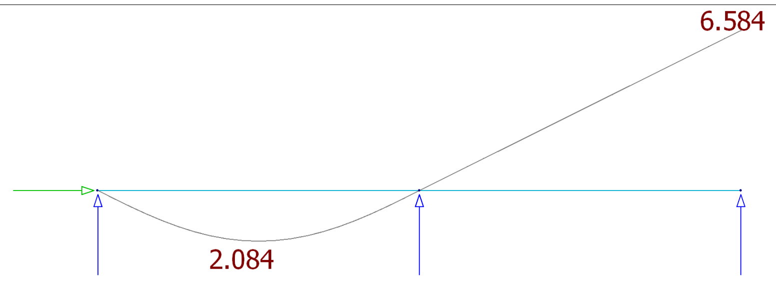


Figure 7.1.2.12 – The deflection of the beam when the right support only bear compression (partial load)

The extremum of the shear force without sign:

$$V = \frac{1}{2} p(L) = \frac{1}{2} 10 \cdot 2 = 10\,\text{kN}$$

The relevant results with FEM-Design:

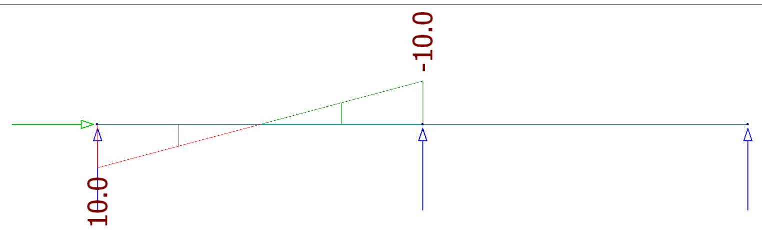


Figure 7.1.2.13 – The shear diagram of the beam when the right support only bear compression (partial load)

The extremum of the bending moment without sign:

$$M_{max} = \frac{1}{8} p L^2 = \frac{1}{8} 10 \cdot 2^2 = 5.0\,\text{kNm}$$

The relevant results with FEM-Design:

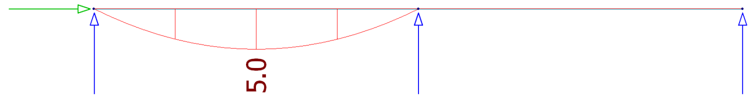


Figure 7.1.2.14 – The bending moment diagram of the beam when the right support only bear compression (partial load)

The differences between the calculated results by hand and by FEM-Design are less than 2%.

7.2 Cracked section analysis by reinforced concrete elements

7.2.1 Cracked deflection of a simply supported beam

Inputs:

Span length	L_{eff} = 7.2 m
The cross section	Rectangle: b = 300 mm; h = 450 mm
The elastic modulus of concrete	E_{cm} = 31.476 GPa, C25/30
The creep factor	$\varphi_{28} = 2.35$
Effective elastic modulus of concrete	$E_{ceff} = E_{cm}/(1+\varphi_{28}) = 9.396$ GPa
Mean tensile strength	f_{ctm} = 2.565 MPa
Elastic modulus of steel bars	E_s = 200 GPa
Characteristic value of dead load	g_k = 8.5 kN/m
Characteristic value of live load	q_k = 12.0 kN/m
Live load combination factor	$\psi_2 = 0.6$
Diameter of the longitudinal reinforcement	ϕ_l = 18 mm
Diameter of the stirrup reinforcement	ϕ_s = 8 mm
Area of longitudinal reinforcement	$A_l = 5x18^2\pi/4 = 1272.3\ m^2$
Nominal concrete cover	c_{nom} = 20 mm

The cross sectional properties without calculation details:

I. stress stadium second moment of inertia	$I_I = 3.075x10^9\ mm^4$
II. stress stadium second moment of inertia	$I_{II} = 2.028x10^9\ mm^4$
I. stress stadium position of neutral axis	x_I = 256.4 mm
II. stress stadium position of neutral axis	x_{II} = 197.3 mm

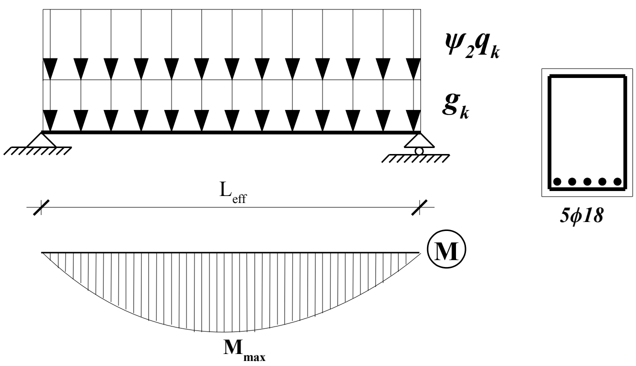


Figure 7.2.1.1 – The simply supported RC beam

The calculation of deflection according to EN 1992-1-1:

The load value for the quasi-permanent load combination:

$$p_{qp}=g_k+\psi_2 q_k=8.5+0.6\cdot 12=15.7\frac{\text{kN}}{\text{m}}$$

The maximum deflection with cross sectional properties in Stadium I. (uncracked):

$$w_{k.I}=\frac{5}{384}\frac{p_{qp}L_{eff}^{4}}{E_{ceff}I_I}=\frac{5}{384}\frac{15.7\cdot 7.2^4}{9396000\cdot 0.003075}=0.01901\,\text{m}=19.01\,\text{mm}$$

The maximum deflection with cross sectional properties in Stadium II. (cracked):

$$w_{k.II}=\frac{5}{384}\frac{p_{qp}L_{eff}^{4}}{E_{ceff}I_{II}}=\frac{5}{384}\frac{15.7\cdot 7.2^4}{9396000\cdot 0.002028}=0.02883\,\text{m}=28.83\,\text{mm}$$

The maximum bending moment under the quasi-permanent load:

$$M_{max}=\frac{1}{8}p_{qp}L_{eff}^{2}=\frac{1}{8}15.7\cdot 7.2^2=101.74\,\text{kNm}$$

The cracking moment with the mean tensile strength:

$$M_{cr}=f_{ctm}\frac{I_I}{h-x_I}=2565\frac{0.003075}{0.45-0.2564}=40.74\,\text{kNm}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta=\max\left[1-0.5\left(\frac{M_{cr}}{M_{max}}\right)^2,0\right]=\max\left[1-0.5\left(\frac{40.74}{101.74}\right)^2,0\right]=0.9198$$

This value is almost 1.0, it means that the final deflection will be closer to the cracked deflection than to the uncracked one.

The final deflection with the aim of interpolation factor:

$$w_k=(1-\zeta)w_{k.I}+\zeta w_{k.II}=(1-0.9198)19.01+0.9198\cdot 28.83=28.04\,\text{mm}$$

First we modelled the beam with beam elements. In FEM-Design we increased the division number of the beam finite elements to five to get the more accurate results.

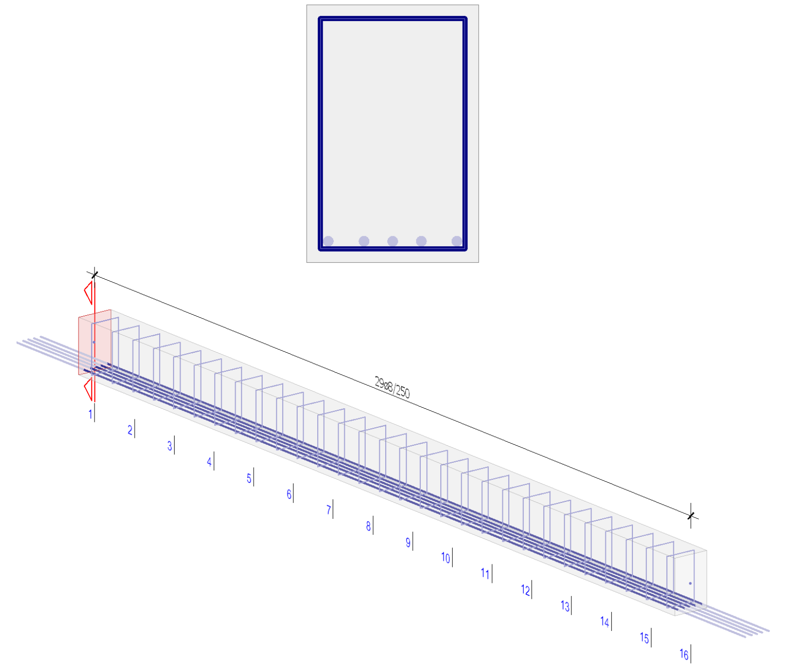


Figure 7.2.1.2 – The cross section of the RC beam in FEM-Design

Fig. 7.2.1.2 shows the applied cross section and reinforcement with the defined input parameters.

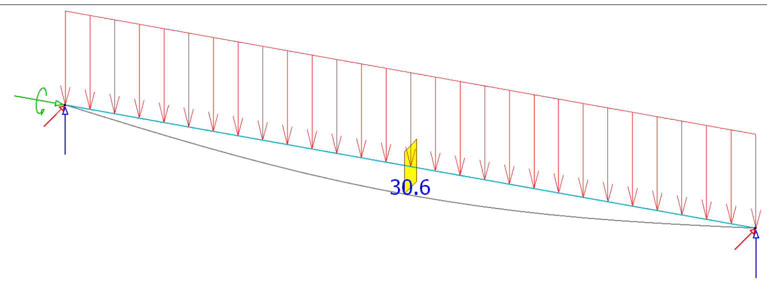


Figure 7.2.1.3 – The deflection of the RC beam in FEM-Design with cracked section analysis

Fig. 7.2.1.3 shows the deflection after the cracked section analysis. The deflection of the beam model in FEM-Design:

$w_{kFEM} = 30.59\ \text{mm}$

The difference is less than 9 %.

Secondly we modelled the beam with shell finite elements. Fig. 7.2.1.4 shows the applied specific reinforcement with the defined input parameters with slab.

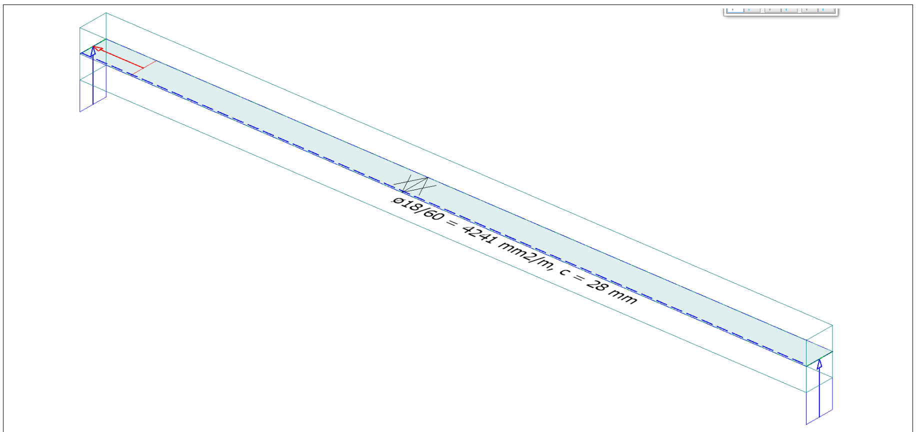


Figure 7.2.1.4 – The specific reinforcement with the shell model in FEM-Design

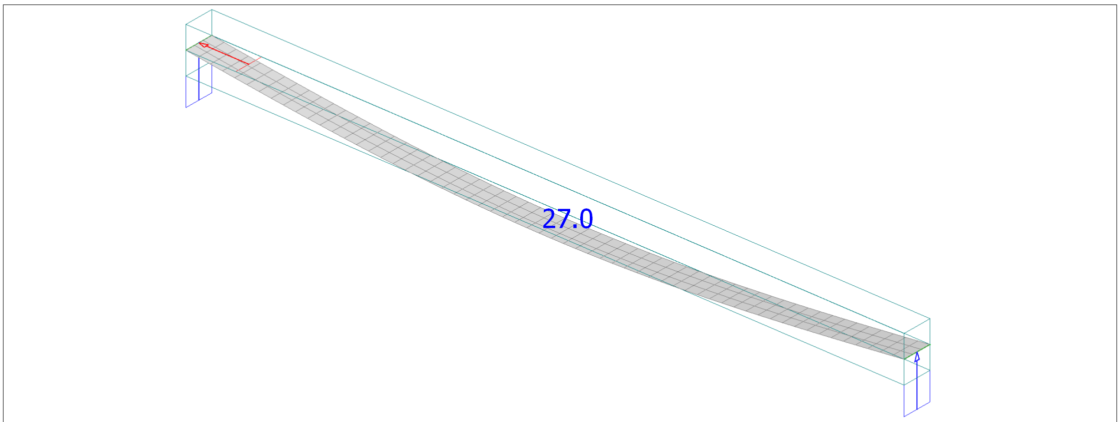


Figure 7.2.1.5 – The deflection of the RC shell model in FEM-Design with cracked section analysis

Fig. 7.2.1.5 shows the deflection and the finite element mesh after the cracked section analysis. The deflection of the shell model in FEM-Design:

$w_{kFEM} = 27.0\,\text{mm}$

The difference is less than 4 %.

7.2.2 Cracked deflection of a cantilever beam

Inputs:

Span length	$L_{eff} = 4$ m
The cross section	Rectangle: b = 300 mm; h = 450 mm
The elastic modulus of concrete	$E_{cm} = 31.476$ GPa, C25/30
The creep factor	$\varphi_{28} = 2.35$
Effective elastic modulus of concrete	$E_{ceff} = E_{cm}/(1+\varphi_{28}) = 9.396$ GPa
Mean tensile strength	$f_{ctm} = 2.565$ MPa
Elastic modulus of steel bars	$E_s = 200$ GPa
Characteristic value of dead load	$g_k = 8.5$ kN/m
Characteristic value of live load	$q_k = 12.0$ kN/m
Live load combination factor	$\psi_2 = 0.6$
Diameter of the longitudinal reinforcement	$\phi_l = 18$ mm
Diameter of the stirrup reinforcement	$\phi_s = 8$ mm
Area of longitudinal reinforcement	$A_l = 5x18^2\pi/4 = 1272.3\ m^2$
Nominal concrete cover	$c_{nom} = 20$ mm

The cross sectional properties without calculation details:

I. stress stadium second moment of inertia	$I_I = 3.075x10^9\ mm^4$
II. stress stadium second moment of inertia	$I_{II} = 2.028x10^9\ mm^4$
I. stress stadium position of neutral axis	$x_I = 256.4$ mm
II. stress stadium position of neutral axis	$x_{II} = 197.3$ mm

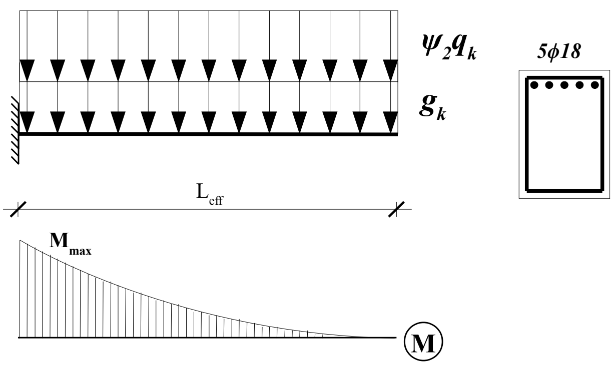


Figure 7.2.2.1 – The cantilever RC beam

The calculation of deflection according to EN 1992-1-1:

The load value for the quasi-permanent load combination:

$$p_{qp}=g_k+\psi_2 q_k=8.5+0.6\cdot 12=15.7\,\frac{\mathrm{kN}}{\mathrm{m}}$$

The maximum deflection with cross sectional properties in Stadium I. (uncracked):

$$w_{k.I}=\frac{1}{8}\frac{p_{qp}L_{eff}^{4}}{E_{ceff}I_I}=\frac{1}{8}\frac{15.7\cdot 4^4}{9396000\cdot 0.003075}=0.01739\,\mathrm{m}=17.39\,\mathrm{mm}$$

The maximum deflection with cross sectional properties in Stadium II. (cracked):

$$w_{k.II}=\frac{1}{8}\frac{p_{qp}L_{eff}^{4}}{E_{ceff}I_{II}}=\frac{1}{8}\frac{15.7\cdot 4^4}{9396000\cdot 0.002028}=0.02637\,\mathrm{m}=26.37\,\mathrm{mm}$$

The maximum bending moment under the quasi-permanent load:

$$M_{max}=\frac{1}{2}p_{qp}L_{eff}^{2}=\frac{1}{2}15.7\cdot 4^2=125.6\,\mathrm{kNm}$$

The cracking moment with the mean tensile strength:

$$M_{cr}=f_{ctm}\frac{I_I}{h-x_I}=2565\frac{0.003075}{0.45-0.2564}=40.74\,\mathrm{kNm}$$

The interpolation factor considering the mixture of cracked and uncracked behaviour:

$$\zeta=\max\left[1-0.5\left(\frac{M_{cr}}{M_{max}}\right)^2,0\right]=\max\left[1-0.5\left(\frac{40.74}{125.6}\right)^2,0\right]=0.9474$$

This value is almost 1.0, it means that the final deflection will be closer to the cracked deflection than to the uncracked one.

The final deflection with the aim of interpolation factor:

$$w_k=(1-\zeta)w_{k.I}+\zeta\,w_{k.II}=(1-0.9474)17.39+0.9474\cdot 26.37=25.90\,\mathrm{mm}$$

We modelled the beam with beam finite elements. In FEM-Design we increased the division number of the beam finite elements to five to get the more accurate results.

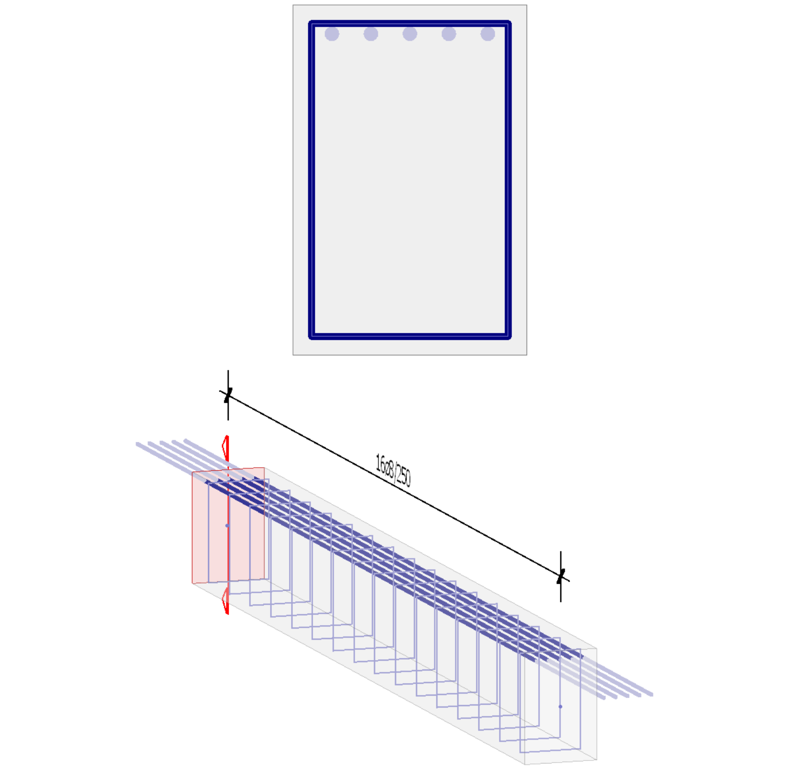


Figure 7.2.2.2 – The cross section of the RC cantilever in FEM-Design

Fig. 7.2.2.2 shows the applied cross section and reinforcement with the defined input parameters.

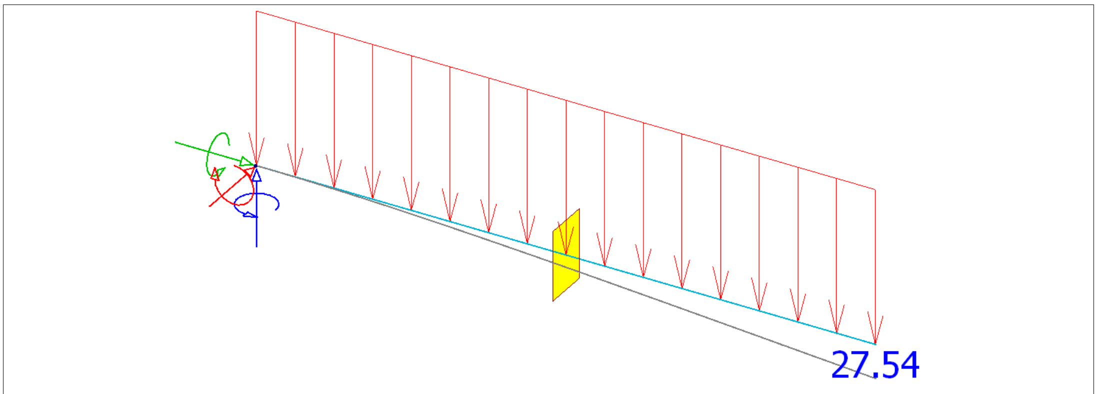


Figure 7.2.2.3 – The deflection of the RC cantilever in FEM-Design with cracked section analysis

Fig. 7.2.2.3 shows the deflection after the cracked section analysis. The deflection of the beam model in FEM-Design:

$$w_{kFEM} = 27.54\,\text{mm}$$

The difference is less than 7 %.

7.3 Nonlinear soil calculation

This chapter goes beyond the scope of this document, therefore additional informations are located in:

FEM-Design – Geotechnical modul in 3D, Theoretical background and verification and validation handbook

http://download.strusoft.com/FEM-Design/inst150x/documents//3dsoilmanual.pdf

8. Cross section editor

8.1 Calculation of a compound cross section

An example for compound cross section is taken from [7] where the authors calculated the cross sectional properties with the *assumption of thin-walled simplifications*. The welded cross section is consisting of U300 and L160x80x12 (DIN) profiles. In the **Section Editor** the *exact cold rolled geometry* was analyzed as it is seen in Figure 8.1.1.

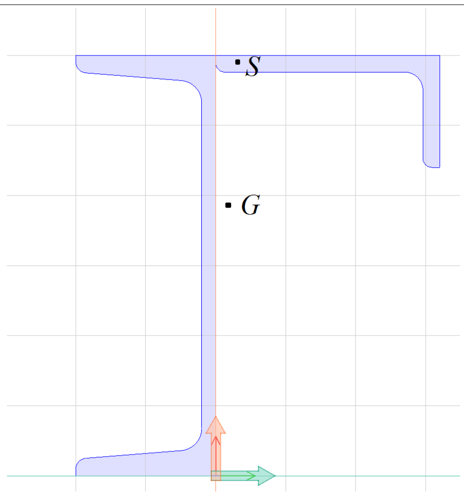


Figure 8.1.1 – The analyzed cross section

The following table contains the results of the two independent calculations with several cross sectional properties.

Notation	Ref. [1]	**Section Editor**
A [cm^2]	86.76	86.30
y_G [cm]	1.210	1.442
z_G [cm]	19.20	19.22
y'_S [cm]	1.39	0.7230
z'_S [cm]	10.06	10.36
I_y [cm^4]	11379.9	11431.2
I_z [cm^4]	4513.3	4372.9
I_{yz} [cm^4]	3013.2	3053.5
I_t [cm^4]	48.83	52.11
I_w [cm^6]	-	203082.0

Table 8.1.1 – The results of the example

9. Design calculations

9.1 Required reinforcement calculation for a slab

In this example we calculate the required reinforcements of a slab due to elliptic and hyperbolic bending conditions. First of all the applied reinforcement is orthogonal and then the applied reinforcement is non-orthogonal. We calculate the required reinforcement with hand calculation and then compare the results with FEM-Design values.

Inputs:

The thickness	h = 200 mm
The elastic modulus of concrete	E_{cm} = 33 GPa, C30/37
The Poisson's ratio of concrete	ν = 0.2
The design value of compressive strength	f_{cd} = 20 MPa
Elastic modulus of steel bars	E_s = 200 GPa
The design value of yield stress of steel bars	f_{yd} = 434.8 MPa
Diameter of the longitudinal reinforcement	ϕ_l = 10 mm
Nominal concrete cover	c_x = 20 mm; c_y = 30 mm
Effective heights	d_x = 175 mm; d_y = 165 mm

I.) Elliptic bending

In the first case the bending condition is an elliptic bending. In FEM-Design the model is a slab with statically determinant support system and specific moment loads at its edges for the pure internal force state (see Fig. 9.2.2.1).

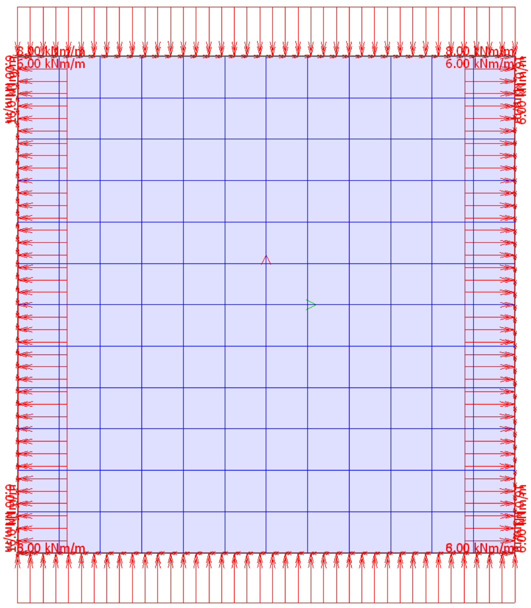


Figure 9.2.2.1 – The slab with the edge loads for pure stress state

Fig. 9.2.2.2 shows the constant internal forces in the slab due to the loads. Fig. 9.2.2.3 shows the principal moments and their directions based on the FEM-Design calculation. According to the pure stress state the principal moments and the directions are the same in each elements.

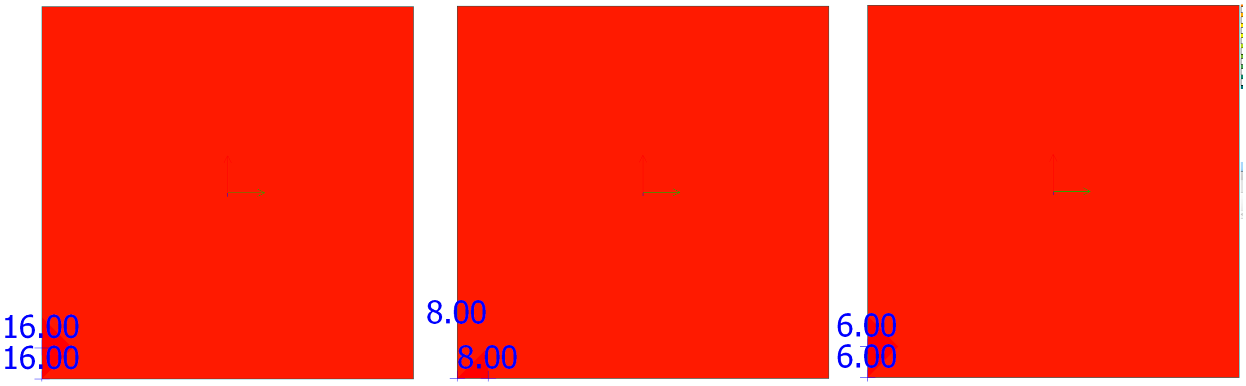


Figure 9.2.2.2 – The m_x, m_y, and m_{xy} internal forces in the slab [kNm]

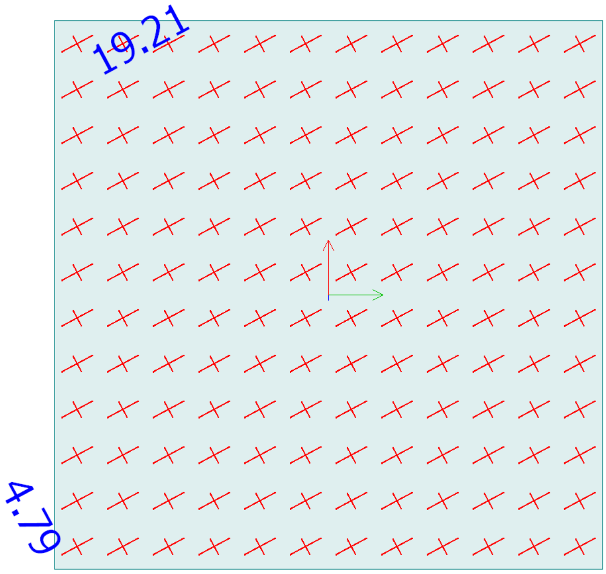


Figure 9.2.2.3 – The m_1 and m_2 principal moments and their directions in the slab [kNm]

First of all the reinforcement is orthogonal and the hand calculation and the comparison are the following:

1. Orthogonal reinforcement (φ=90°)

The reinforcement is orthogonal and their directions concide with the local system (x=ξ, y=ϑ=η).

The moments in the slab (tensor of the applied moments):

$$m_x = m_\xi = +16\ \mathrm{kNm/m}$$

$$m_y = m_\vartheta = m_\eta = +8\ \mathrm{kNm/m}$$

$$m_{xy} = m_{\xi\vartheta} = m_{\xi\eta} = +6\ \mathrm{kNm/m}$$

The first invariant of the tensor: $m_x + m_y = +24\,\text{kNm/m}$

The calculation of the principal moments and their directions:

$$m_1 = \frac{m_x + m_y}{2} + \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^{\ 2}} = \frac{16+8}{2} + \sqrt{\left(\frac{16-8}{2}\right)^2 + 6^2} = 19.21\,\text{kNm/m}$$

$$m_2 = \frac{m_x + m_y}{2} - \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^{\ 2}} = \frac{16+8}{2} - \sqrt{\left(\frac{16-8}{2}\right)^2 + 6^2} = 4.79\,\text{kNm/m}$$

$$\alpha_0 = \arctan\frac{m_1 - m_x}{m_{xy}} = \arctan\frac{19.21-16}{6} = 28.15^{\circ}$$

Compare these results with Fig. 9.2.2.3. The difference is 0 %.

The design moments (according to [9][10]) if the reinforcement (ξ,η) is orthogonal and their directions concide with the local co-ordinate system (x,y):

Case a)

$$m_{ud\,\xi} = m_{\xi} - m_{\vartheta}\frac{\cos\varphi}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1-2\cos\varphi}{\sin\varphi} = 16 - 8\frac{\cos 90^{\circ}}{1+\cos 90^{\circ}} + 6\frac{1-2\cos 90^{\circ}}{\sin 90^{\circ}} = +22\,\text{kNm/m}$$

$$m_{ud\,\eta} = m_{\vartheta}\frac{1}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1}{\sin\varphi} = 8\frac{1}{1+\cos 90^{\circ}} + 6\frac{1}{\sin 90^{\circ}} = +14\,\text{kNm/m}$$

This is a valid solution! Because $m_{ud\,\xi} + m_{ud\,\eta} = +36\,\text{kNm/m} > m_x + m_y = +24\,\text{kNm/m}$

$m_{ud\,\xi} = +22\,\text{kNm/m}$ $m_{ud\,\eta} = +14\,\text{kNm/m}$

Case b)

$$m_{ud\,\xi} = m_{\xi} + m_{\vartheta}\frac{\cos\varphi}{1-\cos\varphi} - m_{\xi\vartheta}\frac{1+2\cos\varphi}{\sin\varphi} = 16 + 8\frac{\cos 90^{\circ}}{1-\cos 90^{\circ}} - 6\frac{1+2\cos 90^{\circ}}{\sin 90^{\circ}} = +10\,\text{kNm/m}$$

$$m_{ud\,\eta} = m_{\vartheta}\frac{1}{1-\cos\varphi} - m_{\xi\vartheta}\frac{1}{\sin\varphi} = 8\frac{1}{1-\cos 90^{\circ}} - 6\frac{1}{\sin 90^{\circ}} = +2\,\text{kNm/m}$$

Invalid solution! Because $m_{ud\,\xi} + m_{ud\,\eta} = +12\,\text{kNm/m} < m_x + m_y = +24\,\text{kNm/m}$

Case ξ)

$$m_{ud\,\xi} = m_{\xi} - \frac{m_{\xi\vartheta}^2}{m_{\vartheta}} = 16 - \frac{6^2}{8} = +11.5\,\text{kNm/m}$$

$$m_{ud\,\eta} = 0$$

Invalid solution! Because $m_{ud\,\xi} + m_{ud\,\eta} = +11.5\,\text{kNm/m} < m_x + m_y = +24\,\text{kNm/m}$

Case η)

$$m_{ud\,\xi} = 0$$

$$m_{ud\,\eta} = \frac{m_\xi m_\vartheta - m_{\xi\vartheta}^2}{m_\xi \sin^2\varphi + m_\vartheta \cos^2\varphi - m_{\xi\vartheta} \sin 2\varphi} = \frac{16 \cdot 8 - 6^2}{16 \cdot \sin^2 90^\circ + 8 \cdot \cos^2 90^\circ - 6 \cdot \sin(2 \cdot 90^\circ)} = +5.75 \frac{\text{kNm}}{\text{m}}$$

Invalid solution! Because $m_{ud\,\xi} + m_{ud\,\eta} = +5.75\,\text{kNm/m} < m_x + m_y = +24\,\text{kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.4 and 9.2.2.5. The difference between the hand and FE calculation is 0%.

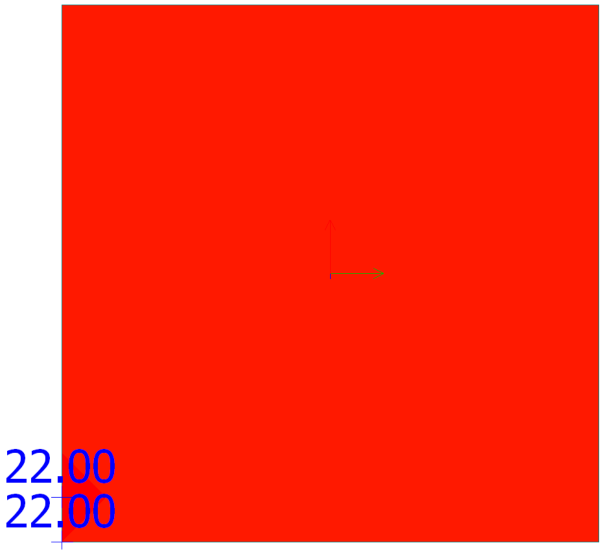


Figure 9.2.2.4 – The $m_{ud\,\xi}$ design moment for elliptic bending with orthogonal reinforcement [kNm]

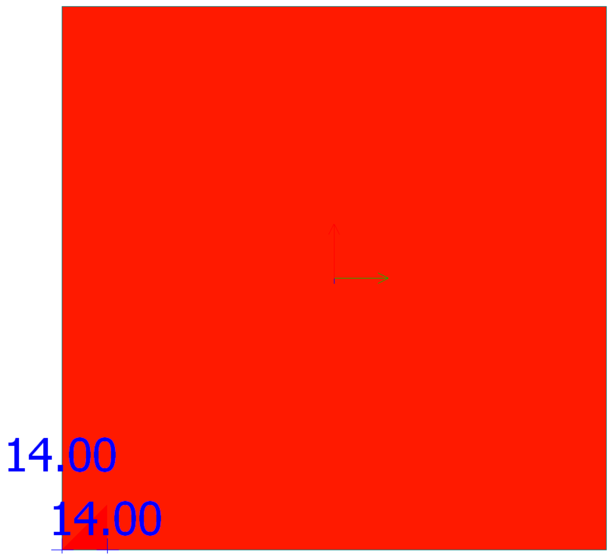


Figure 9.2.2.5 – The $m_{ud\,\eta}$ design moment for elliptic bending with orthogonal reinforcement [kNm]

Calculation of the required reinforcement based on the valid design moments:

In x (ξ) direction:

Sum of the moments:

$$m_{ud\,\xi}=f_{cd}\,x_c\left(d_x-\frac{x_c}{2}\right)\quad;\quad 22000=20\,x_c\left(175-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=6.403\,\mathrm{mm}$$

Sum of the forces:

$$x_c\,f_{cd}=a_{s\xi}\,f_{yd}\quad;\quad 6.403\cdot 20=a_{s\xi}\,434.8\quad;\quad a_{s\xi}=0.2945\,\mathrm{mm^2/mm}=294.5\,\mathrm{mm^2/m}$$

Fig. 9.2.2.6 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

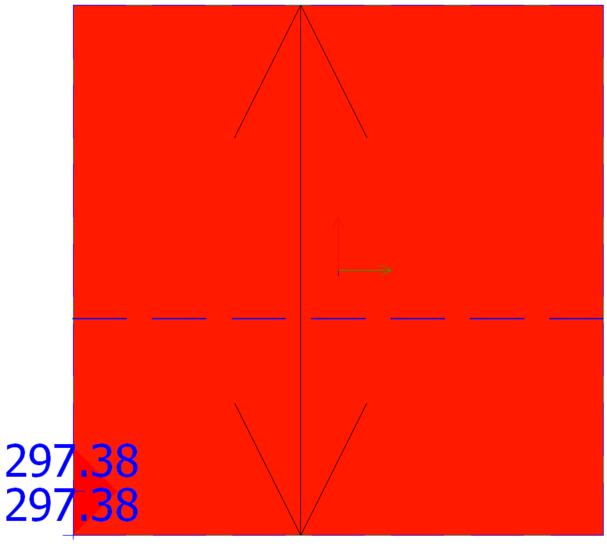


Figure 9.2.2.6 – The $a_{s\xi}$ required reinforcement at the bottom for elliptic bending with orthogonal reinforcement [mm²/m]

In y (η) direction:

Sum of the moments:

$$m_{ud\,\eta}=f_{cd}\,x_c\left(d_y-\frac{x_c}{2}\right)\quad;\quad 14000=20\,x_c\left(165-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=4.298\,\mathrm{mm}$$

Sum of the forces:

$$x_c\,f_{cd}=a_{s\eta}\,f_{yd}\quad;\quad 4.298\cdot 20=a_{s\eta}\,434.8\quad;\quad a_{s\eta}=0.1977\,\mathrm{mm^2/mm}=197.7\,\mathrm{mm^2/m}$$

Fig. 9.2.2.7 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

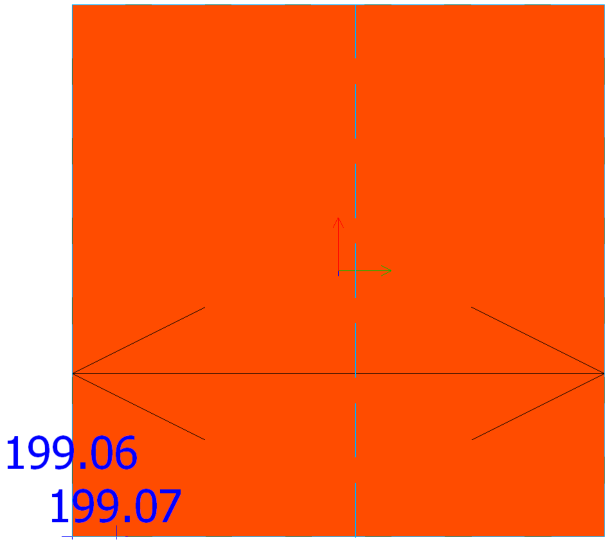


Figure 9.2.2.7 – The $a_{s\eta}$ required reinforcement at the bottom for elliptic bending with orthogonal reinforcement [mm²/m]

Secondly the reinforcement is non-orthogonal and the hand calculation and the comparison are the following:

2. Non-orthogonal reinforcement (φ=75° between ξ and η)

The reinforcement is non-orthogonal and the ξ direction concides with the local x direction. Thus y=ϑ. The angle between the ξ directional reinforcement and η directional reinforcement is φ=75°.

The moments in the slab (tensor of the applied moments):

$$m_x = m_\xi = +16\,\text{kNm/m}$$

$$m_y = m_\vartheta = m_\eta = +8\,\text{kNm/m}$$

$$m_{xy} = m_{\xi\vartheta} = +6\,\text{kNm/m}$$

The first invariant of the tensor: $m_x + m_y = +24\,\text{kNm/m}$

The design moments (according to [9][10]) if the reinforcement (ξ,η) is non-orthogonal:

Case a)

$$m_{ud\,\xi} = m_\xi - m_\vartheta \frac{\cos\varphi}{1+\cos\varphi} + m_{\xi\vartheta} \frac{1-2\cos\varphi}{\sin\varphi} = 16 - 8\frac{\cos 75^\circ}{1+\cos 75^\circ} + 6\frac{1-2\cos 75^\circ}{\sin 75^\circ} = +17.35\,\text{kNm/m}$$

$$m_{ud\,\eta} = m_\vartheta \frac{1}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1}{\sin\varphi} = 8\frac{1}{1+\cos 75^\circ} + 6\frac{1}{\sin 75^\circ} = +12.57\,\text{kNm/m}$$

This is a valid solution! Because $m_{ud\,\xi} + m_{ud\,\eta} = +29.92\,\text{kNm/m} > m_x + m_y = +24\,\text{kNm/m}$

$m_{ud\,\xi} = +17.35\,\text{kNm/m}$ $m_{ud\,\eta} = +12.57\,\text{kNm/m}$

Case b)

$$m_{ud\,\xi}=m_\xi+m_\vartheta\frac{\cos\varphi}{1-\cos\varphi}-m_{\xi\vartheta}\frac{1+2\cos\varphi}{\sin\varphi}=16+8\frac{\cos 75^o}{1-\cos 75^o}-6\frac{1+2\cos 75^o}{\sin 75^o}=+9.37\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=m_\vartheta\frac{1}{1-\cos\varphi}-m_{\xi\vartheta}\frac{1}{\sin\varphi}=8\frac{1}{1-\cos 75^o}-6\frac{1}{\sin 75^o}=+4.58\,\mathrm{kNm/m}$$

Invalid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+13.95\,\mathrm{kNm/m}<m_x+m_y=+24\,\mathrm{kNm/m}$

Case ξ)

$$m_{ud\,\xi}=m_\xi-\frac{m_{\xi\vartheta}^2}{m_\vartheta}=16-\frac{6^2}{8}=+11.5\,\mathrm{kNm/m}$$

$$m_{ud\,\eta}=0$$

Invalid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+11.5\,\mathrm{kNm/m}<m_x+m_y=+24\,\mathrm{kNm/m}$

Case η)

$$m_{ud\,\xi}=0$$

$$m_{ud\,\eta}=\frac{m_\xi m_\vartheta-m_{\xi\vartheta}^2}{m_\xi\sin^2\varphi+m_\vartheta\cos^2\varphi-m_{\xi\vartheta}\sin 2\varphi}=\frac{16\cdot 8-6^2}{16\cdot\sin^2 75^o+8\cdot\cos^2 75^o-6\cdot\sin(2\cdot 75^o)}=+7.38\,\frac{\mathrm{kNm}}{\mathrm{m}}$$

Invalid solution! Because $m_{ud\,\xi}+m_{ud\,\eta}=+7.38\,\mathrm{kNm/m}<m_x+m_y=+24\,\mathrm{kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.8 and 9.2.2.9. The difference between the hand and FE calculation is 0%.

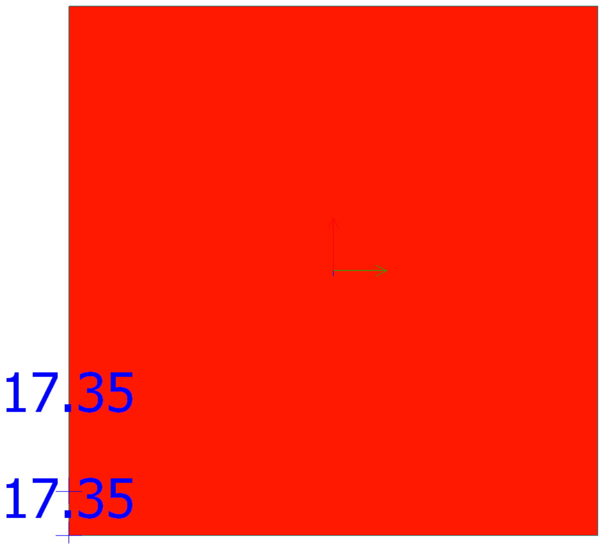


Figure 9.2.2.8 – The $m_{ud\,\xi}$ design moment for elliptic bending with non-orthogonal reinforcement [kNm]

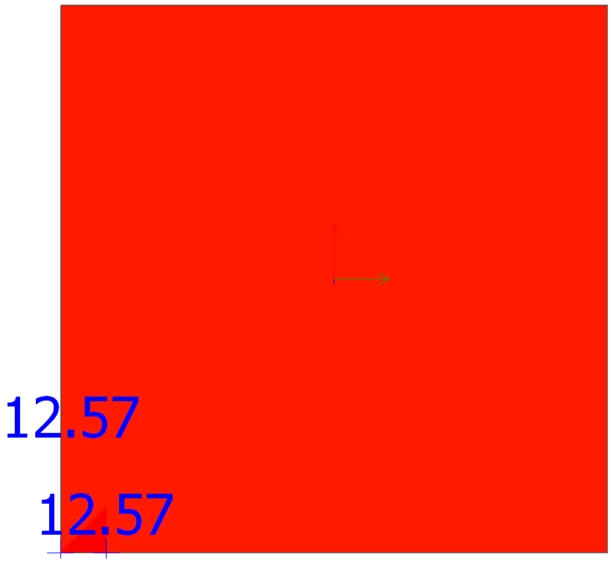


Figure 9.2.2.9 – The $m_{ud\,\eta}$ design moment for elliptic bending with non-orthogonal reinforcement [kNm]

<u>Calculation of the required reinforcement based on the valid design moments:</u>

<u>In x (ξ) direction:</u>

Sum of the moments:

$$m_{ud\,\xi}=f_{cd}\,x_c\left(d_x-\frac{x_c}{2}\right)\quad;\quad 17350=20\,x_c\left(175-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=5.029\,\mathrm{mm}$$

Sum of the forces:

$$x_c\,f_{cd}=a_{s\xi}\,f_{yd}\quad;\quad 5.029\cdot 20=a_{s\xi}\,434.8\quad;\quad a_{s\xi}=0.2313\,\mathrm{mm^2/mm}=231.3\,\mathrm{mm^2/m}$$

Fig. 9.2.2.10 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

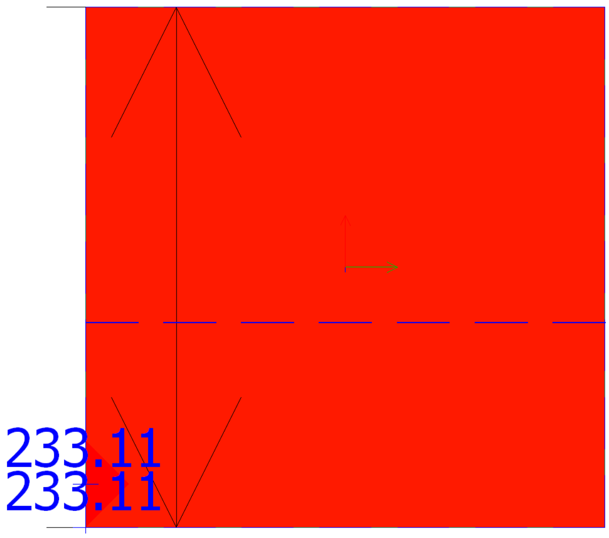


Figure 9.2.2.10 – The $a_{s\xi}$ required reinforcement at the bottom for elliptic bending with non-orthogonal reinforcement [$\mathrm{mm^2/m}$]

In η direction:

Sum of the moments:

$$m_{ud\eta}=f_{cd}x_c\left(d_y-\frac{x_c}{2}\right) \quad ; \quad 12570=20x_c\left(165-\frac{x_c}{2}\right) \quad ; \quad x_c=3.854\,\mathrm{mm}$$

Sum of the forces:

$$x_c f_{cd}=a_{s\eta}f_{yd} \quad ; \quad 3.854\cdot 20=a_{s\eta}434.8 \quad ; \quad a_{s\eta}=0.1773\,\mathrm{mm^2/mm}=177.3\,\mathrm{mm^2/m}$$

Fig. 9.2.2.11 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

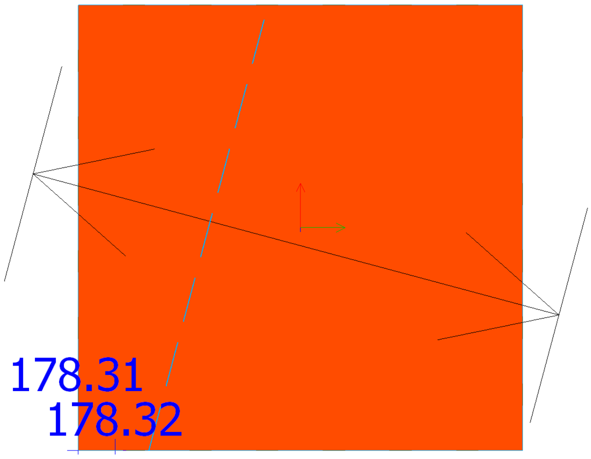


Figure 9.2.2.11 – The $a_{s\eta}$ required reinforcement at the bottom for elliptic bending with non-orthogonal reinforcement [mm²/m]

II.) Hyperbolic bending

In the second case the bending condition is a hyperbolic bending. In FEM-Design the model is a slab with statically determinant support system and specific moment loads at its edges for the pure internal force state (see Fig. 9.2.2.12).

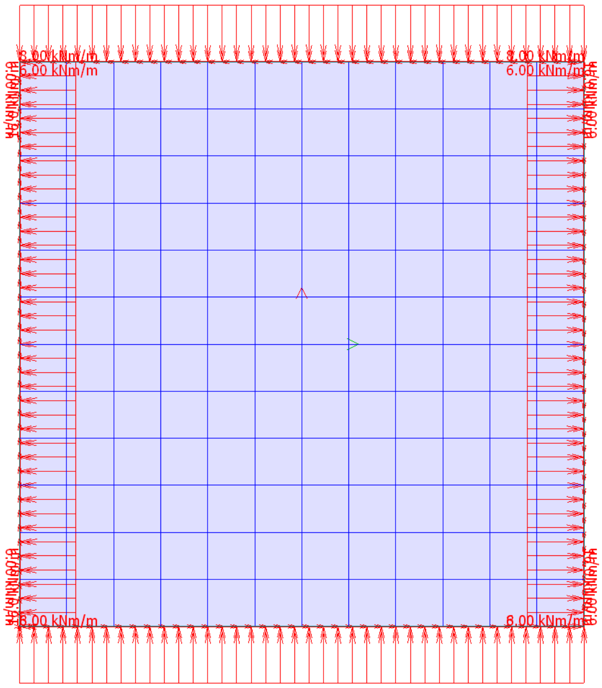


Figure 9.2.2.12 – The slab with the edge loads for pure stress state

Fig. 9.2.2.13 shows the constant internal forces in the slab due to the loads. Fig. 9.2.2.14 shows the principal moments and their directions based on the FEM-Design calculation. According to the pure stress state the principal moments and the directions are the same in each elements.

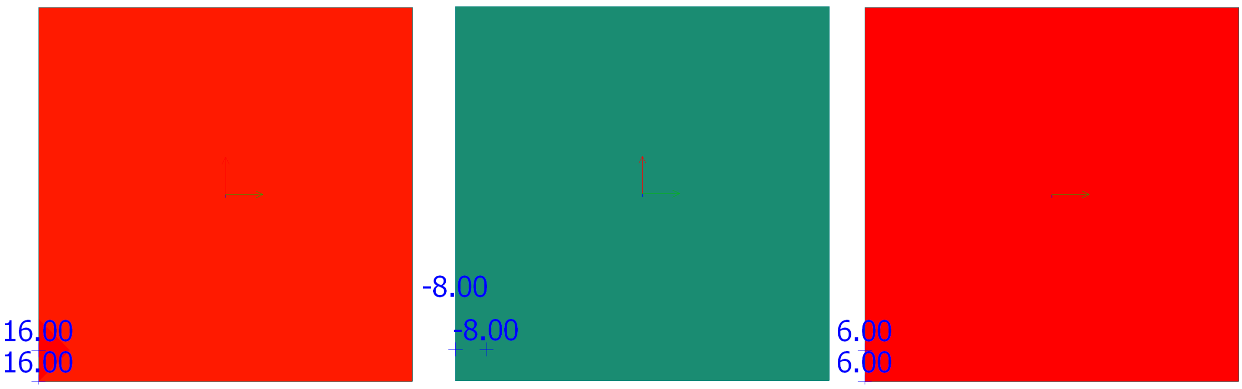


Figure 9.2.2.13 – The m_x, m_y, and m_{xy} internal forces in the slab [kNm]

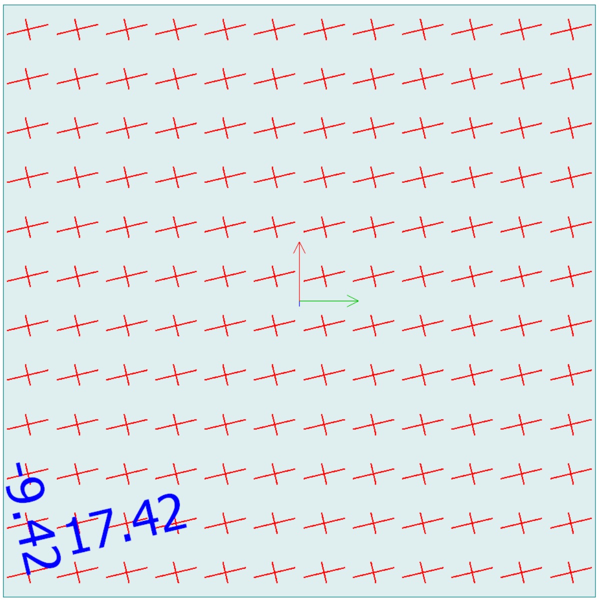


Figure 9.2.2.14 – The m_1 and m_2 principal moments and their directions in the slab [kNm]

Firstly the reinforcement is orthogonal and the hand calculation and the comparison are the following:

1. <u>Orthogonal reinforcement</u>

The reinforcement is orthogonal and their directions concide with the local system (x=ξ, y=ϑ=η).

The moments in the slab (tensor of the applied moments):

$$m_x = m_\xi = +16\,\mathrm{kNm/m}$$

$$m_y = m_\vartheta = m_\eta = -8\,\mathrm{kNm/m}$$

$$m_{xy} = m_{\xi\vartheta} = m_{\xi\eta} = +6\,\mathrm{kNm/m}$$

The first invariant of the tensor: $m_x + m_y = +8\,\mathrm{kNm/m}$

The calculation of the principal moments and their directions:

$$m_1 = \frac{m_x+m_y}{2} + \sqrt{\left(\frac{m_x-m_y}{2}\right)^2 + m_{xy}^{\ 2}} = \frac{16+(-8)}{2} + \sqrt{\left(\frac{16-(-8)}{2}\right)^2 + 6^2} = 17.42\,\mathrm{kNm/m}$$

$$m_2 = \frac{m_x+m_y}{2} - \sqrt{\left(\frac{m_x-m_y}{2}\right)^2 + m_{xy}^{\ 2}} = \frac{16+(-8)}{2} - \sqrt{\left(\frac{16-(-8)}{2}\right)^2 + 6^2} = -9.42\,\mathrm{kNm/m}$$

$$\alpha_0 = \arctan\frac{17.42-16}{6} = \arctan\frac{19.21-16}{6} = 13.32^\circ$$

Compare these results with Fig. 9.2.2.14. The difference is 0 %.

<u>The design moments</u> (according to [9][10]) if the reinforcement (ξ,η) is non-orthogonal:

Case a)

$$m_{ud\,\xi} = m_\xi - m_\vartheta \frac{\cos\varphi}{1+\cos\varphi} + m_{\xi\vartheta} \frac{1-2\cos\varphi}{\sin\varphi} = 16 + 8 \frac{\cos 90^o}{1+\cos 90^o} + 6 \frac{1-2\cos 90^o}{\sin 90^o} = +22\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = m_\vartheta \frac{1}{1+\cos\varphi} + m_{\xi\vartheta} \frac{1}{\sin\varphi} = -8 \frac{1}{1+\cos 90^o} + 6 \frac{1}{\sin 90^o} = -2\,\mathrm{kNm/m}$$

Invalid solution! Because their have different signs.

Case b)

$$m_{ud\,\xi} = m_\xi + m_\vartheta \frac{\cos\varphi}{1-\cos\varphi} - m_{\xi\vartheta} \frac{1+2\cos\varphi}{\sin\varphi} = 16 - 8 \frac{\cos 90^o}{1-\cos 90^o} - 6 \frac{1+2\cos 90^o}{\sin 90^o} = +10\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = m_\vartheta \frac{1}{1-\cos\varphi} - m_{\xi\vartheta} \frac{1}{\sin\varphi} = -8 \frac{1}{1-\cos 90^o} - 6 \frac{1}{\sin 90^o} = -14\,\mathrm{kNm/m}$$

Invalid solution! Because their have different signs.

Case ξ)

$$m_{ud\,\xi} = m_\xi - \frac{m_{\xi\vartheta}^2}{m_\vartheta} = 16 - \frac{6^2}{-8} = +20.5\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = 0$$

This is a valid solution at the bottom!

$m_{ud\,\xi} = +20.5\,\mathrm{kNm/m}$ $m_{ud\,\eta} = 0\,\mathrm{kNm/m}$

Case η)

$$m_{ud\,\xi} = 0$$

$$m_{ud\,\eta} = \frac{m_\xi m_\vartheta - m_{\xi\vartheta}^2}{m_\xi \sin^2\varphi + m_\vartheta \cos^2\varphi - m_{\xi\vartheta} \sin 2\varphi} = \frac{16\cdot(-8)-6^2}{16\cdot\sin^2 90^o + (-8)\cdot\cos^2 90^o - 6\cdot\sin(2\cdot 90^o)} = -10.25 \frac{\mathrm{kNm}}{\mathrm{m}}$$

This is a valid solution at the top!

$m_{ud\,\xi} = 0\,\mathrm{kNm/m}$ $m_{ud\,\eta} = -10.25\,\mathrm{kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.15 and 9.2.2.16. The difference between the hand and FE calculation is 0%.

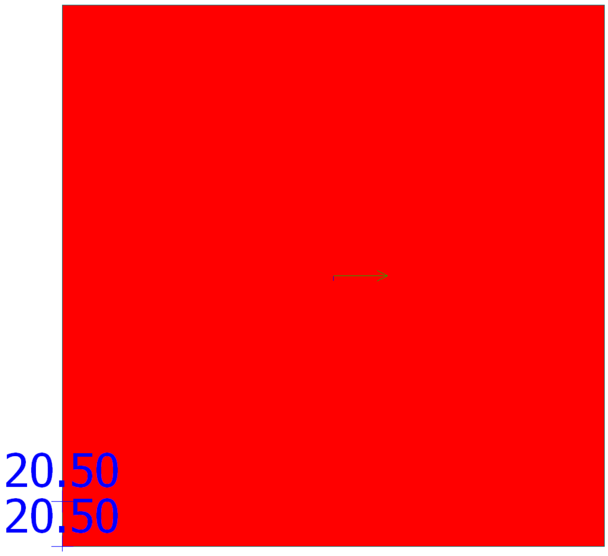


Figure 9.2.2.15 – The $m_{ud\,\xi}$ design moment for hyperbolic bending with orthogonal reinforcement [kNm]

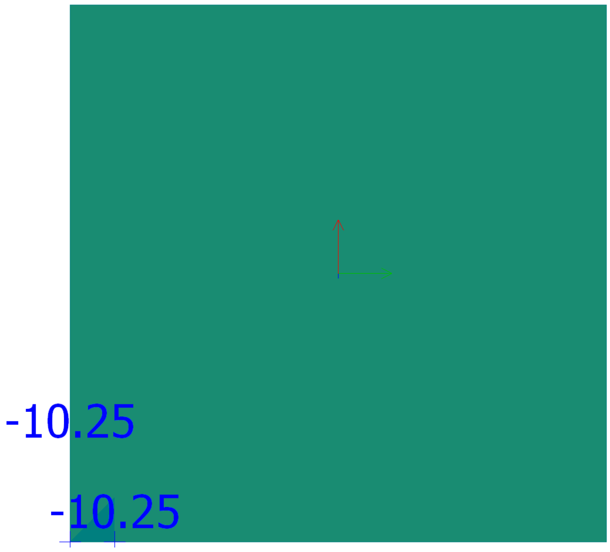


Figure 9.2.2.16 – The $m_{ud\,\eta}$ design moment for hyperbolic bending with orthogonal reinforcement [kNm]

Calculation of the required reinforcement based on the valid design moments:

In x (ξ) direction at the bottom:

Sum of the moments:

$$m_{ud\,\xi}=f_{cd}\,x_c\left(d_x-\frac{x_c}{2}\right)\quad;\quad 20500=20\,x_c\left(175-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=5.959\,\mathrm{mm}$$

Sum of the forces:

$$x_c\,f_{cd}=a_{s\xi}\,f_{yd}\quad;\quad 5.959\cdot 20=a_{s\xi}\,434.8\quad;\quad a_{s\xi}=0.2741\,\mathrm{mm^2/mm}=274.1\,\mathrm{mm^2/m}$$

Fig. 9.2.2.17 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

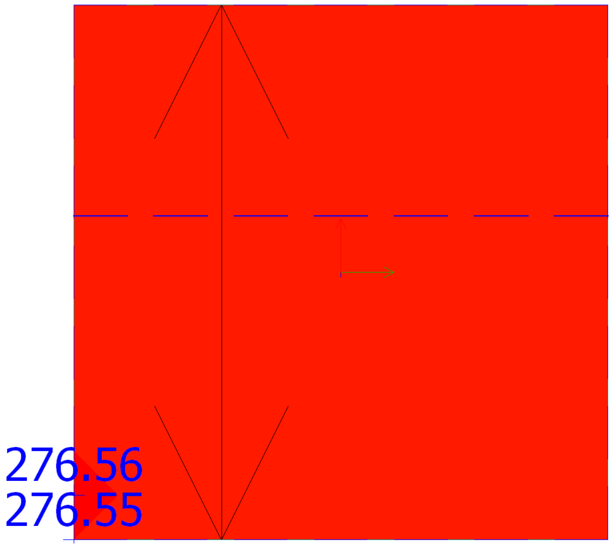


Figure 9.2.2.17 – The $a_{s\xi}$ required reinforcement at the bottom for hyperbolic bending with orthogonal reinforcement [mm²/m]

In y (η) direction at the top:

Sum of the moments:

$$m_{ud\eta}=f_{cd}x_c\left(d_y-\frac{x_c}{2}\right)\quad;\quad 10250=20x_c\left(165-\frac{x_c}{2}\right)\quad;\quad x_c=3.136\,\mathrm{mm}$$

Sum of the forces:

$$x_c f_{cd}=a_{s\eta}f_{yd}\quad;\quad 3.136\cdot 20=a_{s\eta}434.8\quad;\quad a_{s\eta}=0.1443\,\mathrm{mm^2/mm}=144.3\,\mathrm{mm^2/m}$$

Fig. 9.2.2.18 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

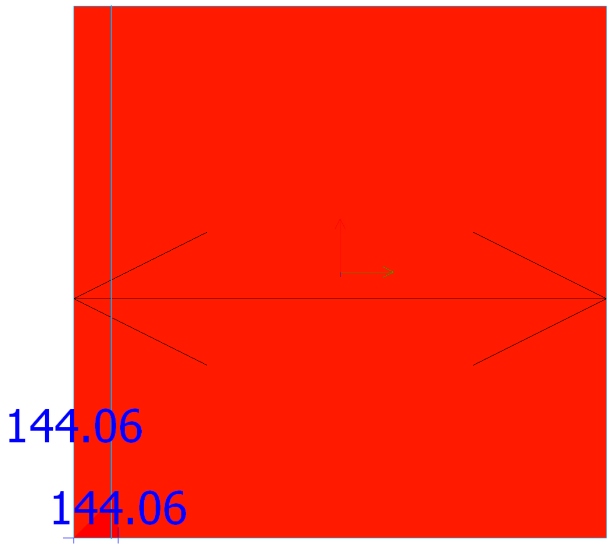


Figure 9.2.2.18 – The $a_{s\eta}$ required reinforcement at the top for hyperbolic bending with orthogonal reinforcement [mm²/m]

Secondly the reinforcement is non-orthogonal and the hand calculation and the comparison are the following:

2. Non-orthogonal reinforcement (φ=75° between ξ and η)

The reinforcement is non-orthogonal and the ξ direction concides with the local x direction. Thus y=ϑ.

The moments in the slab (tensor of the applied moments):

$$m_x = m_\xi = +16\,\mathrm{kNm/m}$$

$$m_y = m_\vartheta = m_\eta = -8\,\mathrm{kNm/m}$$

$$m_{xy} = m_{\xi\vartheta} = +6\,\mathrm{kNm/m}$$

The first invariant of the tensor: $m_x + m_y = +8\,\mathrm{kNm/m}$

The design moments (according to the theory book) if the reinforcement (ξ,η) is orthogonal and their directions concide with the local co-ordinate system (x,y):

Case a)

$$m_{ud\,\xi} = m_\xi - m_\vartheta \frac{\cos\varphi}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1-2\cos\varphi}{\sin\varphi} = 16 + 8\frac{\cos 75^\circ}{1+\cos 75^\circ} + 6\frac{1-2\cos 75^\circ}{\sin 75^\circ} = +20.64\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = m_\vartheta \frac{1}{1+\cos\varphi} + m_{\xi\vartheta}\frac{1}{\sin\varphi} = -8\frac{1}{1+\cos 75^\circ} + 6\frac{1}{\sin 75^\circ} = -0.144\,\mathrm{kNm/m}$$

Invalid solution! Because their have different signs.

Case b)

$$m_{ud\,\xi} = m_\xi + m_\vartheta \frac{\cos\varphi}{1-\cos\varphi} - m_{\xi\vartheta}\frac{1+2\cos\varphi}{\sin\varphi} = 16 - 8\frac{\cos 75^\circ}{1-\cos 75^\circ} - 6\frac{1+2\cos 75^\circ}{\sin 75^\circ} = +3.78\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = m_\vartheta \frac{1}{1-\cos\varphi} - m_{\xi\vartheta}\frac{1}{\sin\varphi} = -8\frac{1}{1-\cos 75^\circ} - 6\frac{1}{\sin 75^\circ} = -17.01\,\mathrm{kNm/m}$$

Invalid solution! Because their have different signs.

Case ξ)

$$m_{ud\,\xi} = m_\xi - \frac{m_{\xi\vartheta}^2}{m_\vartheta} = 16 - \frac{6^2}{-8} = +20.5\,\mathrm{kNm/m}$$

$$m_{ud\,\eta} = 0$$

This is a valid solution at the bottom!

$m_{ud\,\xi} = +20.5\,\mathrm{kNm/m}$ $m_{ud\,\eta} = 0\,\mathrm{kNm/m}$

Case η)

$m_{ud\,\xi} = 0$

$$m_{ud\,\eta} = \frac{m_\xi m_\vartheta - m_{\xi\vartheta}^2}{m_\xi \sin^2\varphi + m_\vartheta \cos^2\varphi - m_{\xi\vartheta} \sin 2\varphi} = \frac{16 \cdot (-8) - 6^2}{16 \cdot \sin^2 75^\circ + (-8) \cdot \cos^2 75^\circ - 6 \cdot \sin(2 \cdot 75^\circ)} = -14.40 \frac{\text{kNm}}{\text{m}}$$

<u>This is a valid solution at the top!</u>

$m_{ud\,\xi} = 0\,\text{kNm/m}$ $m_{ud\,\eta} = -14.40\,\text{kNm/m}$

The results of the design moments based on FEM-Design are in Fig. 9.2.2.19 and 9.2.2.20. The difference between the hand and FE calculation is 0%.

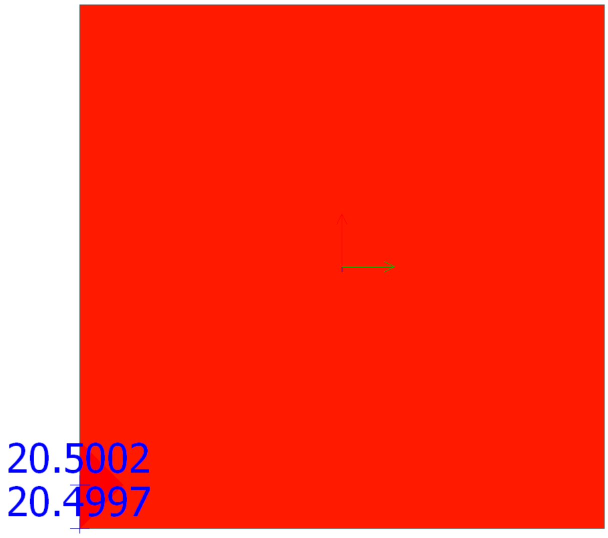


Figure 9.2.2.19 – The $m_{ud\,\xi}$ design moment for hyperbolic bending with non-orthogonal reinforcement [kNm]

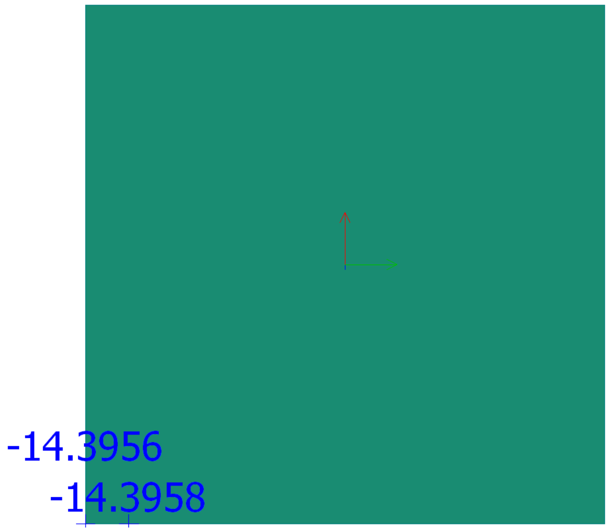


Figure 9.2.2.20 – The $m_{ud\,\eta}$ design moment for hyperbolic bending with non-orthogonal reinforcement [kNm]

Calculation of the required reinforcement based on the valid design moments:

In x (ξ) direction at the bottom:

Sum of the moments:

$$m_{ud\,\xi}=f_{cd}\,x_c\left(d_x-\frac{x_c}{2}\right)\quad;\quad 20500=20\,x_c\left(175-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=5.959\,\mathrm{mm}$$

Sum of the forces:

$$x_c\,f_{cd}=a_{s\xi}\,f_{yd}\quad;\quad 5.959\cdot 20=a_{s\xi}\,434.8\quad;\quad a_{s\xi}=0.2741\,\mathrm{mm^2/mm}=274.1\,\mathrm{mm^2/m}$$

Fig. 9.2.2.21 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

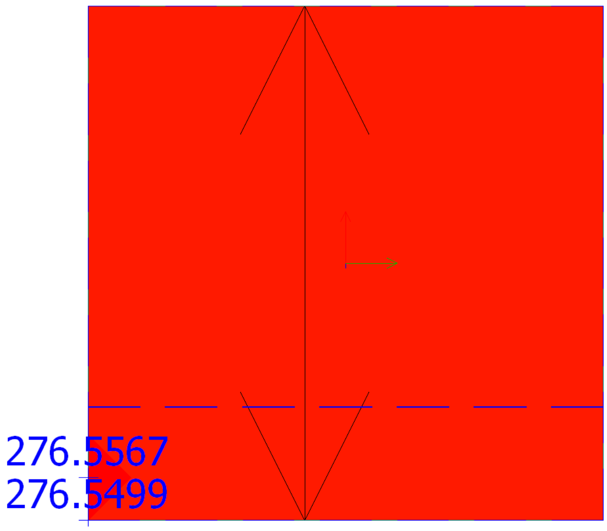


Figure 9.2.2.21 – The $a_{s\xi}$ required reinforcement at the bottom for hyperbolic bending with non-orthogonal reinforcement [mm^2/m]

In η direction at the top:

Sum of the moments:

$$m_{ud\,\eta}=f_{cd}\,x_c\left(d_y-\frac{x_c}{2}\right)\quad;\quad 14400=20\,x_c\left(165-\frac{x_c}{2}\right)\quad;\quad \mathrm{x_c}=4.423\,\mathrm{mm}$$

Sum of the forces:

$$x_c\,f_{cd}=a_{s\eta}\,f_{yd}\quad;\quad 4.423\cdot 20=a_{s\eta}\,434.8\quad;\quad a_{s\eta}=0.2034\,\mathrm{mm^2/mm}=203.4\,\mathrm{mm^2/m}$$

Fig. 9.2.2.22 shows the required reinforcement in the relevant direction based on FEM-Design calculation. The difference is less than 1%.

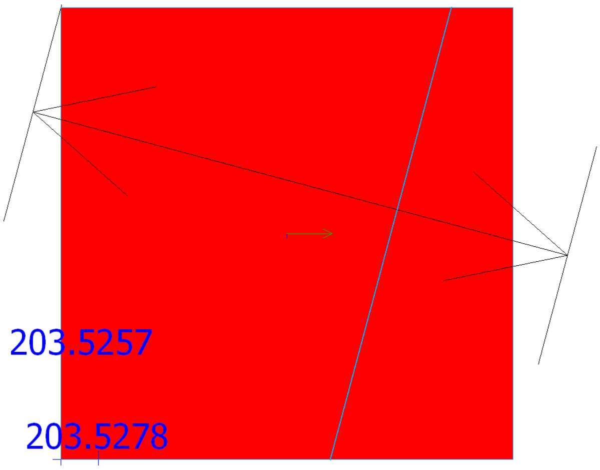


Figure 9.2.2.22 – The $a_{s\eta}$ *required reinforcement at the top for hyperbolic bending with non-orthogonal reinforcement* [mm²/m]

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Notes